Kerr Black Hole Scattering: An On-Shell Approach to Conserved Quantities and Integrability

based on arXiv:2508.10761

Dogan Akpinar
G. Brown, R. Gonzo, and M. Zeng

Amplitudes, Strong-Field Gravity and Resummation @ Nordita
17th April 2026

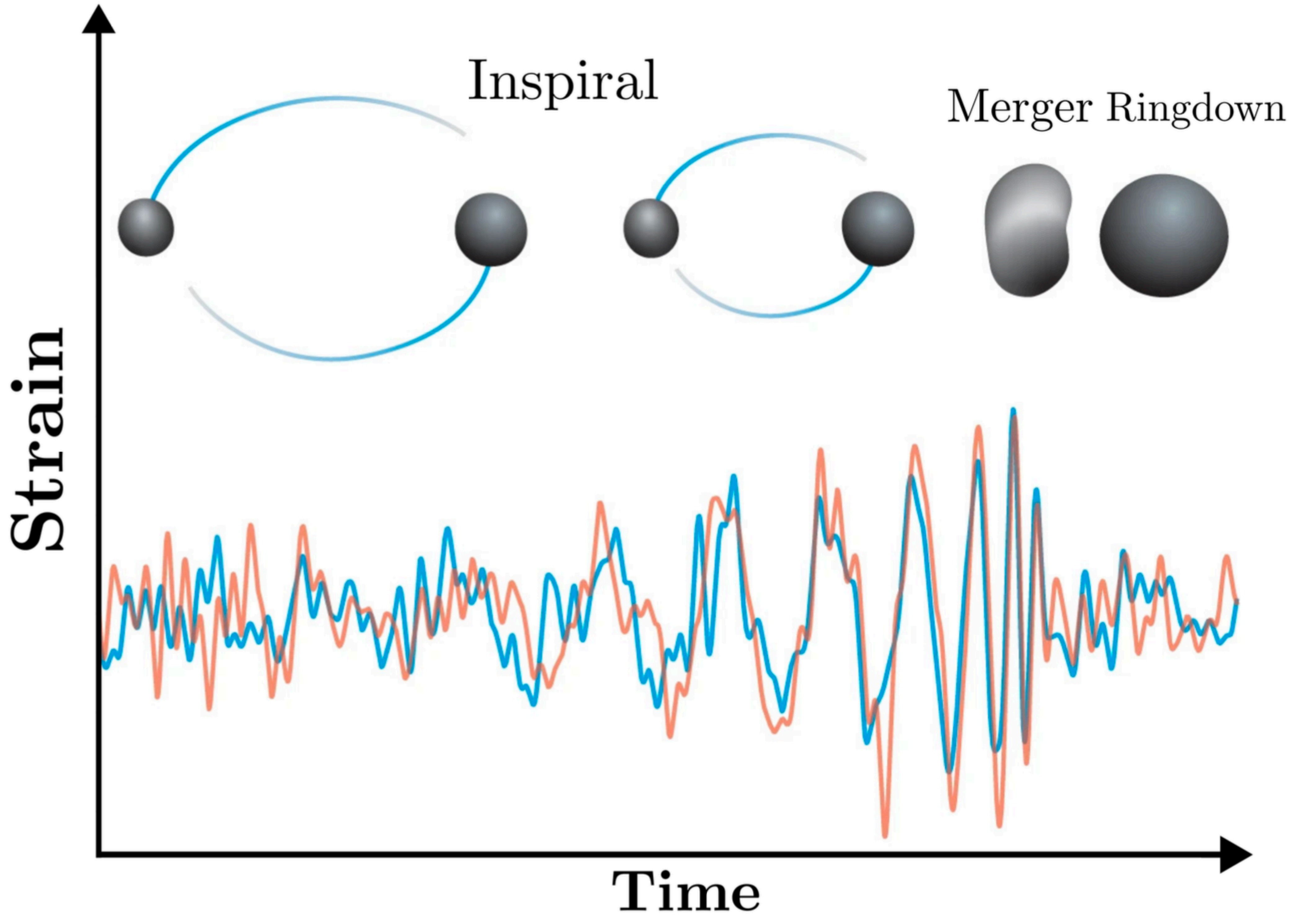
Outline

- Background and Motivation
- An introduction to Dirac Brackets
- Spinning Probe in a Kerr Background and Asymptotic Integrability
- Spin-Shift Symmetry and Bootstrapping the Radial Action
- (Backup slide: Beyond the Probe Limit, Beyond Kerr)

Based on a talk by Graham Brown

Gravitational Waves

The discovery of GWs and the advent of highly anticipated future detectors have called for advances in our theoretical understanding of the classical two-body problem in GR



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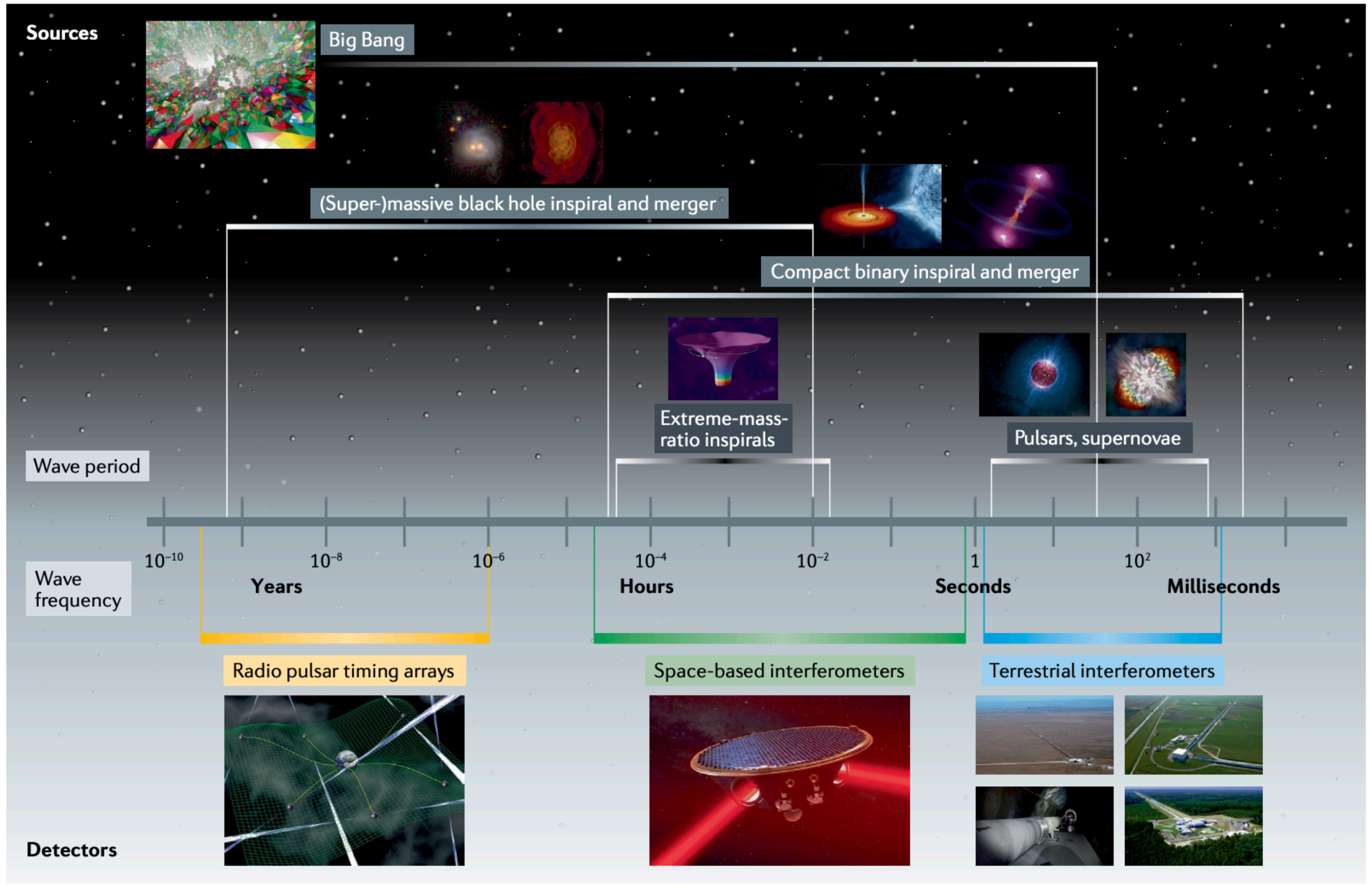

[Bailes et al., Nature Rev.Phys. 3 (2021) 5, 344-366]

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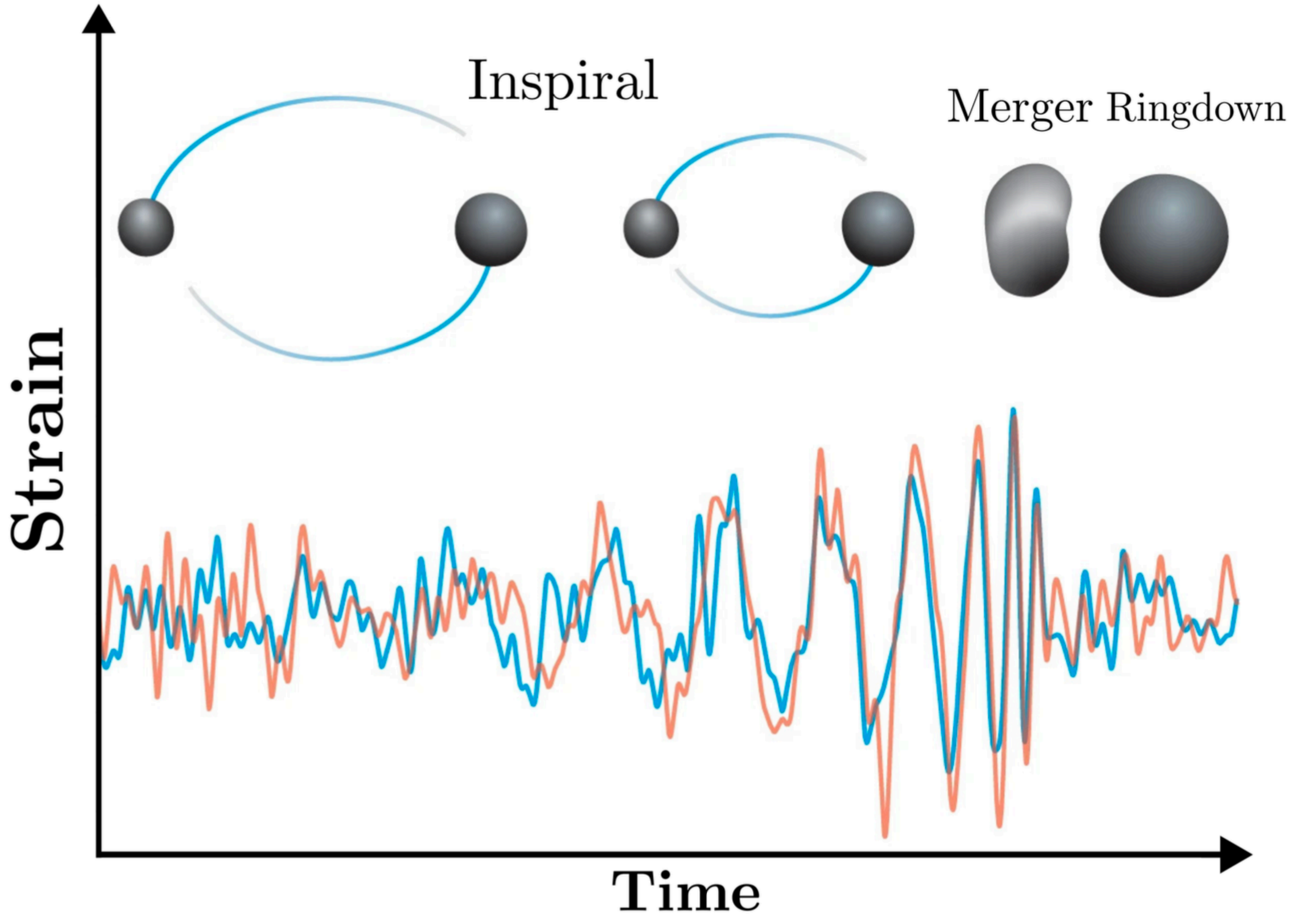
→ Amplitude and worldline methods have emerged as powerful frameworks for the inspiral dynamics!



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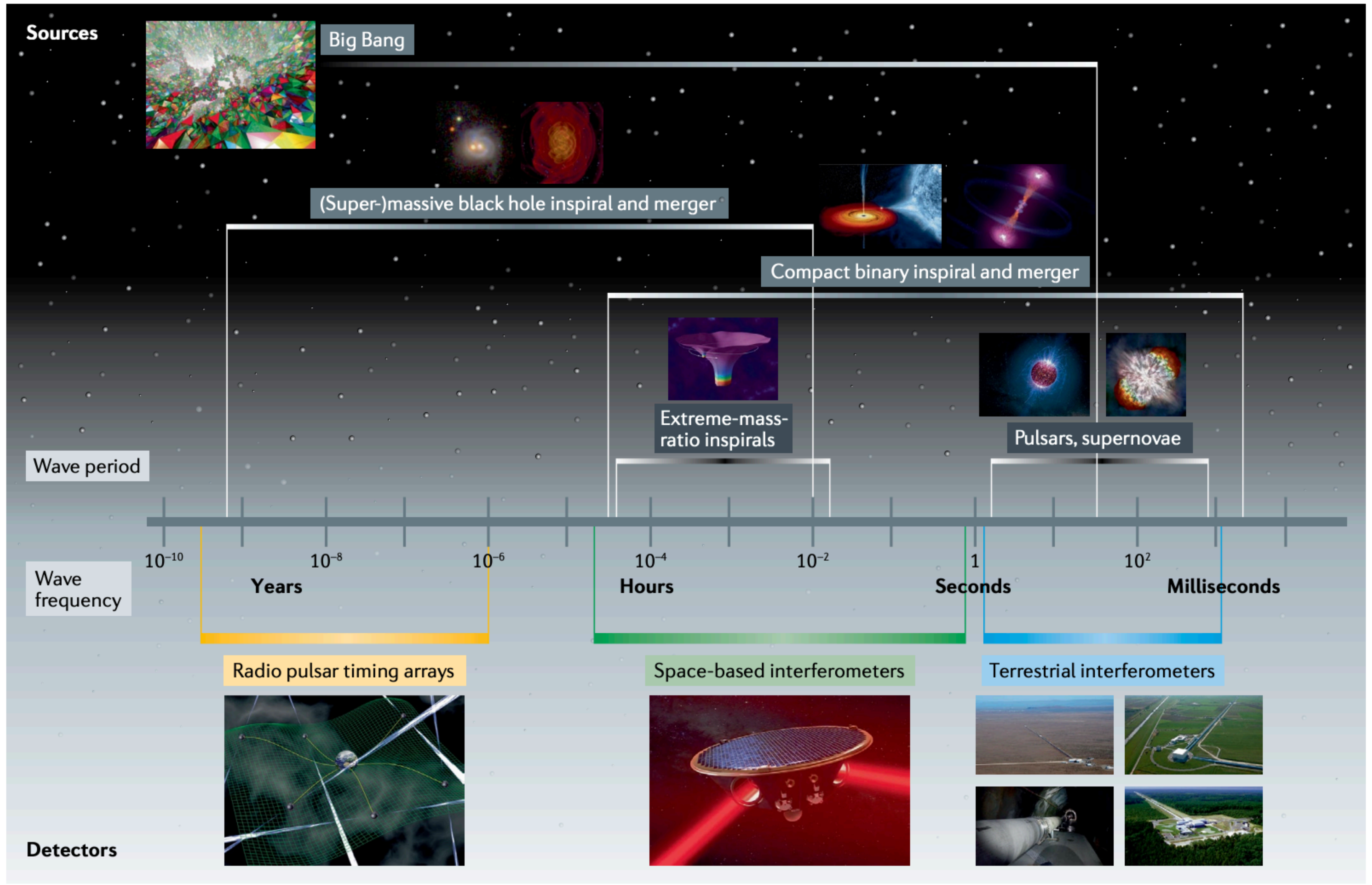

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Leveraging PM Data

Post-Minkowskian (PM): an expansion in Newton's constant G

New results:

High PM computations of scattering observables with spin

$$\Delta p_i^\mu, \Delta s_i^\mu, \ldots$$

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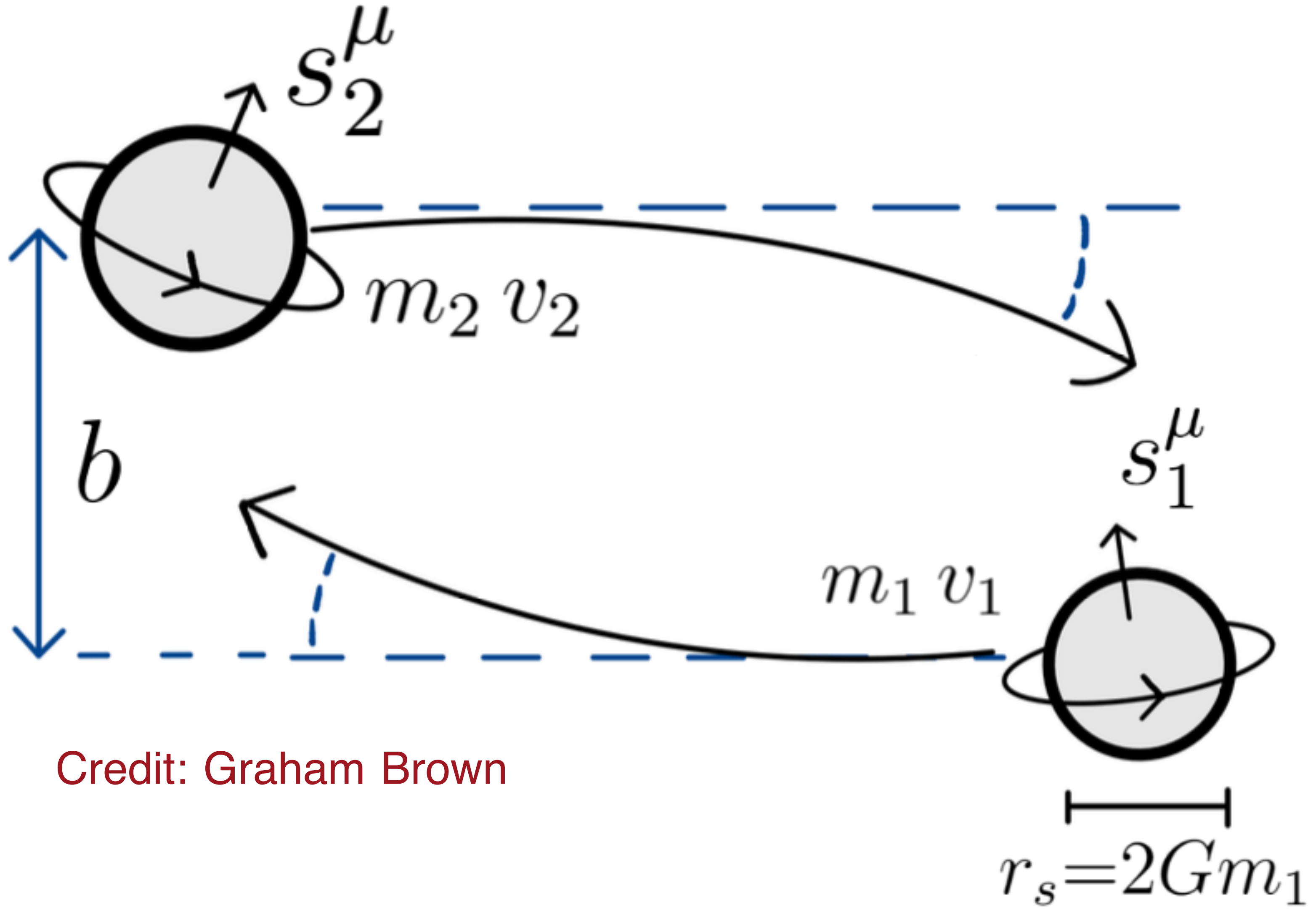

Credit: Graham Brown

New machinery:

- KMOC $\langle \Delta \hat{Q} \rangle = \langle \Psi | \hat{S}^\dagger [\hat{Q}, \hat{S}] | \Psi \rangle$
- Exp. representation of the S-matrix

$$\hat{S} = \exp(i\hat{N})$$

- Dirac brackets $\{f, g\}_{\mathrm{DB}}$
- ……

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Unexpected symmetry (spin-shift symmetry):

$$\mathcal{M}(v_i, s_i, q) \to \mathcal{M}(v_i, s_i, q) \text{ under } s_i^\mu \to s_i^\mu + \xi q^\mu \text{ through } \mathcal{O}(s^4)$$

Velocities | Spins | Momentum transfer

[Bern et al., '22] [Aoude et al., '22]
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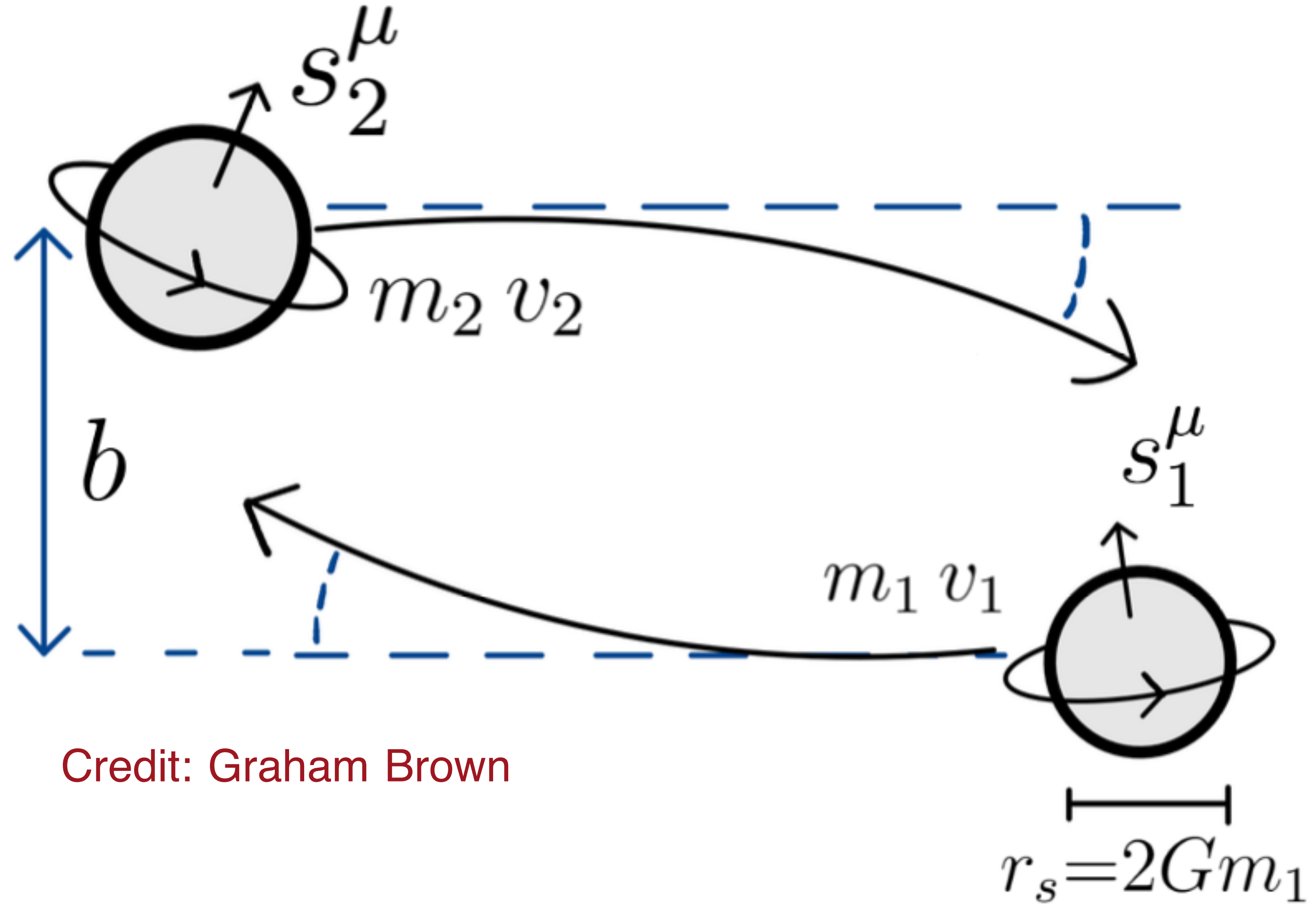

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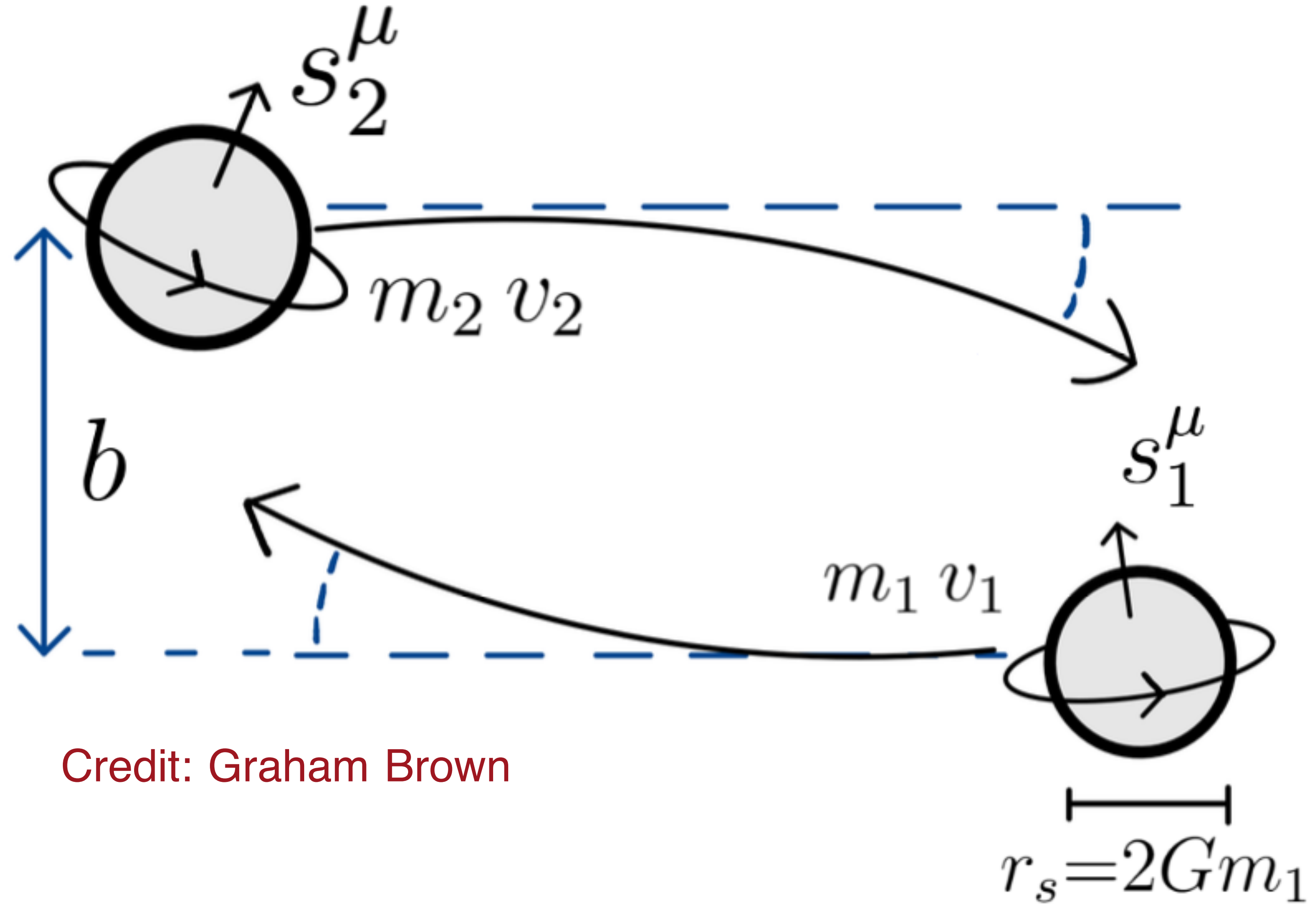

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Can we study conservation laws and integrability for Kerr BH Scattering?!

An Introduction to Dirac Brackets

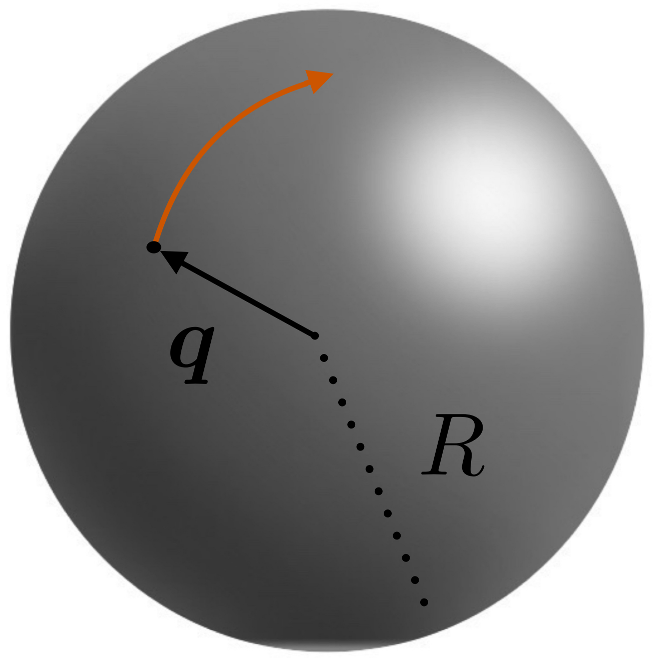
Constrains us to the physical phase space

- Consider some system with Hamiltonian H subject to constraints ϕ_i

- Define $H' = H + u_i\phi_i$ which is weakly equivalent to the original Hamiltonian $H' \approx H$

$\phi_i = 0$

Example:



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$$\dot{f} = \{f, H'\} \approx \{f, H\} + \{f, \phi_i\} u_i \quad \text{with} \quad \dot{\phi}_i = 0 \approx \{\phi_i, H\} + \underbrace{\{\phi_i, \phi_j\}}_{M_{ij}} u_j$$

such that

$$u_i = -M_{ij}^{-1}\{\phi_j, H\}$$

$$\dot{f} \approx \{f, H\} - \{f, \phi_i\} M_{ij}^{-1} \{\phi_j, H\} \equiv \{f, H\}_{\mathrm{DB}}$$

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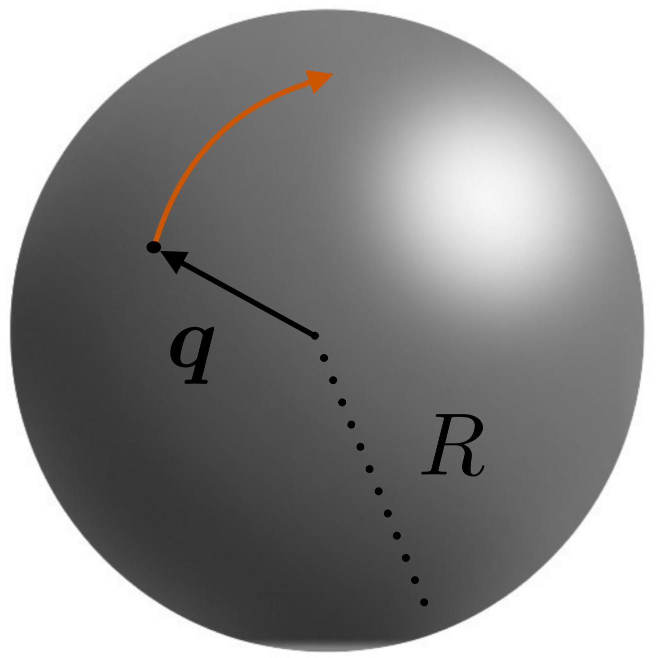

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Defines evolution on the physical phase space!

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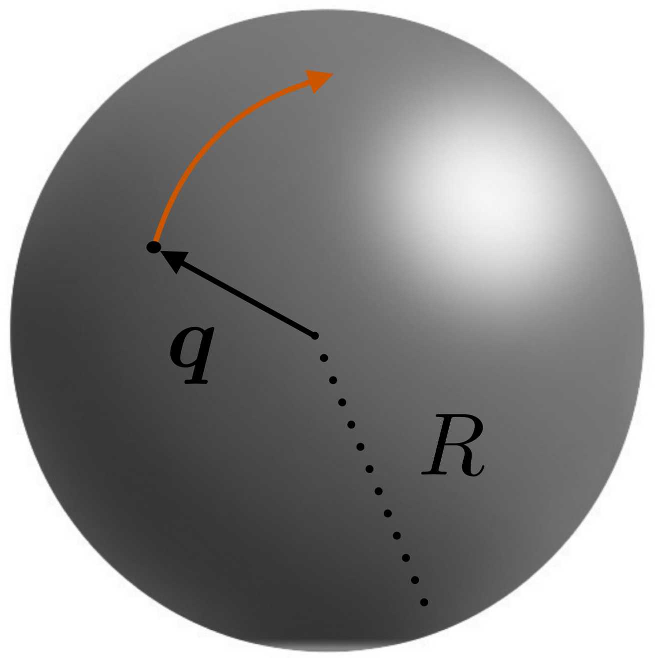

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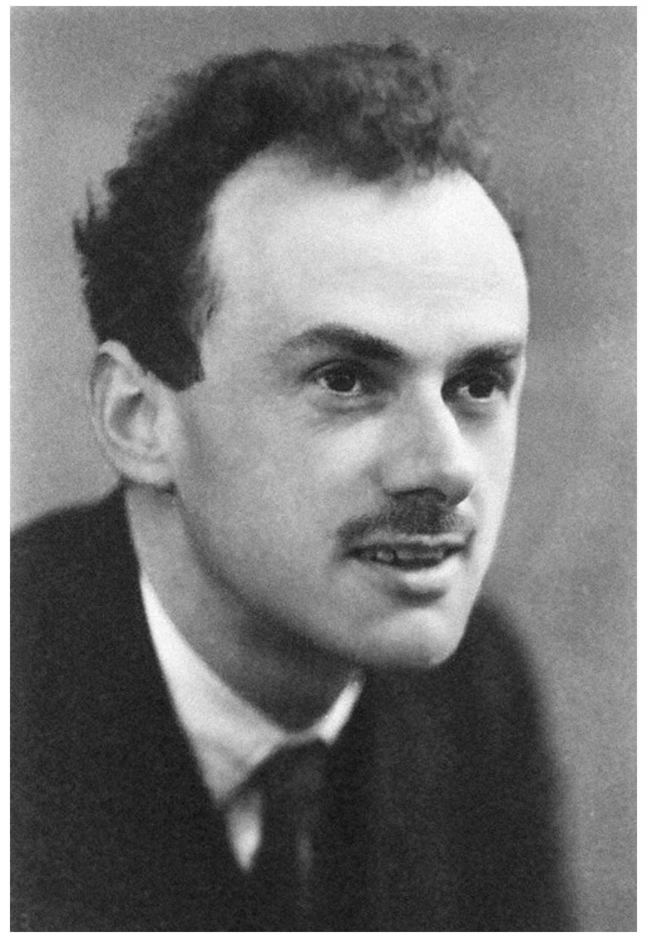
Dirac Brackets: Key Takeaways



1. A function Q is said to be conserved if $\dot{Q} = 0 \iff \{Q, H\}_{\mathrm{DB}} = 0$

2. Brackets act through derivatives: $\{q, f(p)\}_{\mathrm{DB}} = \{q, p\}_{\mathrm{DB}} \frac{d}{dp} f(p)$

3. Time evolution follows from iterated brackets

$$\dot{f} = \{f, H\}_{\mathrm{DB}} \xrightarrow{\text{Solution}} f(t_1) = \exp((t_1 - t_0)\{\cdot, H\}_{\mathrm{DB}}) f(t_0)$$

4. Calculate the change:

$$\Delta f = f(t_1) - f(t_0) = [\exp((t_1 - t_0)\{\cdot, H\}_{\mathrm{DB}}) - 1] f(t_0)$$
$$= \sum_{n=1}^{\infty} \frac{(t_1 - t_0)^n}{n!} \{\{\{f, H\}_{\mathrm{DB}}, H\}_{\mathrm{DB}}, \ldots, H\}_{\mathrm{DB}}$$

Iterative Dirac bracket structure!

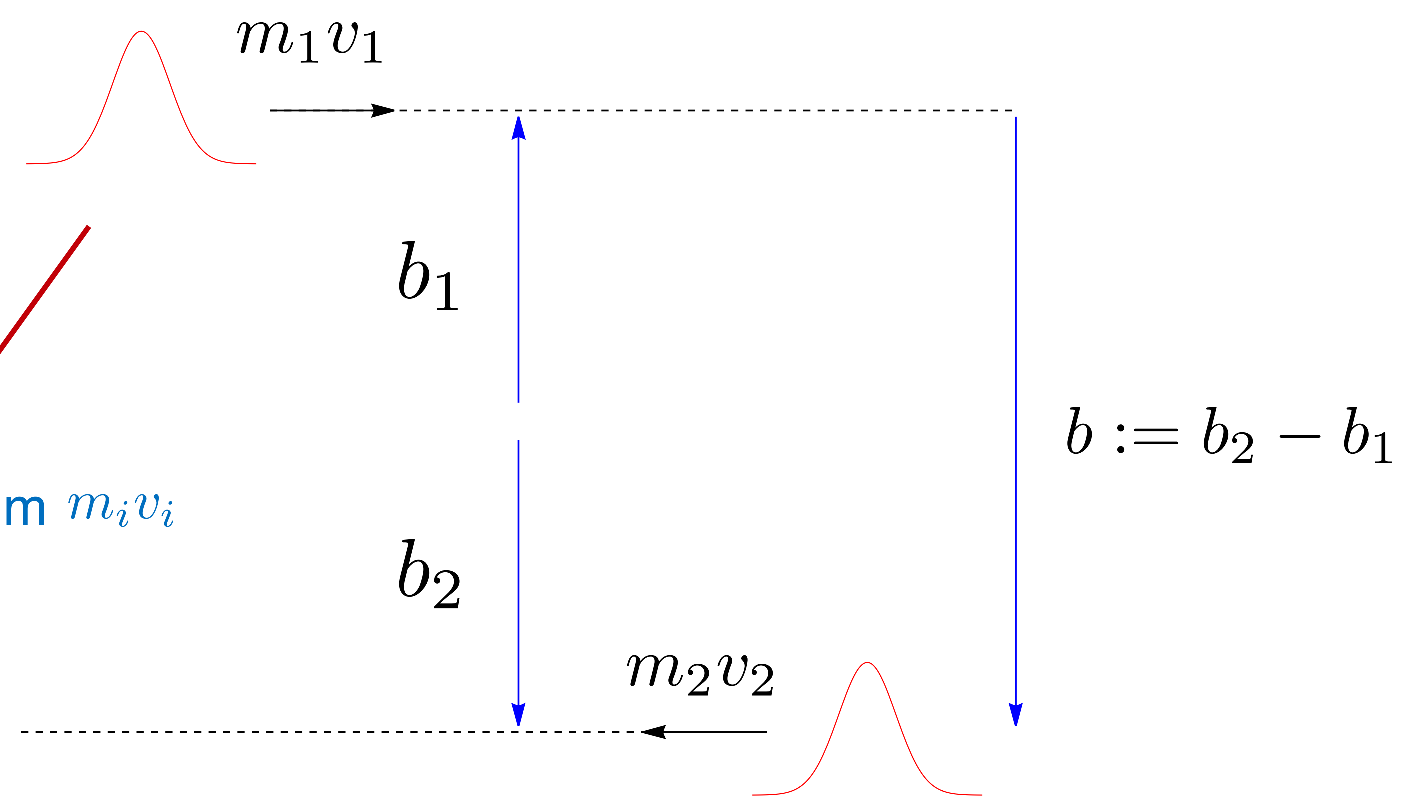
A QFT Perspective on Scattering

- Our starting point is the KMOC formalism [Kosower, Maybee, O'Connell, '18] [Maybee, O'Connell, Vines, '19]…..

$$|\psi\rangle_{\text{in}} := \int \mathrm{d}\Phi(p_1)\mathrm{d}\Phi(p_2)e^{i(p_1\cdot b_1+p_2\cdot p_2)}\phi(p_1)\phi(p_2)|p_1,p_2\rangle$$

$$\mathrm{d}\Phi(p_i) := \frac{\mathrm{d}^D p_i}{(2\pi)^{D-1}}\delta^{(+)}(p_i^2-m_i^2)$$

$\phi_i(p_i)$ peaks around the classical momentum $m_i v_i$



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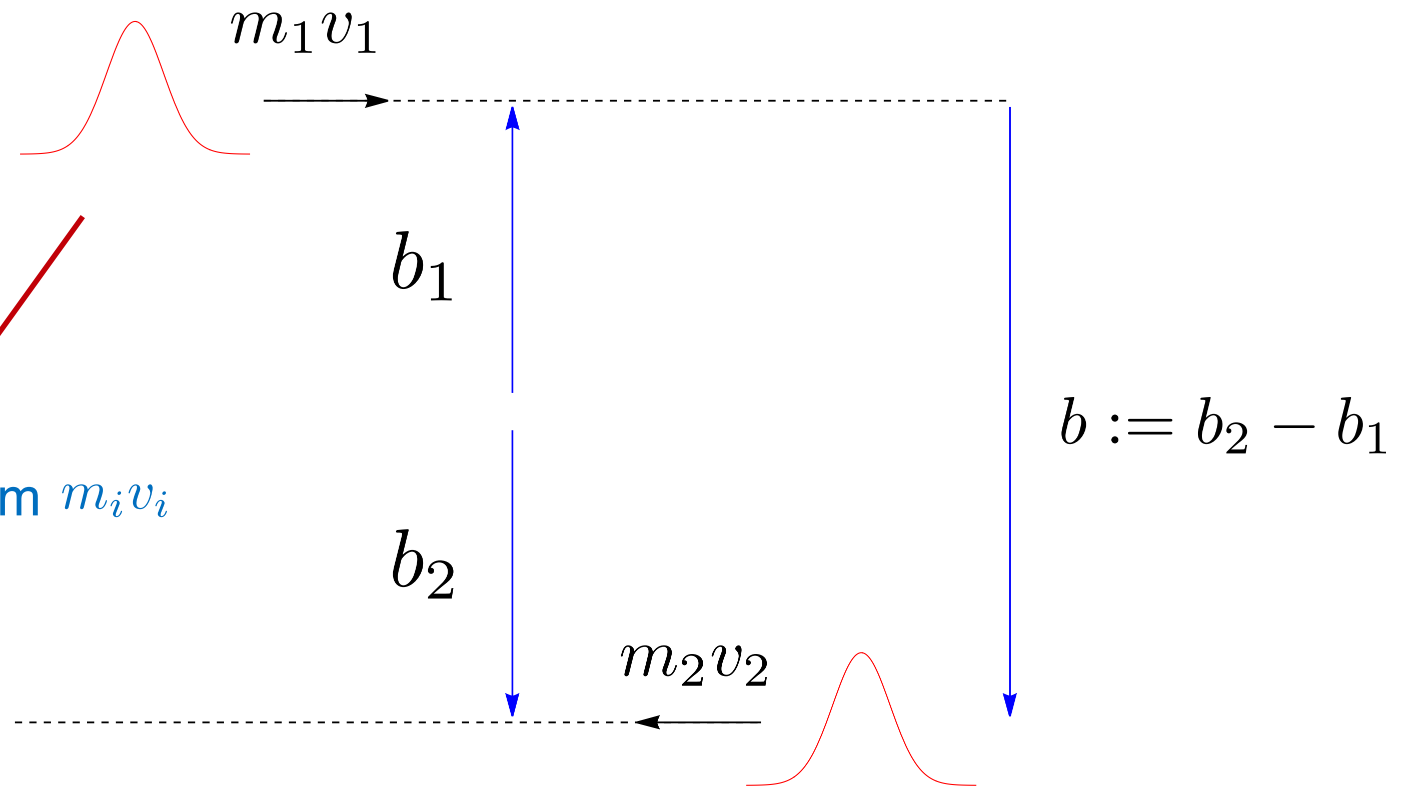

- Interested in the change of observables between past and future infinity

$$\langle\Delta\hat{O}\rangle := {}_{\text{out}}\langle\psi|\hat{O}|\psi\rangle_{\text{out}} - {}_{\text{in}}\langle\psi|\hat{O}|\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\psi|\hat{S}^\dagger[\hat{O},\hat{S}]|\psi\rangle_{\text{in}}$$

$$|\psi\rangle_{\text{out}} = \hat{S}|\psi\rangle_{\text{in}}$$

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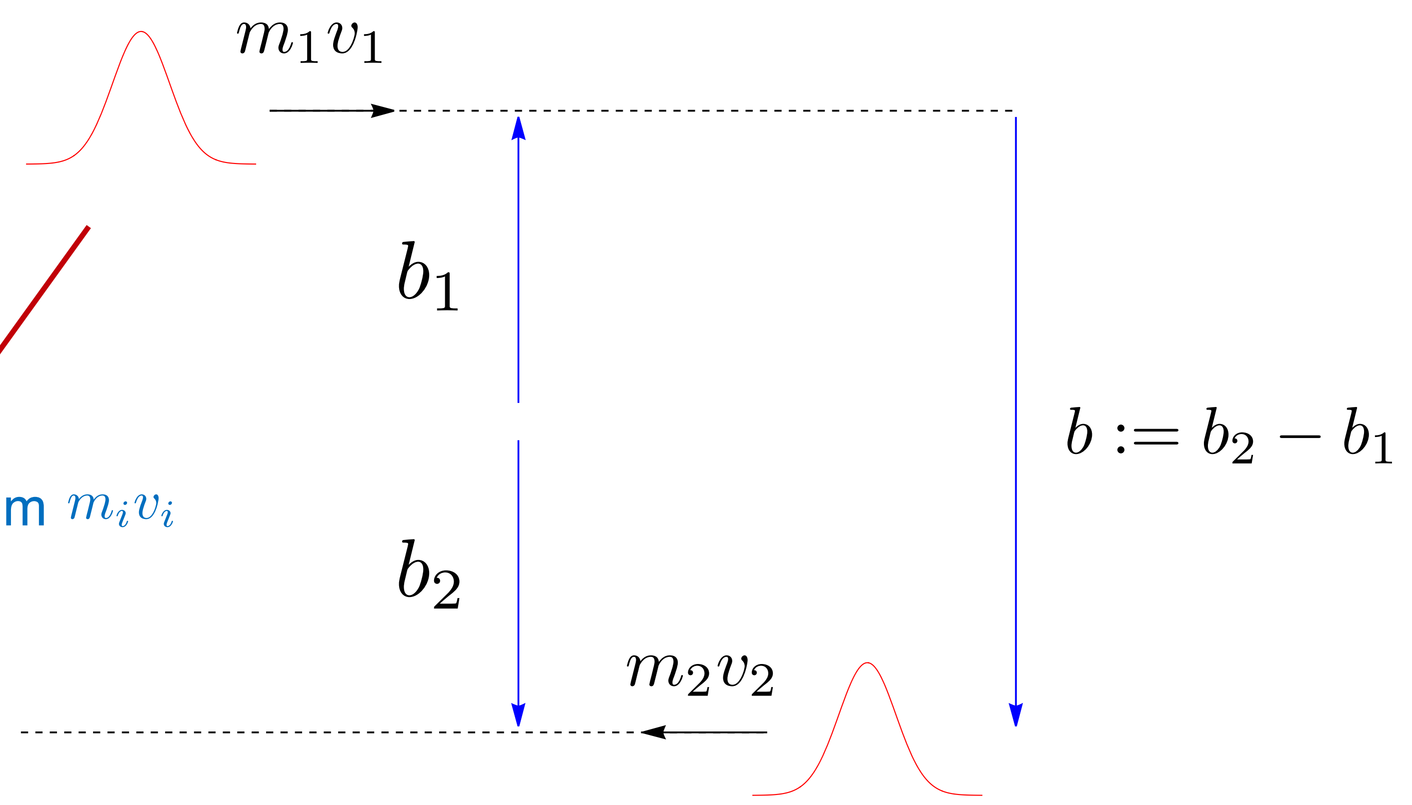

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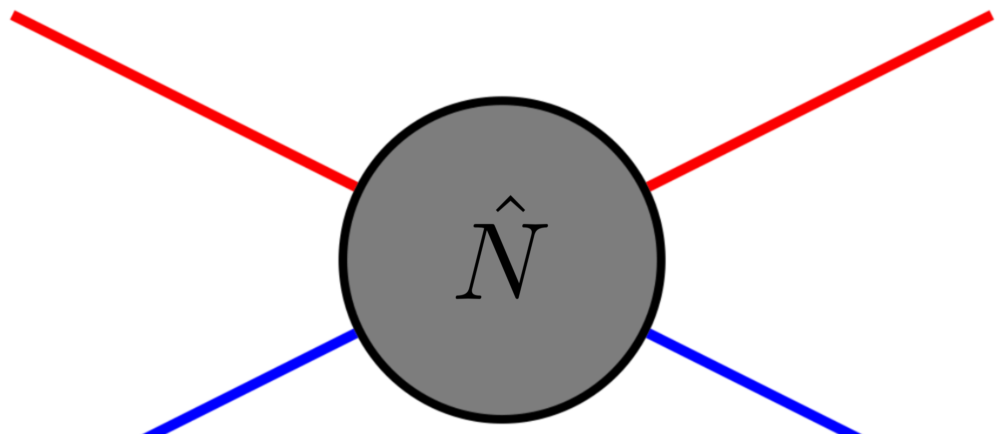
- Choose exponential form of the S-matrix [Damgaard, Plante, Vanhove, 21'] [Damgaard, Hansen, Plante, Vanhove, 23']

$$\hat{S} = \exp(i\hat{N}) \implies \langle\Delta\hat{O}\rangle = \langle\psi|\sum_{n=1}^{\infty}\frac{(-i)^n}{\hbar^n n!}\underbrace{[\hat{N},[\hat{N},\ldots[\hat{N}}_{n\text{ times}},\hat{O}]]]|\psi\rangle$$

The Classical Limit: Radial Action & Dirac Brackets

- Consider the conservative $2 \to 2$ scattering



$$p_1^\mu = \bar{p}_1^\mu + \hbar\frac{\bar{q}^\mu}{2} \qquad p_1'^\mu = \bar{p}_1^\mu - \hbar\frac{\bar{q}^\mu}{2}$$

$$p_2^\mu = \bar{p}_2^\mu - \hbar\frac{\bar{q}^\mu}{2} \qquad p_2'^\mu = \bar{p}_2^\mu + \hbar\frac{\bar{q}^\mu}{2}$$

Classical momentum

Momentum transfer

$$I_r(\{v_i, s_i, b\}) = \int \hat{\mathrm{d}}^D q\, \hat{\delta}\left(2\bar{p}_1 \cdot q\right) \hat{\delta}\left(2\bar{p}_2 \cdot q\right) \times \mathrm{e}^{i(q\cdot b)/\hbar} \langle p_1' p_2' | \hat{N} | p_1 p_2 \rangle \Big|_{\hbar \to 0}$$

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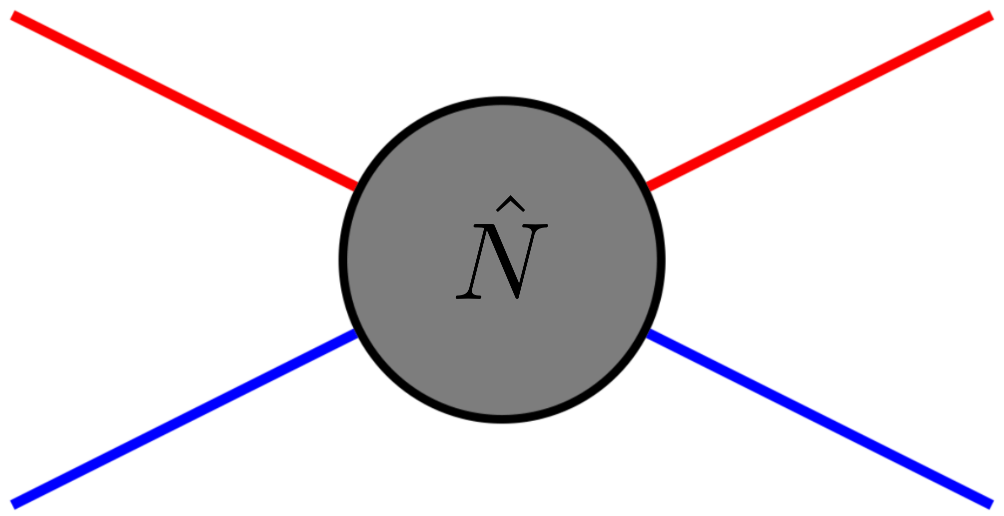

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Previous work on brackets: [Bern, Luna, Roiban, Shen, Zeng, '20] [Kosmopoulos, Luna, '21][Gatica, '21][Luna, Moynihan, O'Connell, Ross, '23]

- "Dequantise" commutators $[f(\cdot), g(\cdot)] \overset{\hbar \to 0}{\longrightarrow} i\hbar \{f(\cdot), g(\cdot)\}_{\mathrm{DB}}$

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[Gonzo, Shi, 24'] [Kim, Kim, Lee, '24]
Generalisation to radiative case [Alessio, Gonzo, Shi, '25] [Kim, '25] [Kim, Lee, Lee, '25]…..

Relativistic Spinning Particles

- The key ingredients that we need are I_r and $\{\cdot,\cdot\}_{\rm DB}$ — Let us start with the latter!

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Free trajectory:

$$x_i^\mu(\tau) = v_i^\mu \tau + b_i^\mu$$

$i = 1, 2$

Poisson brackets

$$\{b_i^\mu, v_i^\mu\} = -\eta^{\mu\nu}/m_i$$

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Constraints:

On-shell

$$v_i^2 = 1,\ b \cdot v_i = 0$$

Transverse

$$S_i^{\mu\nu} v_{i\nu} = 0$$

Spin supplementary condition (SSC)

$$\Lambda_{i,0}^{\ \mu} - v_i^\mu = 0$$

Body-fixed frame

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- The Dirac brackets that fall out are very simple:

$b^\mu := b_2^\mu - b_1^\mu$

Variables: $\{v_i^\mu, s_i^\mu, b^\mu\}$

$$\{b^\mu, v_i^\nu\}_{\rm DB} = -\,\mathrm{sgn}_i \frac{(y^2-1)b^\mu b^\nu + l^\mu l^\nu}{m_i b^2 (y^2-1)}\,,$$

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$$\{s_i^\mu, s_i^\nu\}_{\rm DB} = \frac{\epsilon^{\mu\nu}{}_{\rho\sigma} v_i^\rho s_i^\sigma}{m_i}\,,$$

$y = v_1\cdot v_2$

$$S_i^{\mu\nu} = m_i \epsilon^{\mu\nu}{}_{\rho\sigma} v_i^\rho s_i^\sigma$$

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These brackets were used to compute state-of-the-art results for scattering observables! [DA, Febres Cordero, Kraus, Smirnov, Zeng, '25]

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NOTE: In the probe limit $m_2 \gg m_1$ we ignore anything with a factor $1/m_2$

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Spinning Probe in Kerr

An interesting problem to study is the dynamics of a spinning probe in a Kerr background ($m_2 \gg m_1$).

There is a lot of work in the GR literature incorporating spin order by order (spin multipole expansion):

- Geodesics in Kerr: $\mathcal{O}(s_1^0 s_2^\infty)$
  [Carter, '68]

$$\pi_\mu = p_\mu + \mathcal{O}(S)$$

Conserved quantities:

1. Energy

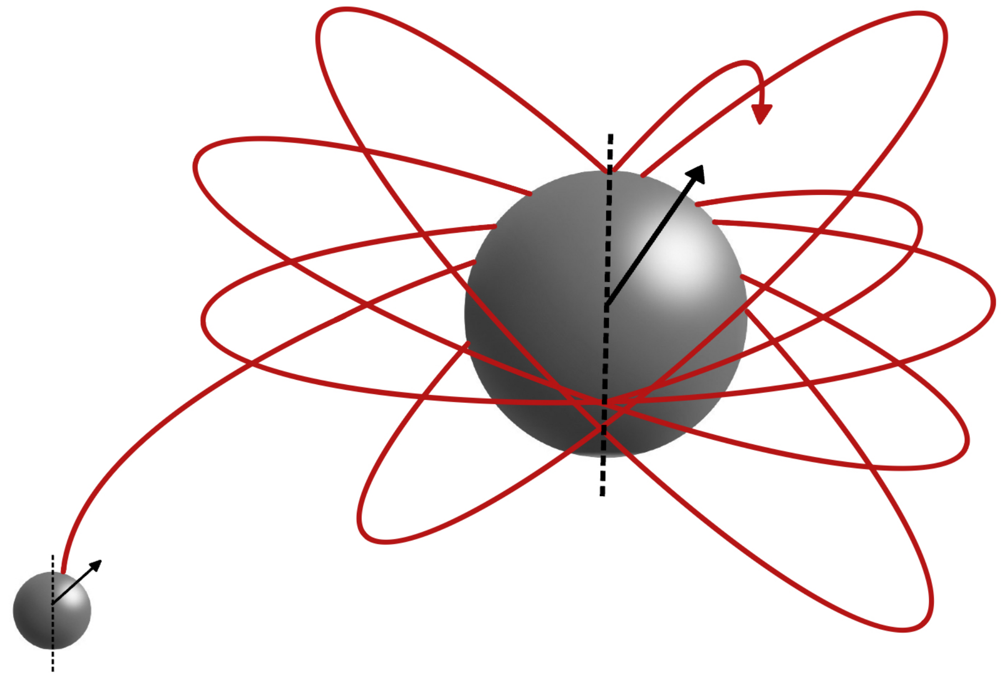
$$E = -\pi_\mu \xi^\mu_{(t)}$$

2. Azimuthal angular momentum

$$L = \pi_\mu \xi^\mu_{(\phi)}$$

3. Carter constant

$$\mathcal{K} = Y_{\mu\rho} Y^\rho{}_\nu p^\mu p^\nu$$



Spinning Probe in Kerr

An interesting problem to study is the dynamics of a spinning probe in a Kerr background ($m_2 \gg m_1$).

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- Geodesics in Kerr: $\mathcal{O}(s_1^0 s_2^\infty)$
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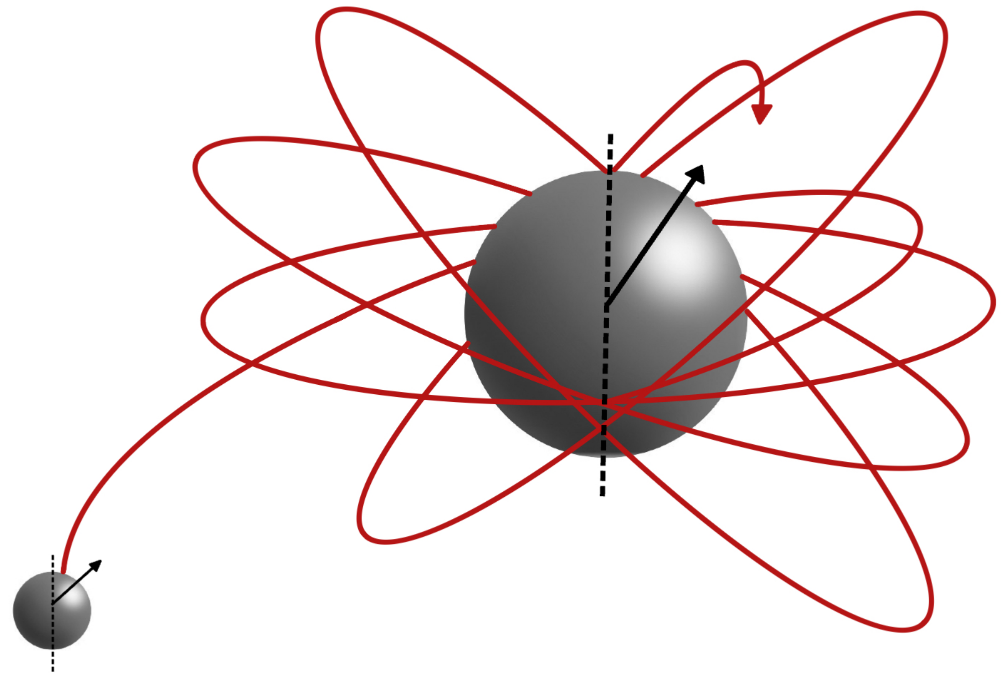

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$$\mathcal{K} = Y_{\mu\rho} Y^\rho{}_\nu p^\mu p^\nu + 4 \xi^\lambda_{(t)} \epsilon_{\lambda\mu\sigma[\rho} Y_{\nu]}{}^\sigma S^{\mu\nu} p^\rho$$

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$$Q_Y = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} S_{\mu\nu} Y_{\alpha\beta}$$

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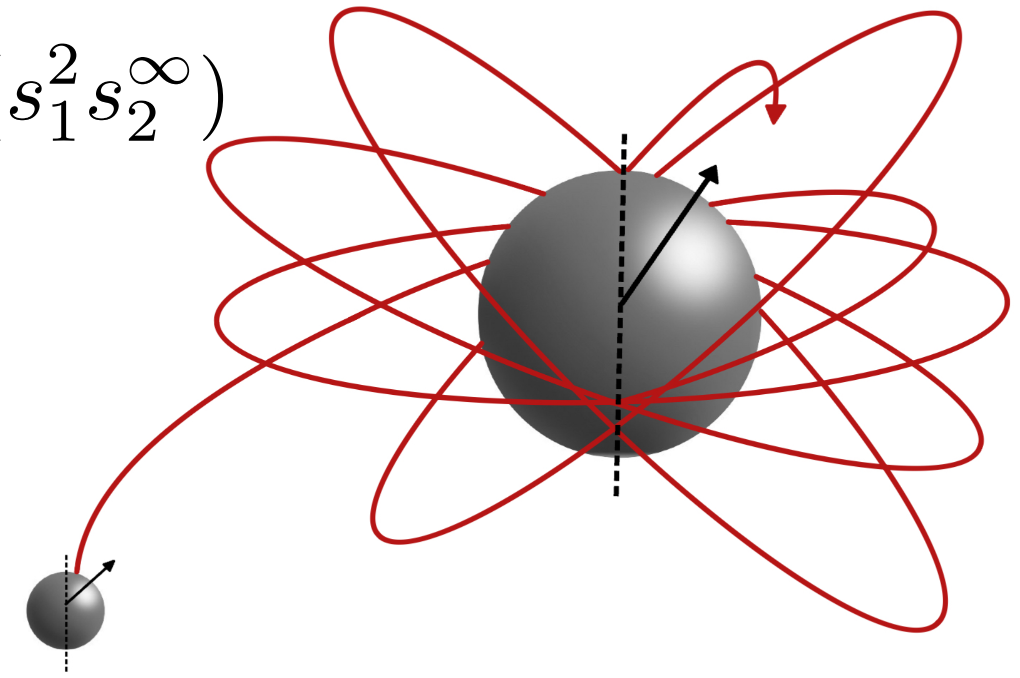

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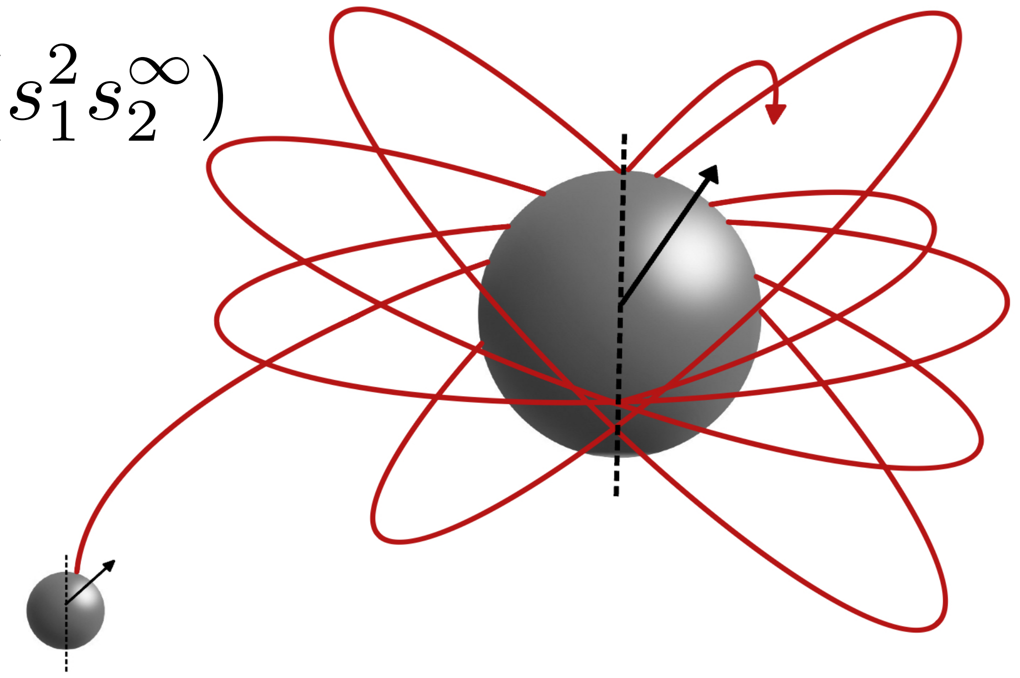
Dynamics of a quadrupolar test particle in Kerr is integrable!!



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Asymptotic Conserved Quantities

- To make a connection with scattering, take an asymptotic limit
  [Gonzo, Shi, '24] [Vines, '23]

$$l^\mu = \epsilon^{\mu\nu\rho\sigma} b_\nu v_{1\rho} v_{2\sigma}$$

$$E, L, \mathcal{K}, Q_Y \xrightarrow{r \to \infty}$$

$$\frac{E}{m_1} = y,$$

$$\tilde{L} = l \cdot s_2 - y(s_1 \cdot s_2) + (v_2 \cdot s_1)(v_1 \cdot s_2)\,,$$

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Not one-to-one!

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- Caveat: this is a weaker notion of conservation, as it only applies to scattering!

Collecting the Available PM Data

To test for asymptotic conservation, we use the following PM results:

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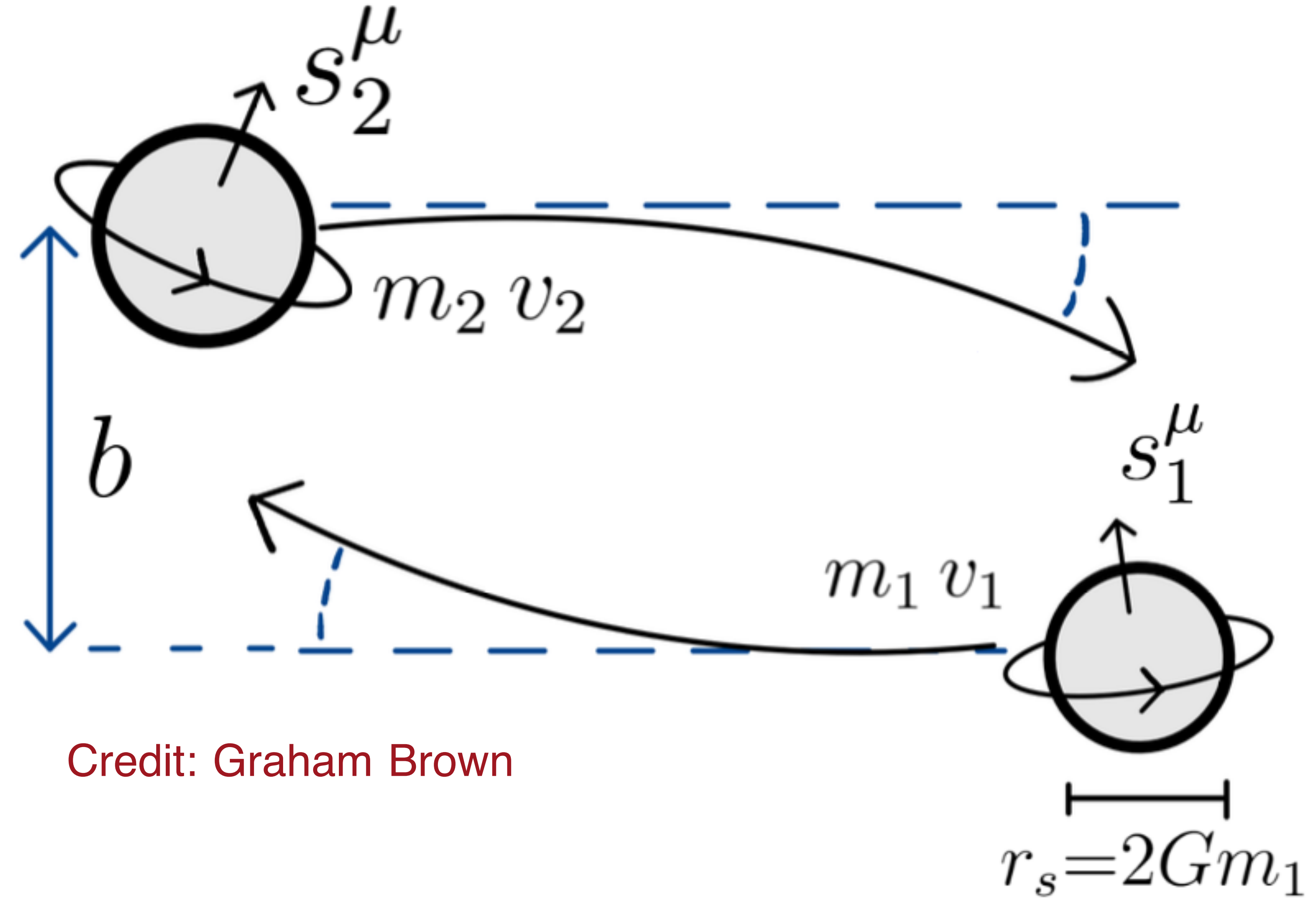

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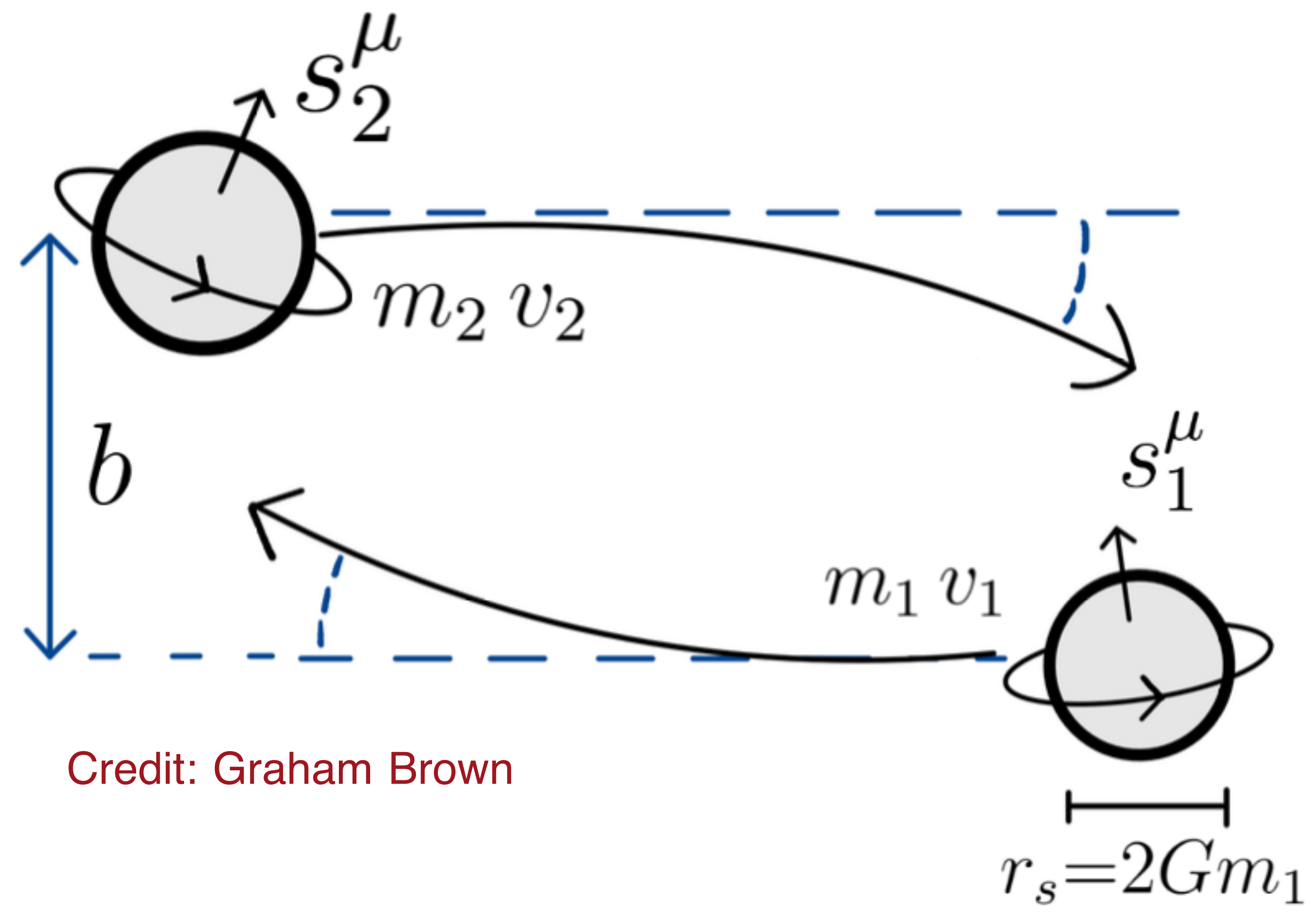

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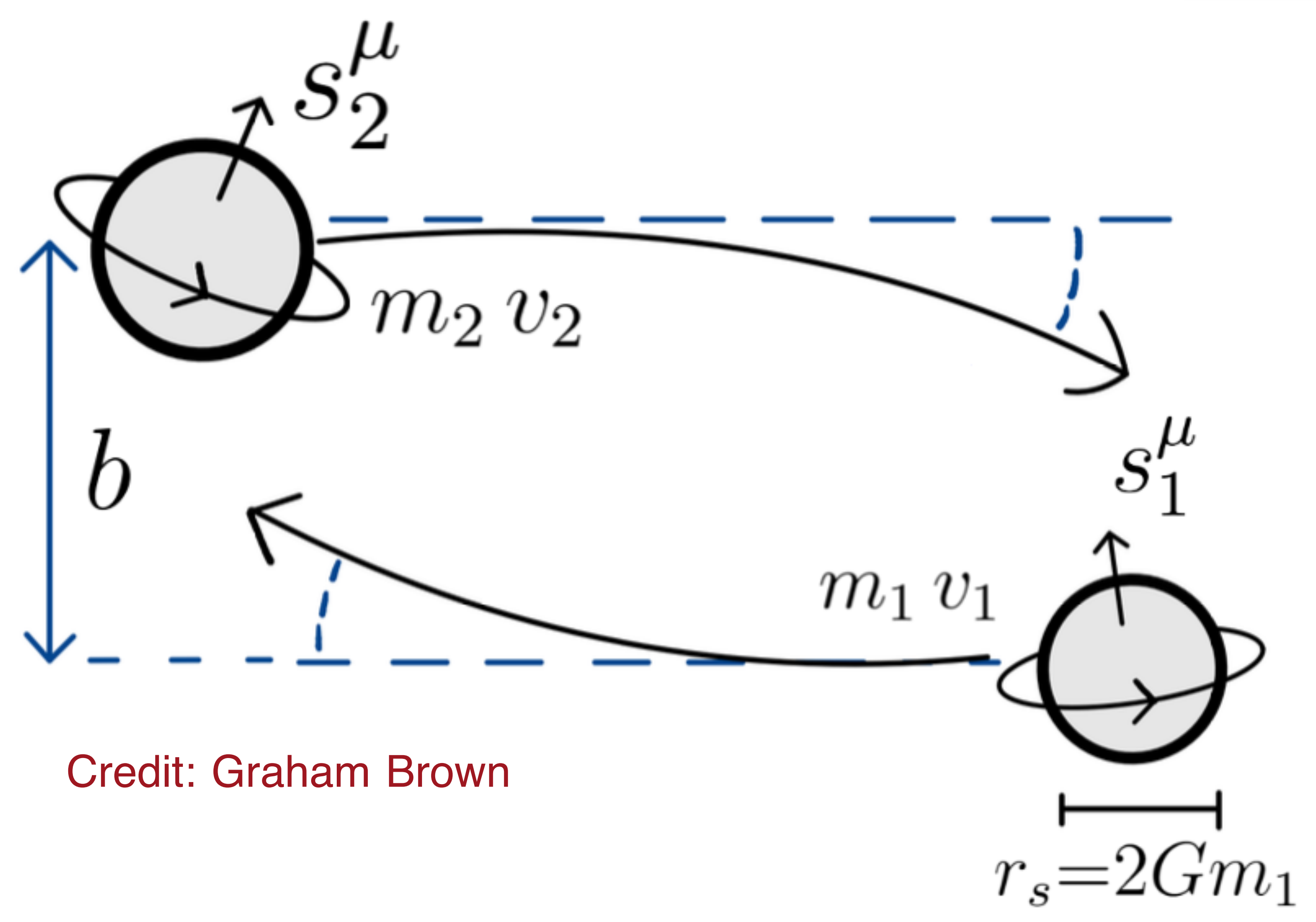

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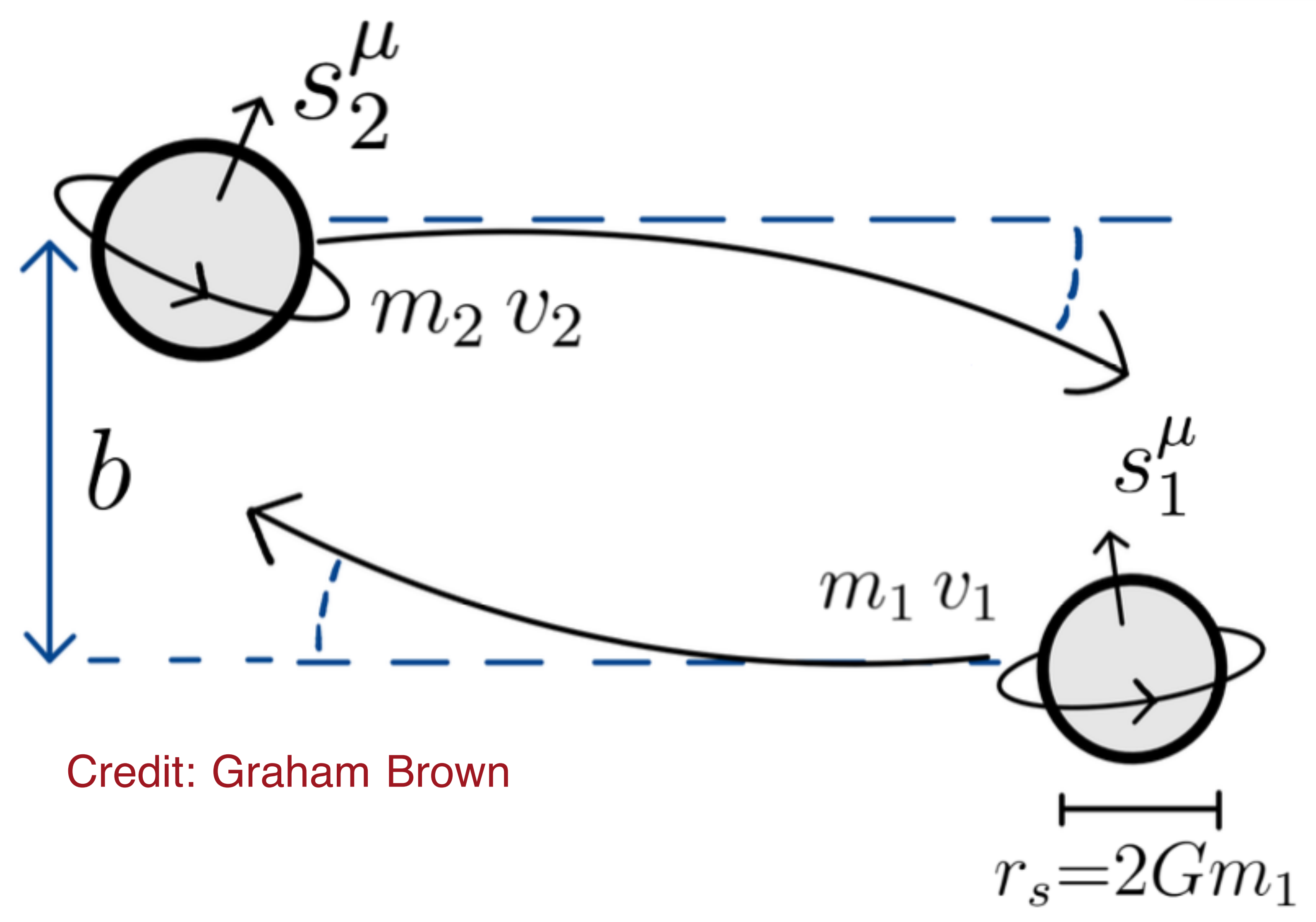

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Fixes coefficients uniquely

Conservation in the Probe Limit

Probe limit					
$\tilde{L}, Q_Y, Q$	$\mathcal{O}(s^2)$	$\mathcal{O}(s^3)$	$\mathcal{O}(s^4)$	$\mathcal{O}(s^5)$	$\mathcal{O}(s^\infty)$
G^1	✓	✓	✓	✓	✓
G^2	✓	✓	✓	X	?
G^3	✓	✓	✓	?	?
G^4	✓	✓	?	?	?
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- We find conservation at:

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Two orders higher than previously known!!

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Can we say anything about integrability?

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It turns out that we can! To do this, we need to count degrees of freedom

$\{v_1^\mu, v_2^\mu, s_1^\mu, s_2^\mu, b^\mu\}$ − (8 + 4) constraints = 8 d.o.f

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Spin magnitude, etc.

Translation invariance

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Spin-Shift Symmetry

Exhibited by spinning four-point amplitudes that holds to quartic order in spin and 3PM, for probe motion.

[Bern, Kosmopoulos, Luna, Roiban, Teng, '22] [Aoude, Haddad, Helset, '22] [DA, Febres Cordero, Kraus, S. Ruf, Zeng, '24] [DA, Febres Cordero, Kraus, Smirnov, Zeng, '25]

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Differential operator

$$q^\mu \frac{\partial}{\partial s_i^\mu} \mathcal{M}_4(q, s_1, s_2) = 0$$

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Exhibited by spinning four-point amplitudes that holds to quartic order in spin and 3PM, for probe motion.

[Bern, Kosmopoulos, Luna, Roiban, Teng, '22] [Aoude, Haddad, Helset, '22] [DA, Febres Cordero, Kraus, S. Ruf, Zeng, '24] [DA, Febres Cordero, Kraus, Smirnov, Zeng, '25]

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Differential operator

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$\int \mathrm{d}^{D-2} q \, e^{ib\cdot q}$ Fourier transform to position space

$$\hat{O}_{\mathrm{SS}} I_r(\{v_i, s_i, b\}) = 0, \;\; \hat{O}_{\mathrm{SS}} = \Pi^{\mu\nu} \frac{\partial}{\partial b^\mu} \frac{\partial}{\partial s_i^\nu}$$

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Probe limit					
$\tilde{L}, Q_Y, Q$	$\mathcal{O}(s^2)$	$\mathcal{O}(s^3)$	$\mathcal{O}(s^4)$	$\mathcal{O}(s^5)$	$\mathcal{O}(s^\infty)$
G^1	✓	✓	✓	✓	✓
G^2	✓	✓	✓	X	?
G^3	✓	✓	✓	?	?
G^4	✓	✓	?	?	?
G^5	✓	?	?	?	?

Curiosity: holds everywhere integrability holds…

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G^1	✓	✓	✓	✓	✓
G^2	✓	✓	✓	X	?
G^3	✓	✓	✓	?	?
G^4	✓	✓	?	?	?
G^5	✓	?	?	?	?

Curiosity: holds everywhere integrability holds…

Holds to all orders in perturbation theory!!

What is Spin-Shift Symmetry?

- We can ask the question: is there a conserved quantity that generates spin-shift symmetry?

$$\hat{O}_{\mathrm{SS}} I_r(\{v_i, s_i, b\}) \stackrel{?}{=} \{\mathcal{O}, I_r\}_{\mathrm{DB}}$$

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- Our guess is that it is a gauge symmetry, i.e. first class constraint, much like for polarisation vectors

$$\epsilon^\mu(k) \to \epsilon^\mu(k) + \zeta k^\mu$$

From the amplitudes perspective

$$s^\mu(p) \sim \epsilon^\mu{}_{\nu\rho\sigma} p^\nu \bar{\varepsilon}^\rho \bar{\varepsilon}^\sigma$$

Bootstrapping the Radial Action

- Can we bootstrap the radial action by demanding conservation of $\{\tilde{L}, Q_Y, Q\}$ + spin-shift symmetry?

  ↳ Turns out that the answer is yes for a spinning probe in a Kerr background!!

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Ansatz for I_r at $\mathcal{O}(s^4)$ and any PM order

$$\mathcal{I}_r^{(n)}(b) = \frac{1}{|b|^{(n-1)(1-2\epsilon)}} \sum_{s_1,s_2} \sum_i \alpha^{(s_1,s_2,i)} \mathcal{J}^{(s_1,s_2,i)}$$

70 unfixed coefficients

−

$$\mathcal{Q} \in \{\tilde{L}, Q_Y, Q\}$$

$$\{I_r, \mathcal{Q}\}_{\rm DB} = 0$$

40 constraints

−

$$\hat{O}_{\rm SS} I_r = 0$$

15 constraints

=

$$I_r$$

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=

$$I_r$$

15 unfixed coefficients

Remaining unfixed coefficients can be fixed by aligned-spin scattering data!!

$$v_i \cdot s_j = 0, \quad s_j \cdot b = 0$$

Conclusions

- Defined a new notion of integrability, which holds at 3/4PM to $\mathcal{O}(s^3)$ and at 3PM to $\mathcal{O}(s^4)$, in the probe limit!!

Probe limit					
$\tilde{L}, Q_Y, Q$	$\mathcal{O}(s^2)$	$\mathcal{O}(s^3)$	$\mathcal{O}(s^4)$	$\mathcal{O}(s^5)$	$\mathcal{O}(s^\infty)$
G^1	✓	✓	✓	✓	✓
G^2	✓	✓	✓	X	?
G^3	✓	✓	✓	?	?
G^4	✓	✓	?	?	?
G^5	✓	?	?	?	?

Beyond probe limit (conservative)							
$\tilde{L}, Q_Y, Q$	$\mathcal{O}(s^1)$	$\mathcal{O}(s_1^2 s_2^0)$	$\mathcal{O}(s_1^3 s_2^0)$	$\mathcal{O}(s_1^4 s_2^0)$	$\mathcal{O}(s_1^5 s_2^0)$	$\mathcal{O}(s_1^\infty s_2^0)$	$\mathcal{O}(s_1^1 s_2^1)$
G^1	✓	✓	✓	✓	✓	✓	X
G^2	✓	✓	✓	✓	X	?	X
G^3	✓	X	X	X	?	?	X

- Conjecture: asymptotic integrability holds to $\mathcal{O}(s^4)$ and all orders in PM

- Provided a position space operator for spin-shift symmetry. We find that it holds to all orders in the PM expansion and seem to accompany integrability.

- Conserved quantities + spin-shift symmetry gives strong constraints on the radial action. We can bootstrap the radial action solely in terms of aligned-spin data!!

Future Directions

1. Can we understand the exact origin of spin-shift symmetry?
2. Can we bootstrap other objects on symmetry grounds as we did for the radial action?
3. root-Kerr?
4. Can we test new data?
5. Can we understand what happens beyond the quartic order in spin?

TACK!

Beyond the Probe Limit, Beyond Kerr

Beyond the probe limit:

Conservation is broken very quickly (unless one BH is spinless)

Beyond probe limit (conservative)							
$\tilde{L}, Q_Y, Q$	$\mathcal{O}(s^1)$	$\mathcal{O}(s_1^2 s_2^0)$	$\mathcal{O}(s_1^3 s_2^0)$	$\mathcal{O}(s_1^4 s_2^0)$	$\mathcal{O}(s_1^5 s_2^0)$	$\mathcal{O}(s_1^\infty s_2^0)$	$\mathcal{O}(s_1^1 s_2^1)$
G^1	✓	✓	✓	✓	✓	✓	X
G^2	✓	✓	✓	✓	X	?	X
G^3	✓	X	X	X	?	?	X

Conserved quantities are no longer in involution (unless one BH is spinless)

$$\vec{\mathcal{Q}} = \{\tilde{L}, Q_Y, Q\} \quad \{\mathcal{Q}_i, \mathcal{Q}_j\}_{\mathrm{DB}} \neq 0$$

Asymptotic integrability for one spinless BH at 2PM, beyond the probe limit!!

Beyond Kerr:

Non-minimal spin coupling terms [U. Jakobsen, Mogull, '22]

$$S_{\mathrm{E}}^{(i)} := -m_i\, C_{\mathrm{E},i} \int \mathrm{d}\tau\, R_{a\mu b\nu} \dot{x}_i^\mu \dot{x}_i^\nu \bar{\psi}_i^a \psi_i^b \tilde{P}_{cd} \bar{\psi}_i^c \psi_i^d$$

Probe limit: 3PM (Non-minimal coupling)	$\mathcal{O}(s_1^1 s_2^0)$	$\mathcal{O}(s_1^0 s_2^1)$	$\mathcal{O}(s_1^1 s_2^1)$	$\mathcal{O}(s_1^2 s_2^0)^*$	$\mathcal{O}(s_1^0 s_2^2)^*$
$\tilde{L}$	n/a	✓	✓	n/a	✓
Q_Y	✓	n/a	✓	X	n/a
Q	✓	✓	✓	✓	X

Breaks conservation for non-Kerr values of Wilson coefficients ($C_{\mathrm{E},i} \neq 0$)

[Compère, Druart, Vines, '23]

Restoring integrability beyond the probe limit:

Can be done at 1PM:

$$\mathscr{L} = \frac{m_1^2 (u_2 \cdot a_1)}{\Sigma_{m^2}} + \frac{m_2^2 (u_1 \cdot a_2)}{\Sigma_{m^2}} - \frac{(2 m_1 m_2 + \Sigma_{m^2} y)}{\Sigma_{m^2}} (a_1 \cdot a_2) + (u_2 \cdot a_1)(u_1 \cdot a_2)$$

$$\mathscr{Q}_Y = \frac{m_1^2 (u_2 \cdot a_1)}{\Sigma_{m^2}} + \frac{m_2^2 (u_1 \cdot a_2)}{\Sigma_{m^2}} + \frac{(2 m_1 m_2 + \Sigma_{m^2} y)}{\Sigma_{m^2}} (a_1 \cdot a_2) - (u_2 \cdot a_1)(u_1 \cdot a_2)$$

$$\mathscr{Q} = -l^2 + \frac{2\Delta_{m^2} y}{\Sigma_{m^2}} (l \cdot a_2) - (y^2 - 1) a_2^2 - (u_1 \cdot a_2)^2 - \frac{2\Delta_{m^2} y}{\Sigma_{m^2}} (l \cdot a_1) + 2\left(y^2 + 1 + \frac{4 m_1 m_2 y}{\Sigma_{m^2}}\right)(a_1 \cdot a_2) - 2y(u_2 \cdot a_1)(u_1 \cdot a_2) - (y^2 - 1) a_1^2 - (u_1 \cdot a_2)^2$$

$$\Sigma_{m^2} = m_1^2 + m_2^2 \,, \quad \Delta_{m^2} = m_1^2 - m_2^2$$

Reduces to previous results in the probe limit!

A sense of perturbative integrability?

[Tanay, Stein, Ghersi, '20]