NORDITA
The Nordic Institute for Theoretical Physics

Gravitational Sommerfeld Effects:

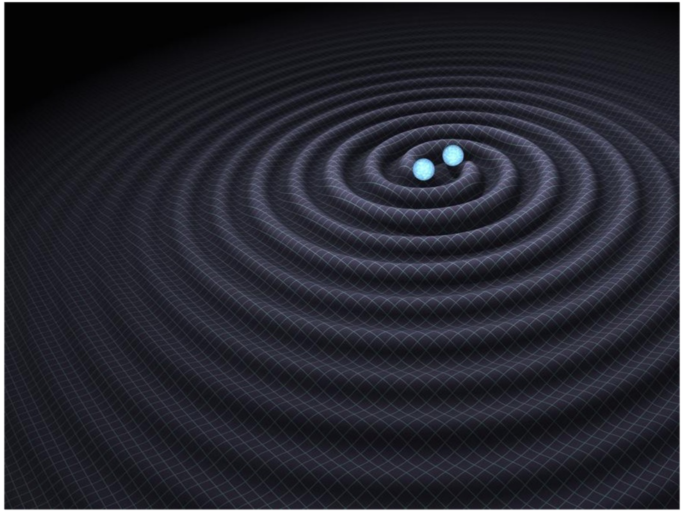
formalism, renormalization and perturbation to $O(G^{10})$

Chih-Hao Chang (Nat'l Taiwan Univ.)

w/ Chia-Hsien Shen and Zihan Zhou

arXiv: 2604.14112

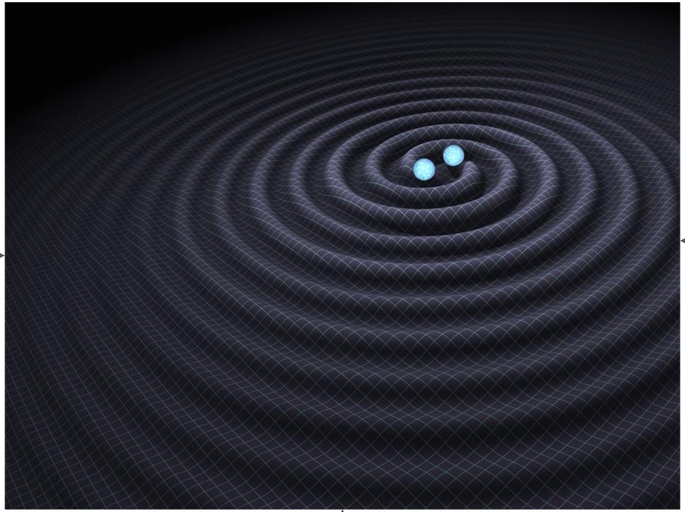
Challenges to High-Precision Waveform



We need the significant improvement on our waveform precision
for current and future experiments

Waveform Modeling: Current Methods

PM/PN

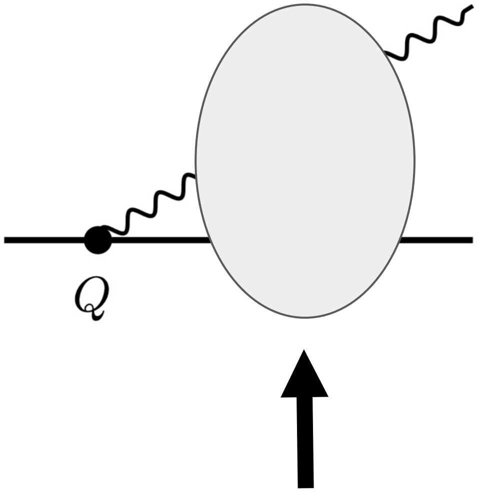


BHPT/SF

EOB

See many great talks in this workshop!

Waveform Modeling: Coulomb Potential Example



All order effects of Coulomb potential $V = \frac{\alpha}{r}$

$$\mathcal{S}_\ell = e^{4\pi\alpha} \frac{\Gamma\left(\ell + 1 - 2i\alpha\right)}{\Gamma(\ell + 1)}$$

Coulomb Sommerfeld Factor

Waveform Modeling: Goals

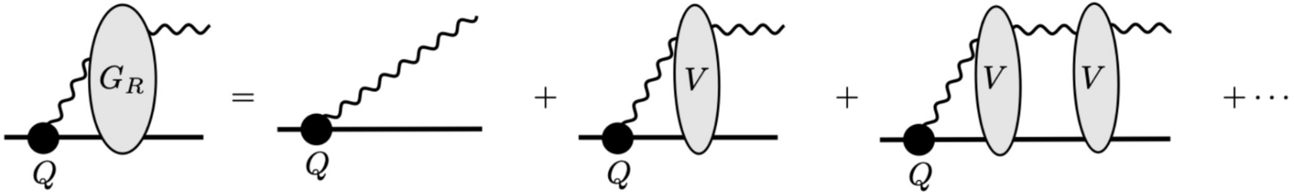
Goals: Achieve the same for generic gravitational sources

Challenges:

- Systematic structures to all orders
- High-precision perturbation Resummation

Sommerfeld Factor

- Localized source: GMω << 1
  - Q can be: binary or any localized source
- Diagrammatic waveform can be written as wave equation [Caron-Huot et al.]



$$\phi = \phi_0 + G_0 V \phi_0 + (G_0 V)^2 \phi_0 + \cdots = \phi_0 + G_0 V \phi$$

$$\left(\nabla^2 + \omega^2\right)\phi = V\phi + Q\delta(\boldsymbol{r})$$

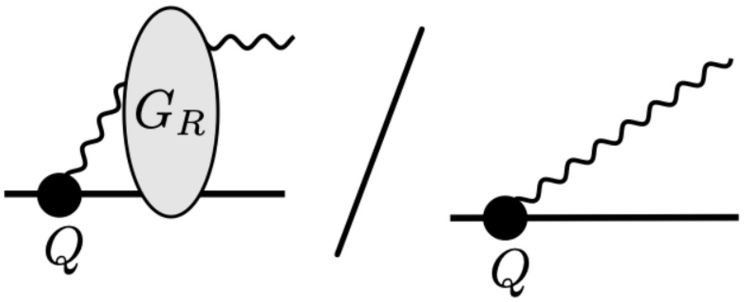
Sommerfeld Factor

- The waveform has a resummed structure

$$\phi = \frac{1}{1 - G_0 V}\phi_0 \quad \longrightarrow \quad \frac{\phi}{\phi_0} = \frac{1}{1 - G_0 V}$$

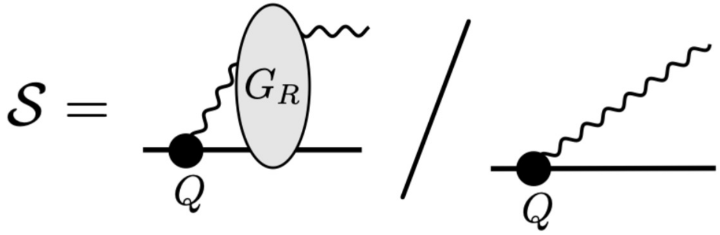
- We define the **Sommerfeld factor**

$\mathcal{S} =$



Sommerfeld Factor: Current Methods

- Traditional PN/MPM: [Blanchet, Damour; …]
- PN EFT: [Rothstein, Goldberger; Goldberger, Ross; …]
- Resummation: [Damour, Nagar; …]
- Synergy w/ BHPT [Ivanov, Li, Parra-Martinez, Zhou]



Structures of the Sommerfeld Factor

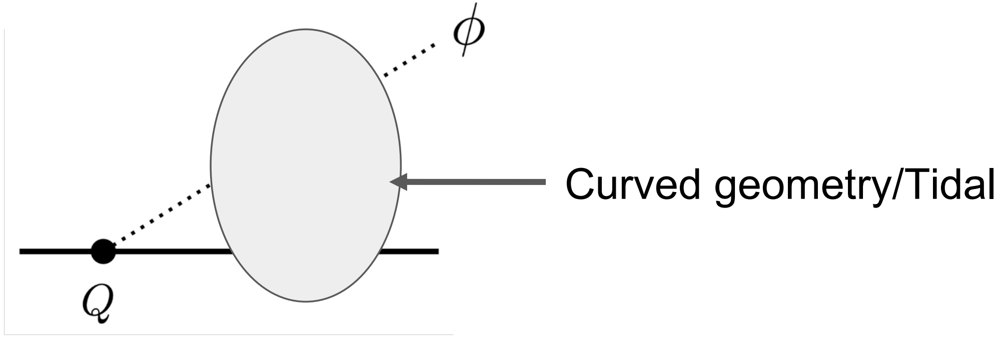
- Universal
  - Birkhoff thm. ⟶ Exterior = Schwarzschild ⟶ universal “tail” effect
  - Logs renormalization group (RG)
  - captured by renormalized angular momentum [Ivanov et al.]
- Non-universal
  - Tidal effects
  - Can be generated by RG

Outline

- Introduction
- **EFT Formalism**
- Sommerfeld Factor
- Renormalization
- Synergy with BHPT
- Resummation
- Conclusion

Scalar Field Model

- We study the waveform of a scalar ϕ
  - We keep the full curved spacetime background and tidal effects on the worldline
  - No gauge ambiguity compared to the gravitational field
  - Do not consider recoil on the worldline



EFT Action of Scalar Toy Model

Bulk **Tidal** **Source**

$$S_{\mathrm{EFT}} = \frac{1}{2}\int d^d x \sqrt{-g}(\nabla\phi)^2 + \sum_{\ell,n} C_{n,\ell} \int d\tau\, \nabla_\ell \phi \partial_\tau^n \nabla_\ell \phi + \sum_\ell \int d\tau\, Q_\ell \nabla_\ell \phi$$

Minimally coupled scalar in Sch. background

Love number

even n: cons.
odd n: diss.

Multipole moments

[Caron-Huot et al.]

New

Important Inputs in the EFT

- G expansion $(r_s = 2GM)$:

$$g_{rr} = \left(1 - \frac{r_s^{d-3}}{r^{d-3}}\right)^{-1} \longrightarrow \sum_{n=0}^{\infty} \left(\frac{r_s^{d-3}}{r^{d-3}}\right)^n$$ ⟶ No horizon!

- dimensional regularization d = 4 - 2ε:

$$\left.\frac{1}{r^{1-2\epsilon}}\right|_{r=0} = 0$$ ⟶ Flat metric at r = 0

  - Regulate the universal IR and UV div. (leads to RG)

EFT Equation of Motion

- We can derive the EOM from the action:

$$(\nabla^2 + \omega^2 + V_{\text{grav}}^{\text{EFT}})\phi = V_{\text{tidal}}\phi + Q_\ell \partial_\ell \delta^{d-1}(\boldsymbol{r})$$

$$\uparrow$$

$$\sum_{n=1}^{\infty} \left(\frac{r_s^{1-2\epsilon}}{r^{1-2\epsilon}} \right)^n \left[\frac{2\epsilon - 1}{r} \frac{d}{dr} + \frac{\ell^2 + \ell + 1 - \epsilon(2\ell + 3) + 2\epsilon^2}{r^2} - (n+1)\omega^2 \right]$$

EFT Equation of Motion

- We can derive the EOM from the action:

$$(\nabla^2 + \omega^2 + V_{\text{grav}}^{\text{EFT}})\phi = V_{\text{tidal}}\phi + Q_\ell \partial_\ell \delta^{d-1}(\boldsymbol{r})$$

$$\sum_\ell \frac{(-1)^\ell}{\ell!} F_\ell(\omega) \partial_\ell \delta^{d-1}(\boldsymbol{r}) \partial_\ell \qquad F_\ell = \sum_n C_{n,\ell}(i\omega)^n$$

Tidal response function
even n: conservative
odd n: dissipative

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Solving the EFT EOM

- Feynman diagram method becomes complicated for higher loops.
- We decompose the scalar field on spherical harmonic:
  - Reduce to a **2nd order ODE**.

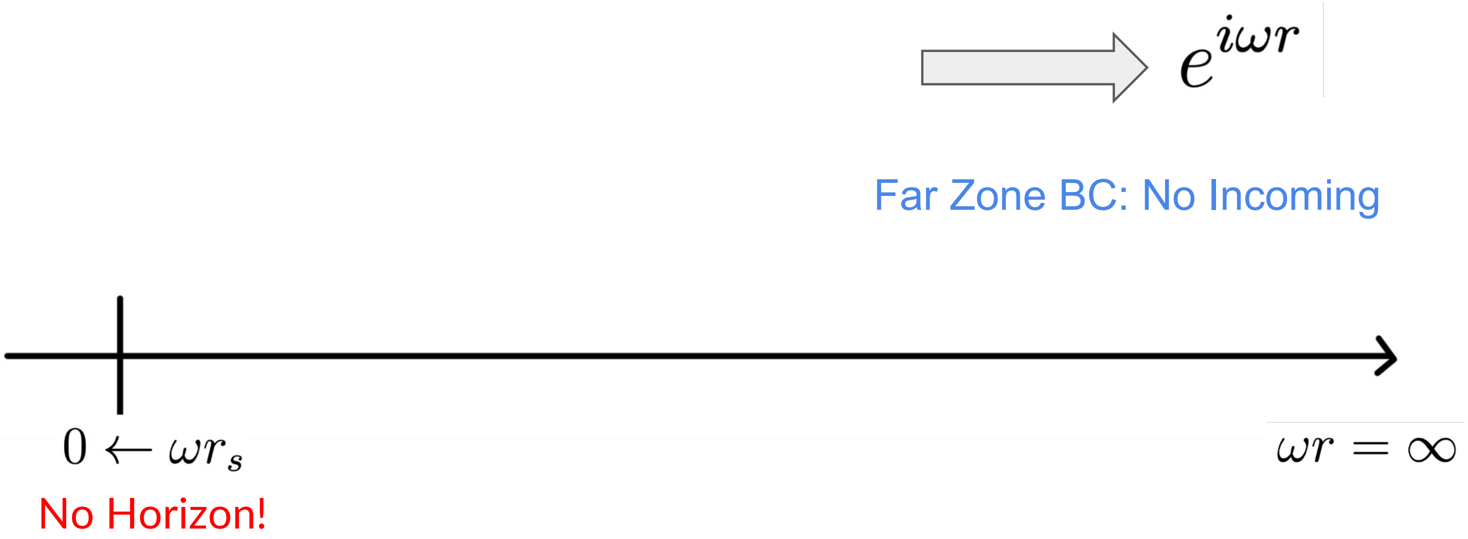
$$\phi(\boldsymbol{r}) = \frac{u_\ell(r)}{r^{1-\epsilon}} Y_\ell$$

$$\downarrow$$

$$\left[\frac{d^2}{dr^2} + \omega^2 - \frac{(\ell-\epsilon)(\ell-\epsilon+1)}{r^2} + \tilde{V}^{\text{EFT}}_{\text{grav}}\right] u_\ell = \tilde{V}_{\text{tidal}} u_\ell + Q_\ell \partial_\ell u_\ell$$

EFT Boundary Conditions

Second order ODE ⟶ Two BCs needed



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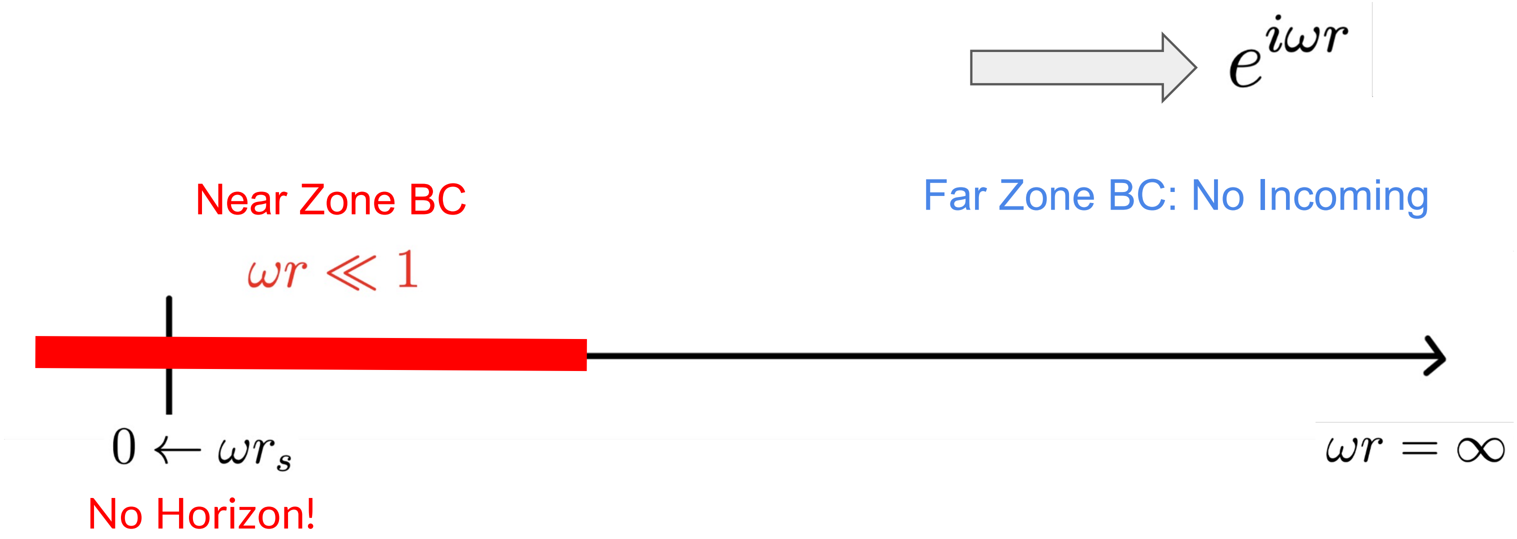


Wave Function Bases

- **Far Zone (FZ)** basis: out and in going mode.
- **Near Zone (NZ)** basis: regular and irregular mode.

Near Zone: $\omega r \to 0$

$$u_\ell^{\text{reg}} \to r^{\ell+1-\epsilon} \qquad u_\ell^{\text{irr}} \to r^{-\ell+\epsilon}$$

$$u_\ell = B_\ell^{\text{reg}} u_\ell^{\text{reg}} + B_\ell^{\text{irr}} u_\ell^{\text{irr}}$$

UV divergent!

Far Zone: $\omega r \to \infty$

$$u_\ell^{+} \to e^{+i\omega r} \qquad u_\ell^{-} \to e^{-i\omega r}$$

$$u_\ell = A_\ell^{+} u_\ell^{+} + A_\ell^{-} u_\ell^{-}$$

No UV divergence

Boundary Conditions

- The Dirac delta functions enforce a boundary condtion **at the origin**, relating regular and irregular mode.

S-wave example:

$$\left(\underline{\nabla^2} + \omega^2 + V_{\mathrm{grav}}^{\mathrm{EFT}}\right)\phi = \underline{F_{\ell=0}\delta^{(d-1)}(\boldsymbol{r})\phi} + Q_{\ell=0}\delta^{(d-1)}(\boldsymbol{r})$$

$$\nabla^2\phi^{\mathrm{irr}} \propto \delta^{(d-1)}(\boldsymbol{r}) \qquad V_{\mathrm{tidal}}\phi^{\mathrm{reg}} \propto \delta^{(d-1)}(\boldsymbol{r})$$

Recall: $\left(\nabla^2 \frac{1}{r^{1-2\epsilon}} \propto \delta^{(d-1)}(\boldsymbol{r})\right)$

Boundary Conditions

- Near Zone BC: Fixed by tidal and multipole moments

$$\frac{1}{c_\ell} B_\ell^{\text{irr}} - F_\ell(\omega) B_\ell^{\text{reg}} = Q_\ell$$

$$c_\ell = \frac{2^\ell}{2\pi^{\frac{3}{2}-\epsilon}} \Gamma\left(\frac{3}{2} - \epsilon + \ell\right)$$

- For Compton scattering: Q = 0

$$\frac{B_\ell^{\text{irr}}}{B_\ell^{\text{reg}}} = c_\ell F_\ell(\omega)$$

[Caron-Huot, Correia, Isabella, Solon]

Solve the Waveform A_ℓ^+

- We have two BCs:

NZ BC $$\frac{1}{c_\ell} B_\ell^{\mathrm{irr}} - F_\ell(\omega) B_\ell^{\mathrm{reg}} = Q_\ell$$

FZ BC $$A_\ell^- = 0$$

- We can relate FZ and NZ bases by **Connection Matrix**. We will discuss how to compute it latter.

$$\boldsymbol{W}^T \begin{pmatrix} \phi^+ \\ \phi^- \end{pmatrix} = \begin{pmatrix} \phi_\ell^{\mathrm{reg}} \\ \phi_\ell^{\mathrm{irr}} \end{pmatrix} \longleftrightarrow \begin{pmatrix} A_\ell^+ \\ A_\ell^- \end{pmatrix} = \boldsymbol{W} \begin{pmatrix} B_\ell^{\mathrm{reg}} \\ B_\ell^{\mathrm{irr}} \end{pmatrix}$$

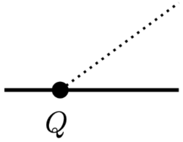
$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

c.c.

Sommerfeld

- Using connection matrix + 2 BCs, we can solve the waveform

$$\phi(r \to \infty) = \sum_{\ell} \mathcal{S}_\ell \times \underline{\frac{i\omega c_\ell}{(2\ell+1)!!} \frac{e^{-i\omega(t-r)}}{r} Q_\ell Y_\ell}$$



- **Sommerfeld factor** is

$$\mathcal{S}_\ell = \frac{(2\ell+1)!!}{2\omega^{\ell+1}(W_{21} + c_\ell F_\ell(\omega) W_{22})}$$

Sommerfeld v.s. Compton

- For Compton scattering, Q = 0, the BC is

$$\frac{B_\ell^{\mathrm{irr}}}{B_\ell^{\mathrm{reg}}} = c_\ell F_\ell(\omega)$$

- We can solve the Compton S matrix

$$\begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

c.c.

$$\hat{S}_\ell = \frac{A_\ell^+}{A_\ell^-} = \frac{W_{11} + c_\ell F_\ell(\omega) W_{12}}{W_{21} + c_\ell F_\ell(\omega) W_{22}} = \frac{W_{21}^* + c_\ell F_\ell(\omega) W_{22}^*}{W_{21} + c_\ell F_\ell(\omega) W_{22}}$$

[Caron-Huot et al.]

Sommerfeld v.s. Compton

- For the **conservative system**, $F_\ell \in \mathbb{R}$,
  the phase of the waveform is **exactly** half of the Compton phase shift!

$$\mathcal{S}_\ell = \frac{(2\ell+1)!!}{2\omega^{\ell+1}} \frac{1}{W_{21} + c_\ell F_\ell(\omega) W_{22}}$$

$$\hat{S}_\ell = \frac{W_{21}^* + c_\ell F_\ell(\omega) W_{22}^*}{W_{21} + c_\ell F_\ell(\omega) W_{22}}$$

$$\mathrm{Arg}(\mathcal{S}_\ell) = \frac{1}{2} \mathrm{Arg}(\hat{S}_\ell)$$

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Renormalization

- Redefine B ⟶ NZ basis is UV finite
- NZ basis can safely takes value d = 4.

$$B_\ell^{\text{reg}} u_\ell^{\text{reg}} + B_\ell^{\text{irr}} u_\ell^{\text{irr}} \longrightarrow \bar{B}_\ell^{\text{reg}} \bar{u}_\ell^{\text{reg}} + \bar{B}_\ell^{\text{irr}} \bar{u}_\ell^{\text{irr}}$$

UV div.

$$\begin{pmatrix} B_\ell^{\text{reg}} \\ B_\ell^{\text{irr}} \end{pmatrix} = \boldsymbol{Z} \begin{pmatrix} \bar{B}_\ell^{\text{reg}} \\ \bar{B}_\ell^{\text{irr}} \end{pmatrix}$$

[Caron-Huot et al.]

UV finite
but components
are μ dependent

Renormalization

- Bare quantities are μ independent ⟶ RG equation.

Do not run ⟶ $$\begin{pmatrix} B_\ell^{\text{reg}} \\ B_\ell^{\text{irr}} \end{pmatrix} = \boldsymbol{Z} \begin{pmatrix} \bar{B}_\ell^{\text{reg}} \\ \bar{B}_\ell^{\text{irr}} \end{pmatrix}$$

RG of B $$\mu \frac{d}{d\mu} \begin{pmatrix} \bar{B}_\ell^{\text{reg}} \\ \bar{B}_\ell^{\text{irr}} \end{pmatrix} = -\boldsymbol{\gamma} \begin{pmatrix} \bar{B}_\ell^{\text{reg}} \\ \bar{B}_\ell^{\text{irr}} \end{pmatrix} \qquad \boldsymbol{\gamma} = \boldsymbol{Z}^{-1} \mu \frac{d}{d\mu} \boldsymbol{Z}$$

[Caron-Huot et al.]

RG of the Multipole Moments

- Bare quantities are μ independent ⟶ RG equation.
- Boudanry condition ⟶ RG of the sources and Tidal.

$$\mu \frac{d}{d\mu} \begin{pmatrix} \bar{B}_\ell^{\text{reg}} \\ \bar{B}_\ell^{\text{irr}} \end{pmatrix} = -\gamma \begin{pmatrix} \bar{B}_\ell^{\text{reg}} \\ \bar{B}_\ell^{\text{irr}} \end{pmatrix}$$

$$\frac{1}{c_\ell} \bar{B}_\ell^{\text{irr}} - \bar{F}_\ell(\omega) \bar{B}_\ell^{\text{reg}} = \bar{Q}_\ell$$

⟹

[Caron-Huot et al.]

$$\mu \frac{d}{d\mu} \bar{F}_\ell = c_\ell \gamma_{12} \bar{F}_\ell^2 + (\gamma_{11} - \gamma_{22}) \bar{F}_\ell - \frac{\gamma_{21}}{c_\ell}$$

$$\boxed{\mu \frac{d}{d\mu} \bar{Q}_\ell = \left(\gamma_{11} + \gamma_{12} c_\ell \bar{F}_\ell\right) \bar{Q}_\ell}$$

Our New Result!

RG of the Multipole Moments

- It is **nonperturbative** in the linearized theory of Q.
- Tidal effects only appears up to linear order.

$$\mu \frac{d}{d\mu}\bar{Q}_\ell = \left(\gamma_{11} + \gamma_{12} c_\ell \bar{F}_\ell\right)\bar{Q}_\ell$$

Universal up to G^{2l+1}
given by renormalized angular momentum [Ivanov et al.]

Nonuniversal part:
Tidal interaction.

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Why BHPT?

- EFT FZ is difficult to computed
- Recycle BHPT by MST method

	BHPT	**EFT**
System	BH only	Generic Localized Source
Dimension	d = 4	d = 4 - 2ε
UV div.	No	Yes
BC	Horizon BC  	Tidal/Q BC 

BHPT

$$\Box_{\rm Sch}\phi = 0$$

Horizon Exist

EFT

$$(\nabla^2 + \omega^2 + V^{\rm EFT}_{\rm grav})\phi = \sout{V_{\rm tidal}\phi + Q_\ell \partial_\ell \delta^{d-1}(\boldsymbol{r})}$$

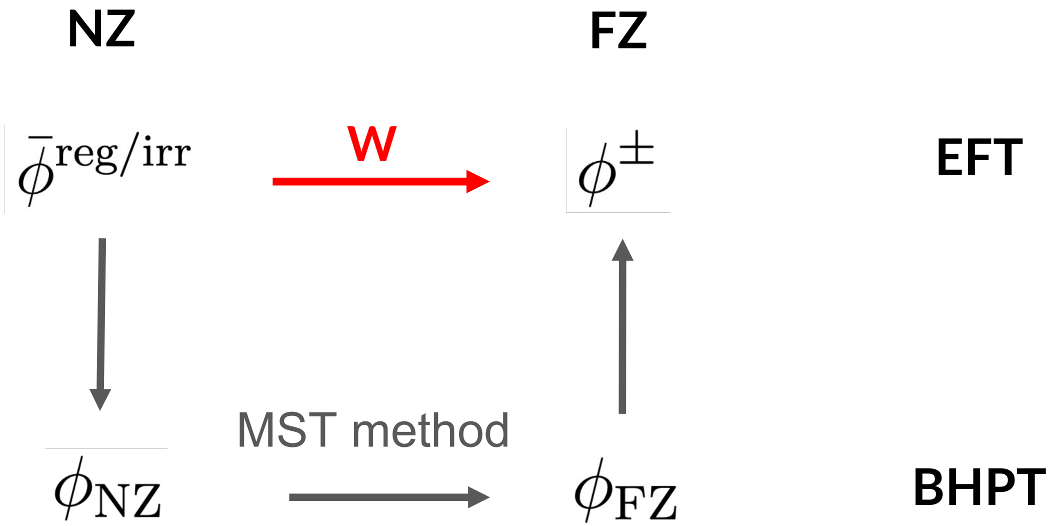
G Expansion/ No Horizon

away from the worldline

- r ≠ 0 ⟶ Tidal and Q vanish
- $V^{\rm EFT}_{\rm grav}$ is derived from bulk action via G expansion
- The EFT homogeneous bases should be the same as the ones in BHPT when
  - **FZ:** $\omega r_s \ll 1$
  - **NZ:** (Point Particle limit) $\omega r_s \ll \omega r$ and $\omega r \ll 1$

Recycle BHPT

- We can use BHPT to compute the connection matrix



Our Results

$$\mathrm{Arg}\,\mathcal{S}_{\ell=0} = x\left[\log\left(\frac{4\omega^2}{\bar{\mu}_{\mathrm{IR}}^2}\right) + 2\gamma_E - 1\right] + x^2\,\frac{11\pi}{3}$$
$$+ x^3\left[\frac{50}{3} - 4\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right) + \frac{22\pi^2}{9} - \frac{8\zeta_3}{3}\right] + x^4\left[\frac{5719\pi}{135} - 8\pi\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\right]$$
$$+ x^5\left[\frac{51728}{135} + \frac{5498\pi^2}{405} - \frac{88\pi^4}{135} + \frac{104\zeta_3}{9} + \frac{32\zeta_5}{5}\right.$$
$$\left. - \left(140 + \frac{16\pi^2}{3}\right)\log\left(\frac{4\omega^2}{\bar{\mu}^2}\right) + \frac{44}{3}\log^2\left(\frac{4\omega^2}{\bar{\mu}^2}\right)\right]$$

$$x \equiv GM\omega$$

For $O(G^{10})$, see our paper

We've checked our results up to $O(G^3)$ with:

- [Ivanov, Li, Parra-Martinez, Zhou. ArXiv: 2401.08752]
- [Caron-Huot, Correia, Isabella, Solon. ArXiv:2503.13593]
- [Correia, Gopalka, Isabella, Wolz. ArXiv:2511.11794]

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See Russo's talk

All-order RG Structure

- We fix the eigenvalues of the anomalous dimension, to all orders beyond universal sector [Ivanov et al.]

$$\mu \frac{d}{d\mu} \begin{pmatrix} R^{\mathrm{reg}} \\ R^{\mathrm{irr}} \end{pmatrix} = \gamma^T \begin{pmatrix} R^{\mathrm{reg}} \\ R^{\mathrm{irr}} \end{pmatrix} \longrightarrow$$

All orders in BHPT

$$R^C_\nu = e^{(\nu-\ell)\log(2\omega r)}\left((2\omega r)^\ell + (2\omega r)^{-\ell-1} + \cdots\right)$$
$$R^C_{-\nu-1} = e^{-(\nu-\ell)\log(2\omega r)}\left((2\omega r)^\ell + (2\omega r)^{-\ell-1} + \cdots\right)$$

The eigenvalues of γ are $\pm(\nu - \ell)$

Exact Solutions to Tidal and Source

- We can solve the RG equation of F/Q exactly:

RG: $\mu \frac{d}{d\mu}\bar{F}_\ell = c_\ell \gamma_{12} \bar{F}_\ell^2 + (\gamma_{11} - \gamma_{22})\bar{F}_\ell - \frac{\gamma_{21}}{c_\ell}$ $\mu \frac{d}{d\mu}\bar{Q}_\ell = \left(\gamma_{11} + \gamma_{12} c_\ell \bar{F}_\ell\right)\bar{Q}_\ell$

$$\bar{Q}(\mu) = \bar{Q}(\mu_0) \frac{\bar{F}_-^* - \bar{F}_+^*}{\bar{F}_-^* - \bar{F}(\mu_0) + \left(\bar{F}(\mu_0) - \bar{F}_+^*\right)\left(\frac{\mu}{\mu_0}\right)^{2(\nu-\ell)}} \left(\frac{\mu}{\mu_0}\right)^{(\nu-\ell)}$$

$$\bar{F}(\mu) = \bar{F}_+^* + \frac{\left(\bar{F}(\mu_0) - \bar{F}_+^*\right)\left(\bar{F}_-^* - \bar{F}_+^*\right)\left(\frac{\mu}{\mu_0}\right)^{2(\nu-\ell)}}{\bar{F}_-^* - \bar{F}(\mu_0) + \left(\bar{F}(\mu_0) - \bar{F}_+^*\right)\left(\frac{\mu}{\mu_0}\right)^{2(\nu-\ell)}}$$

$\bar{F}_\pm^*$ are functions of γ

$\bar{F}(\mu_0)$, $\bar{Q}(\mu_0)$ are reference values at scale $\mu = \mu_0$

See Nagar's talk/ Cipriani's poster

Resummation Proposal: $\mathcal{S}_\ell = |\mathcal{S}_\ell| e^{i\delta_\ell}$

- We propose the factorization of the Sommerfeld factor:

$$|\mathcal{S}| = |\mathcal{S}_{\rm IR}| \times |\mathcal{S}_{\rm run}| \times |\mathcal{S}_{\rm rem}|$$

$$\left|\frac{\Gamma(\nu+1+2iGM\omega)\Gamma(2\ell+2)}{\Gamma(2\nu+2)\Gamma(\ell+1)}\right| e^{\pi GM\omega} \quad |\mathcal{S}_{\rm run}| = \left|\frac{Q(\mu=\omega)}{Q(\mu=1/r_{\rm orb})}\right|$$

[Damour et al.] [Nagar et al.] [Ivanov et al.]

Resummation Proposal: $\mathcal{S}_\ell = |\mathcal{S}_\ell| e^{i\delta_\ell}$

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$$|\mathcal{S}_{\rm run}| = \left|\frac{Q(\mu=\omega)}{Q(\mu = 1/r_{\rm orb})}\right|$$

Improved by our complete RG solutions

calibrated by our high-order data

Resummation Proposal: $\mathcal{S}_\ell = |\mathcal{S}_\ell| e^{i\delta_\ell}$

- We propose the factorization of the Sommerfeld factor:

$$|\mathcal{S}| = |\mathcal{S}_{\rm IR}| \times |\mathcal{S}_{\rm run}| \times |\mathcal{S}_{\rm rem}|$$

- The phase can be calculated by BHPT or through Compton scattering

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Conclusion and Outlook

- We develop the formalism for calculating Sommerfeld factor
  - **All-order formula** in terms of connection matrix
  - Exact relation between **Sommerfeld** and **Compton scattering**
- Combining with **BHPT** to achieve **high-precision** data
- Full RG solutions for waveform **resummation**
- Future work:
  - generalization to gravitational waveform
  - Extension to Kerr Background

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Tack!