Non Linear Memory from Reverse Unitarity

A. Georgoudis, 16/04/26

Amplitudes, Strong-Field Gravity and Resummation

Based on Arxiv: 2506.20733

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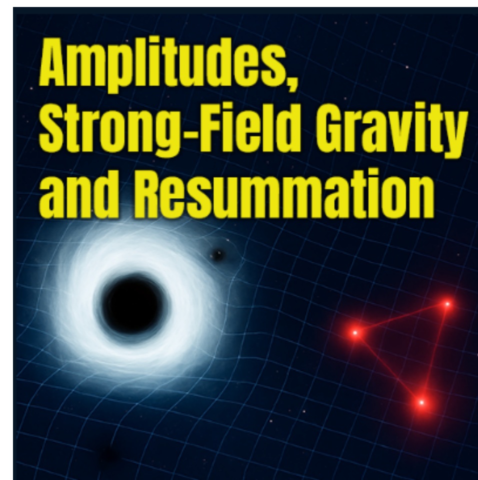
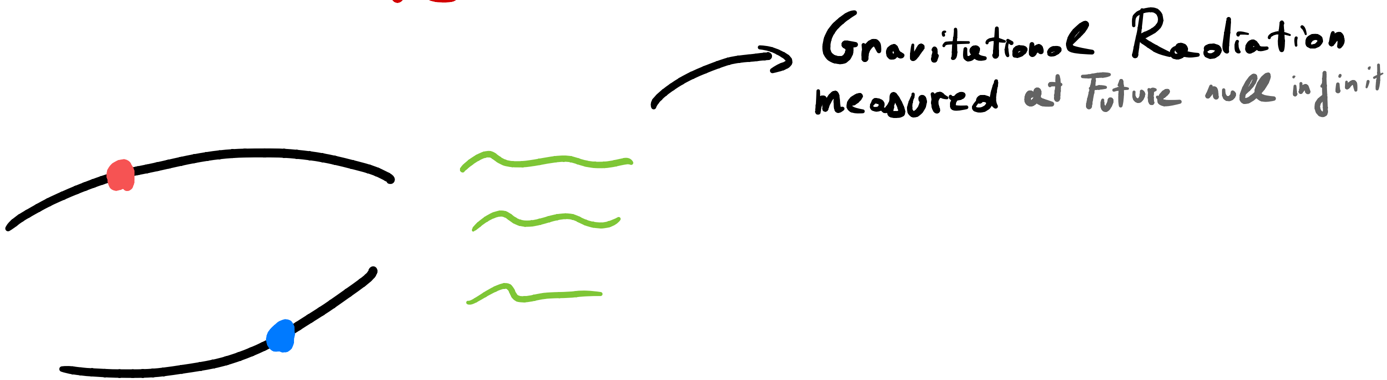


PM WaveForms



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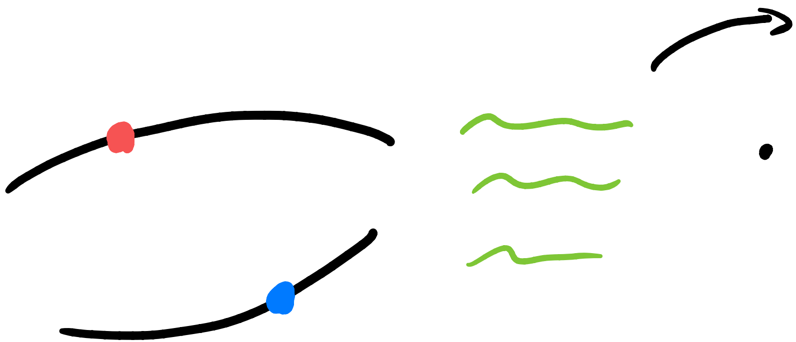
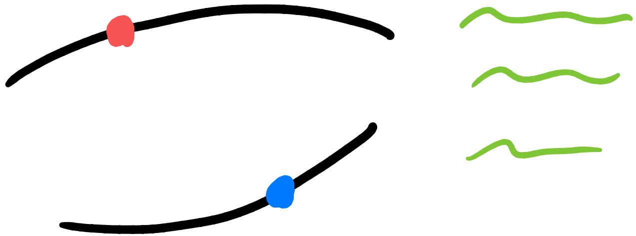


Gravitational Radiation measured at Future null infinity

- $\Delta \mathcal{O} = \langle \text{out} | \mathcal{O} | \text{out} \rangle - \langle \text{in} | \mathcal{O} | \text{in} \rangle$

$$= \langle \text{in} | S^{*} \mathcal{O} S - \mathcal{O} | \text{in} \rangle$$

PM WaveForms



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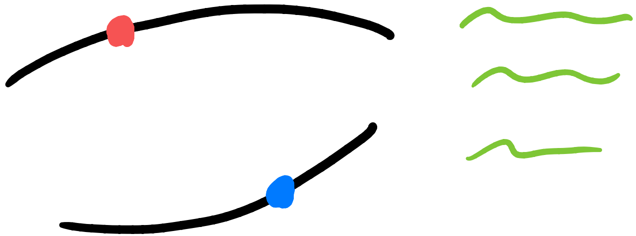
$$= \langle in|\, \mathbb{S}^{*}\mathbb{O}\,\mathbb{S} - \mathbb{O}\,|in\rangle$$

$$|out\rangle = \mathbb{S}\,|in\rangle \qquad \mathbb{S} = 1 + iT \approx e^{i\delta}$$

PM Wave Forms

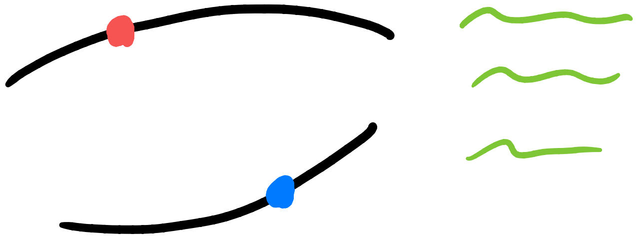


Gravitational Radiation measured at Future null infinity

- $\Delta \Theta \doteq \langle out | \Theta | out \rangle - \langle in | \Theta | in \rangle$
  $= \langle in | S^{*} \Theta S - \Theta | in \rangle$

$|out\rangle = S |in\rangle \qquad S = 1 + iT \approx e^{iN}$

PM WaveForms



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  $= \langle in|S^{*}\mathbb{O}S - \mathbb{O}|in\rangle$

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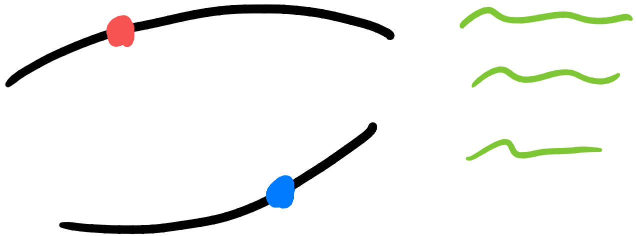
• $h_{\mu\nu} \approx \langle out|\hat{H}_{\mu\nu}|out\rangle \sim \int_0^\infty \frac{d\omega}{2\pi}\, e^{-i\omega u}\, \tilde{W}_{\mu\nu}(\omega n) + c.c.$

PM WaveForms



Gravitational Radiation measured at Future null infinity

- $\Delta \Theta = \langle out|\Theta|out\rangle - \langle in|\Theta|in\rangle$
  $= \langle in|S^{\dagger}\Theta S - \Theta|in\rangle$

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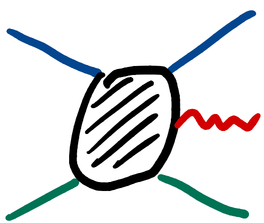
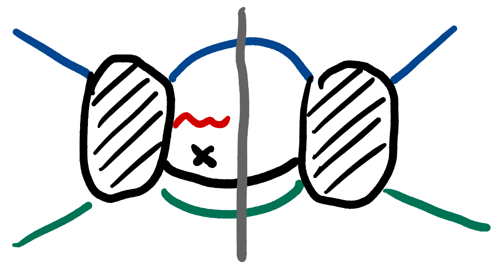
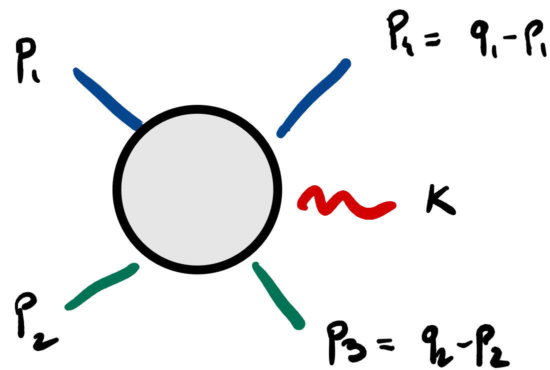
  ↑ $FT[W_{\mu\nu}]$

$W_{\mu\nu} =$  $- i\sum_{X}$ 

Expand in $\frac{GM}{rc^2} \ll 1$ Post Minkowskian regime

WaveForm @ LO

$W_0^{\mu\nu} \sim$



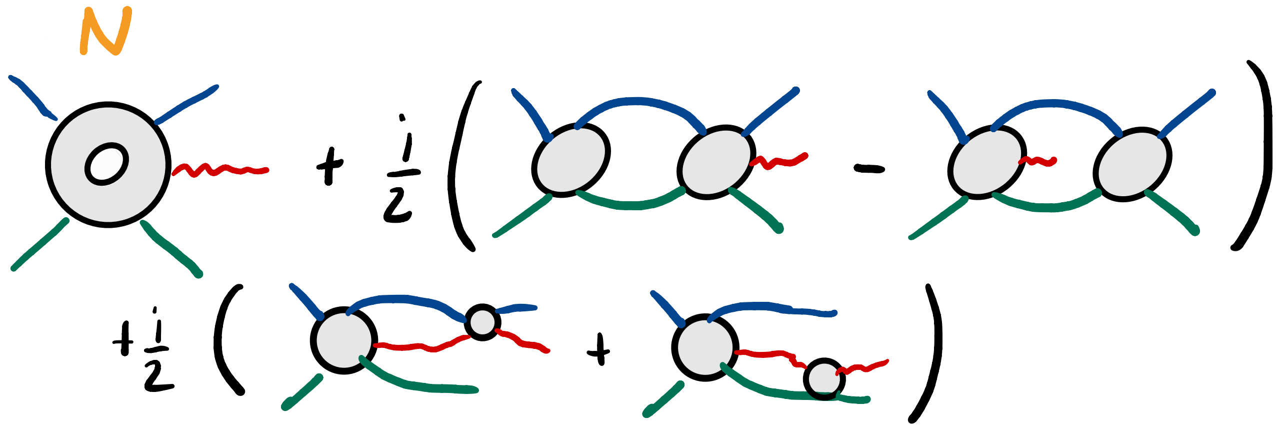
Usefuel Quantities

- $\sigma = -v_1 \cdot v_2$
- $\omega_1 = -v_1 \cdot k$
- $\omega_2 = -v_2 \cdot k$

Several results available in Time / Frequency domain, FT analitically Known.

$$\hookrightarrow \int d^D q_1 \, d^D q_2 \, (2\pi)^D \delta^{(D)}(q_1+q_2+k) \, 2\pi\delta(2p_1\cdot q_1) \, 2\pi\delta(2p_2\cdot q_2) \, e^{ib_1\cdot q_1 + ib_2\cdot q_2} \, f(q_1,q_2,k)$$

WaveForm @NLO

$W_1^{\mu\nu} \sim$ 

WaveForm @NLO

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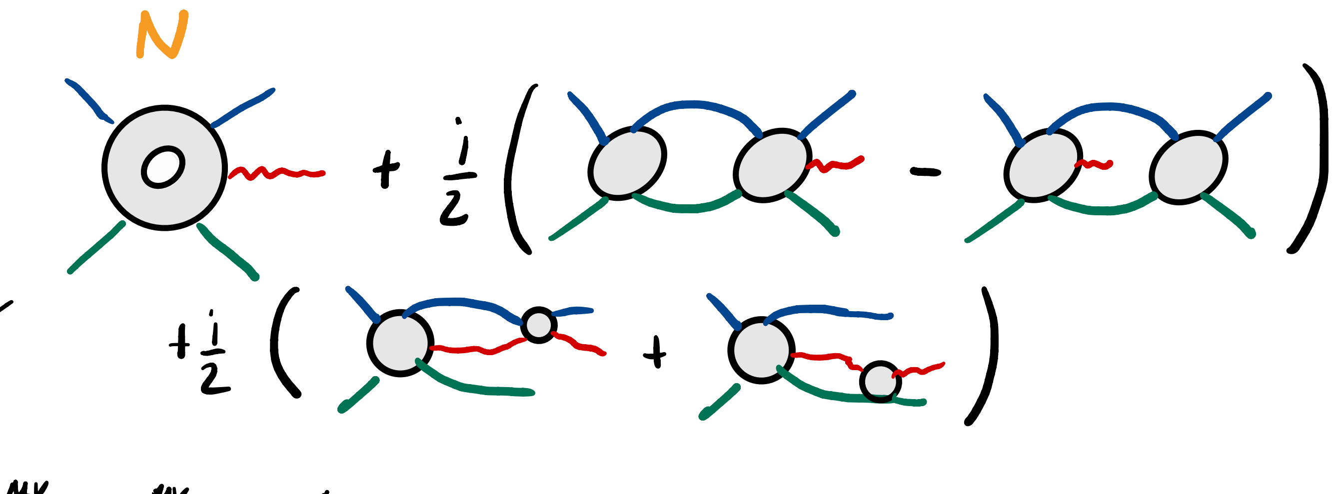
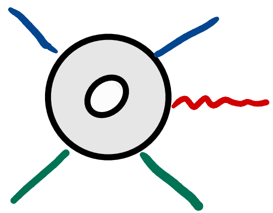
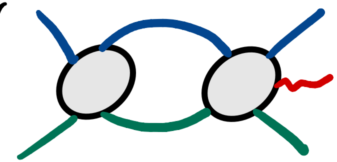
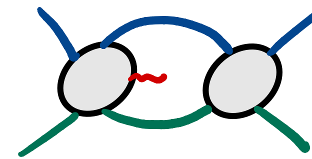


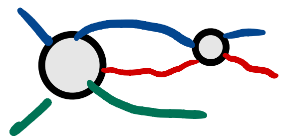
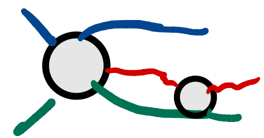
$$B_1^{\mu\nu} = B_{1E}^{\mu\nu} + \underbrace{B_{1O}^{\mu\nu}}_{f(\sigma, w_1, w_2)\, A_0^{\mu\nu}}$$

(even/odd $w_{1,2} \to -w_{1,2}$)

WaveForm @NLO

N

$W_1^{\mu\nu} \sim$  $+ \frac{i}{2}\Big($  $-$  $\Big)$

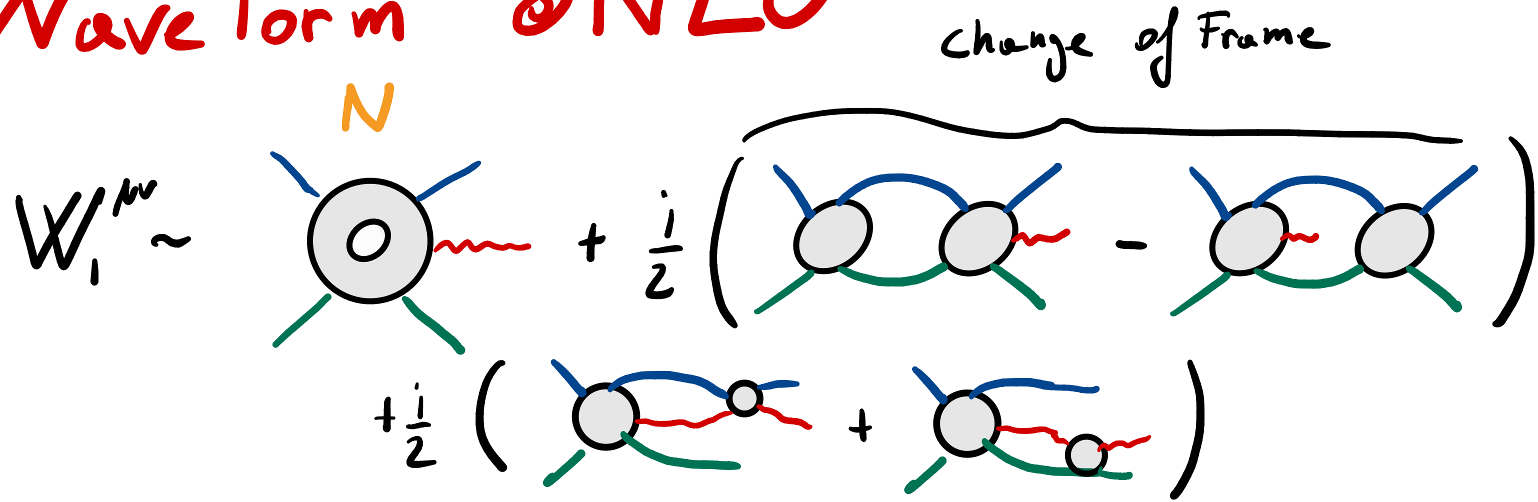
$+ \frac{i}{2}\Big($  $+$  $\Big)$

IR divergence: $C_1^{\mu\nu} \approx \left(-\frac{1}{2\varepsilon} + \log\frac{\omega_1}{\mu}\right) A_0^{\mu\nu} + C_1^{(reg)}$

$$W_1^{\mu\nu} = \underline{e^{-\frac{i}{\varepsilon} G E \omega}} \left(A_0^{\mu\nu} + B_1^{\mu\nu} + \frac{i}{2} C\right)$$

↳ divergence can be absorbed by shift of retarded Time

Waveform @NLO



- Fix BMS frame to match with MPM expansion
- Complicated form in Frequency domain, recent developments

1-Loop Waveform via IBPs directly with FT!

WaveForm @NNLO

Soft WaveForm @NNLO

- What is the simplest interesting effect?

$$\left(\lim_{T\to\infty} w_{\mu\nu}(T,n)\right) - \left(\lim_{T\to-\infty} w_{\mu\nu}(T,n)\right) = \boxed{-F_{\mu\nu}}$$

→ Permanent displacement caused by gravitational radiation.

Wave Form @NNLO $\omega \to 0$

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→ Permanent displacement

- Usually discussed in Frequency domain ($\omega \to 0$ limit)

$$\tilde{w}_{\mu\nu} \sim -\frac{i}{\omega} F_{\mu\nu} + O(\log \omega)$$

Determined by the soft theorem

$$F_{\mu\nu} = f_{\mu\nu} + \delta f_{\mu\nu}$$

$f_{\mu\nu}$: linear

$\delta f_{\mu\nu}$ → non-linear (displacement caused by the radiation itself)

Soft Theorems and Memory

- According to Weinberg the emission of a soft graviton factorizes

$$\mathcal{S}_{s.r.} = \underbrace{e^{i\mathrm{Re}\,2S}}_{\text{Hard scattering}}\; e^{\int_k (F a^\dagger - F^* a)}$$

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$$\mathcal{S}_{s.r.} = \underbrace{e^{i\mathrm{Re}\,2S}}_{\text{Hard scattering}} \; e^{\int_k (F a^\dagger - F^* a)} \leftarrow \text{soft emission}$$

If we plug this in into the definition of waveform $h_{\mu\nu}$

$$h_{\mu\nu} \approx \int_k \left(F_{\mu\nu} e^{ikx} + F^*_{\mu\nu} e^{-ikx} \right) \longrightarrow F_{\mu\nu} = \sum_a \sqrt{8\pi G} \frac{p_a^\mu p_a^\nu}{p_a \cdot k - i0}$$

which we can write

$$\underset{\text{past infinity}}{h_{\mu\nu}} \sim -\frac{4G}{r} \sum_{in} \frac{(K_a^\mu K_a^\nu)_{T.T}}{K_a \cdot n}$$

$$\underset{\text{future infinity}}{h_{\mu\nu}} \sim -\frac{4G}{r} \sum_{in} \frac{(K_a^\mu K_a^\nu)_{T.T}}{K_a \cdot n}$$

Linear memory is sourced by the IN and OUT particles which differ due to the in deflection.

Soft Theorems and Memory II

- What about the graviton? Can it also contribute to the memory effect?

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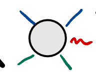
Let us look at **Energy momentum emission spectrum**

$$k^\mu \, dp(k) = k^\mu \left| FT\left(\right. \right.$$  $$\left.\left. \right) \right|^2 2\pi \, \theta(k^0) \, \delta(k^2) \frac{d^D k}{(2\pi)^D} + \mathcal{O}(G^4)$$

Soft Theorems and Memory II

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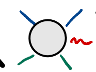
Let us look at Total Radiated Energy-momentum

$$P^\mu = \int k^\mu \, d\rho(k) = \int k^\mu \left| FT\left(\right.\right.$$



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Soft Theorems and Memory II

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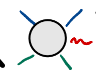
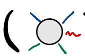
Let us look at number of emitted gravitons

Number of Emitted gravitons $\quad dp(k) = \Big|FT\Big($  $\Big)\Big|^2 2\pi\,\theta(k^0)\,\delta(k^2)\frac{d^D k}{(2\pi)^D} + \mathcal{O}(G^2)$

So the soft gravitons contribution can be written as

$$\tilde{W}^{\mu\nu}_{NL} = \int \sqrt{8\pi G}\,\frac{(K^\mu K^\nu)_{TT}}{K\cdot K_0 - i0}\,dp(k) = \int \frac{\sqrt{8\pi G}}{K\cdot K_0 - i0}(K^\mu K^\nu)_{TT}\Big|FT\Big(\cdots\Big)\Big|^2 2\pi\,\theta(k^0)\,\delta(k^2)\frac{d^D k}{(2\pi)^D}$$



Wave Form @NNLO $\omega \to 0$

- What is the interesting effect?

$$\left(\lim_{T\to\infty} w_{\mu\nu}(T,n)\right) - \left(\lim_{T\to-\infty} w_{\mu\nu}(T,n)\right) = -F_{\mu\nu}$$

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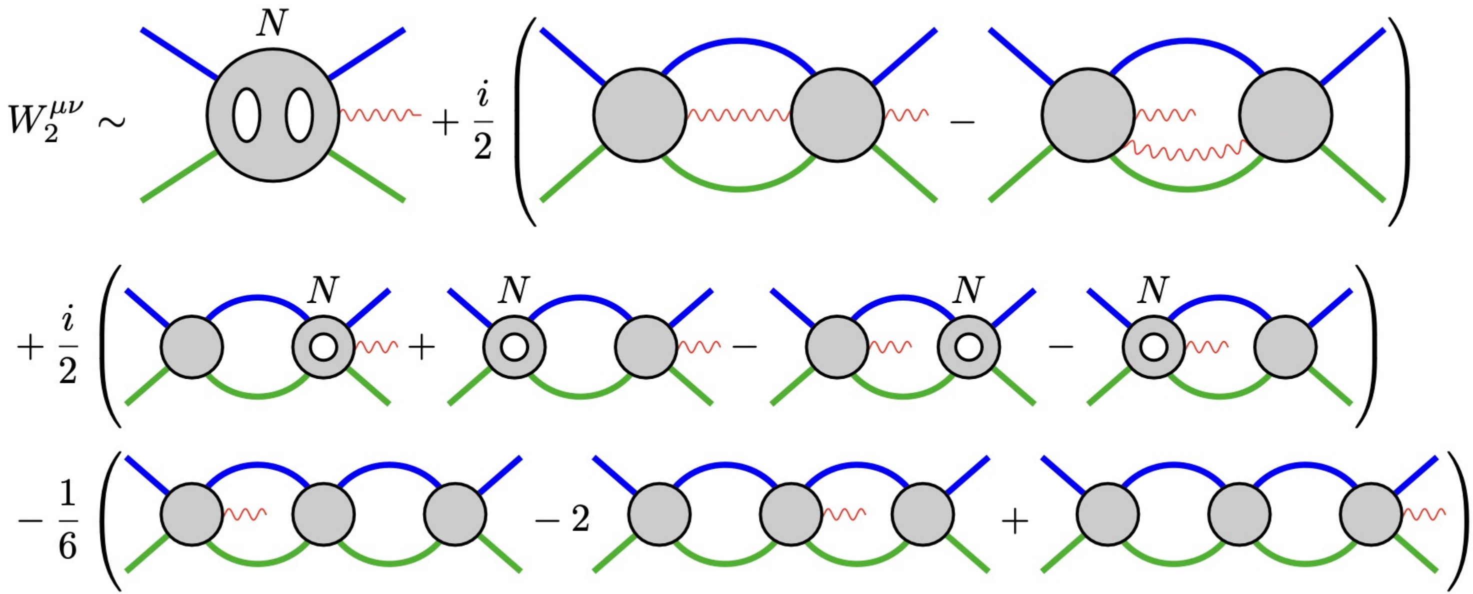
$f_{\mu\nu}$: $\sum_{a=1}^{4} \frac{p_a^\mu p_a^\nu}{p_a \cdot n}$

$\delta f_{\mu\nu}$: $\int \tilde{dk}\, \rho(k) \frac{k^\mu k^\nu}{k\cdot n}$ → First appears at NNL

WaveForm @NNLO

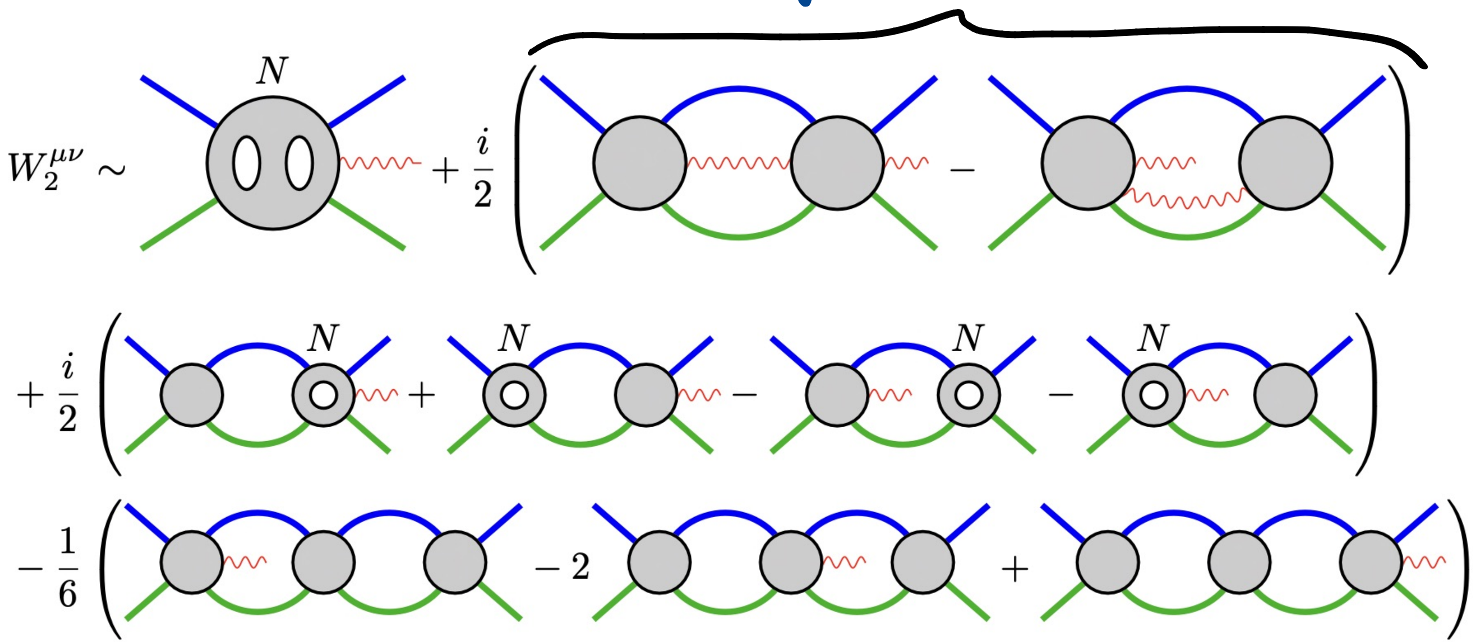
ω->0



WaveForm @NNLO

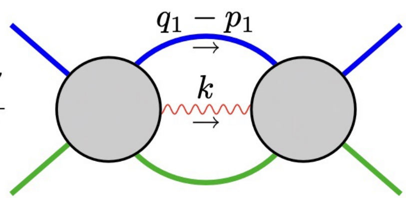
ω->0

Part of Linear + NON-LINEAR (Gauge inv)



NON Linear Memory

$$\delta F^{\mu\nu} = \mathrm{FT}_4 \int_{q_1} \int_k \frac{k^\mu k^\nu}{k \cdot n}$$



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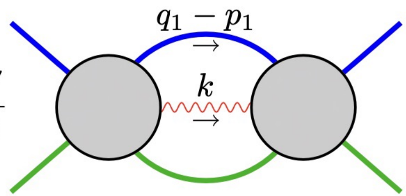
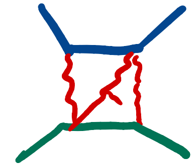
NON Linear Memory

Known 4-point families
but still a function of 3-variables..

$$\delta F^{\mu\nu} = \mathrm{FT}_4 \int_{q_1} \int_k \frac{k^\mu k^\nu}{k \cdot n}$$ ,



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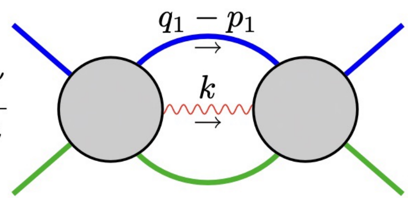

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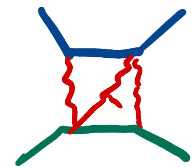


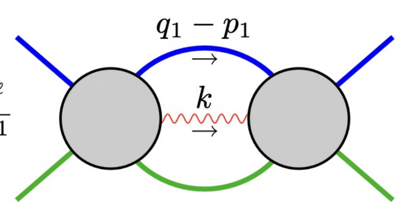
elliptic subsector...

- We can further simplify this by studying directly the multipole decomposition of $\delta F^{\mu\nu}$

$$\delta F^{\ell m} = \oint Y^{\ell m *}_{\pm 2}\,\delta F_{\mu\nu}\,\varepsilon^{\mu}_{\pm}\varepsilon^{\nu}_{\pm}\,d\Omega\,. \;\rightarrow\; \delta F^{\ell m} = -\frac{2\pi}{\ell(\ell-1)}\mathcal{N}^{\ell m}_{2} Y^{\ell m *}_{\mu_1\cdots\mu_\ell}\,\mathrm{FT}_4[\mathbb{I}^{\mu_1\cdots\mu_\ell}]\,,$$

Where :

$$\mathbb{I}^{\mu_1\cdots\mu_\ell} = \int_{q_1}\int_k \frac{k^{\mu_1}\ldots k^{\mu_\ell}}{(-t\cdot k)^{\ell-1}}$$



and we are free to chose a particular R.F. (ex. sitting on p_1)

NON Linear Memory: Multipoles

For each ℓ, m simple dependence on σ $\quad \mathcal{N}_2^{\ell,m}$ is a normalization

$$\delta F^{\ell,m} = \frac{G^3 \pi^2 m_1^2 m_2^2}{b^3(\sigma^2-1)} \, i^{\ell} \mathcal{N}_2^{\ell m} F^{\ell,m}(\sigma) + \ldots$$

$$\hookrightarrow (\sigma^2-1)^{-\frac{\ell}{2}} \left[f_1^{\ell m} + f_2^{\ell m} \log\left(\frac{\sigma+1}{2}\right) + f_3^{\ell m} \frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} \right]$$

f_i polynomials in σ $\left(f_2^{22} = \frac{(\sigma^2-1)^2}{8} \left(35\sigma^4 + 420\sigma^3 - 510\sigma^2 + 2132\sigma + 67\right) \right)$

Match with PN expansion! Also:

$\delta F^{\ell,m} = 0$ if $\ell + m$ is odd

$\delta F^{\ell,-m} = (-1)^{\ell} \delta F^{\ell,m}$ sym of $\rho(k)$ for $b \to -b$

NON Linear Memory II

What happens if we stay in c.o.m frame?

Extra propagator add new scale ($\lambda = \frac{m_2}{m_1}$) and complexity... $(\lambda \, l \cdot u_2 + l \cdot u_1)$

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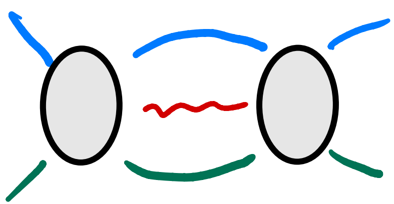
Can we go higher?



NON Linear Memory II

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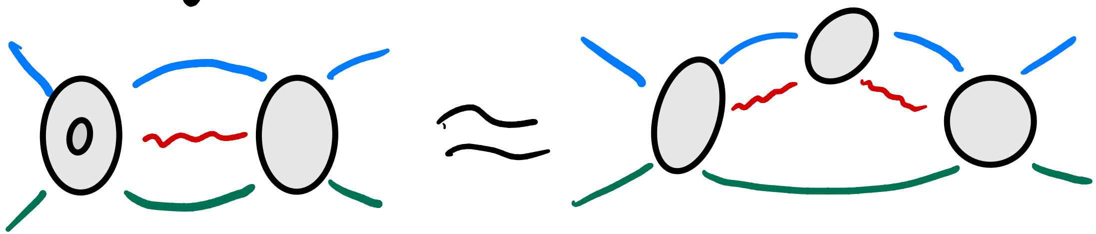
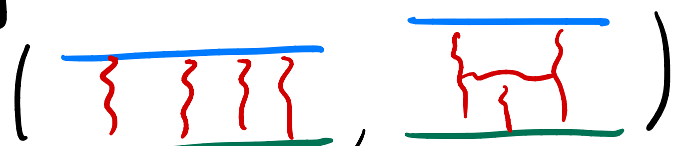
Extra propagator add new scale ($\lambda = \frac{m_2}{m_1}$) and complexity... ($\lambda\, \ell \cdot u_2 + \ell \cdot u_1$)

Can we go higher?

Real part difficult (Except B_{10}!)



New topologies



$\frac{1}{t \cdot K}$

We expect at most L_{12}

Conclusion & Outlook

- Presented computation of the non-linear memory from amplitudes and PM expansion.

- Needed to add higher point cuts. Does this indicates correlation between classical observables?

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- Presented computation of the non-linear memory from amplitudes and PM expansion.

- Needed to add higher point Ampl. Does this indicates correlation between classical observables?

- Compute full NNLO Waveform?
- Understand Memory in detector formalism?
- Squeezed / Non-squeezed Classical gravitational Radiation?