

Metric of a Star

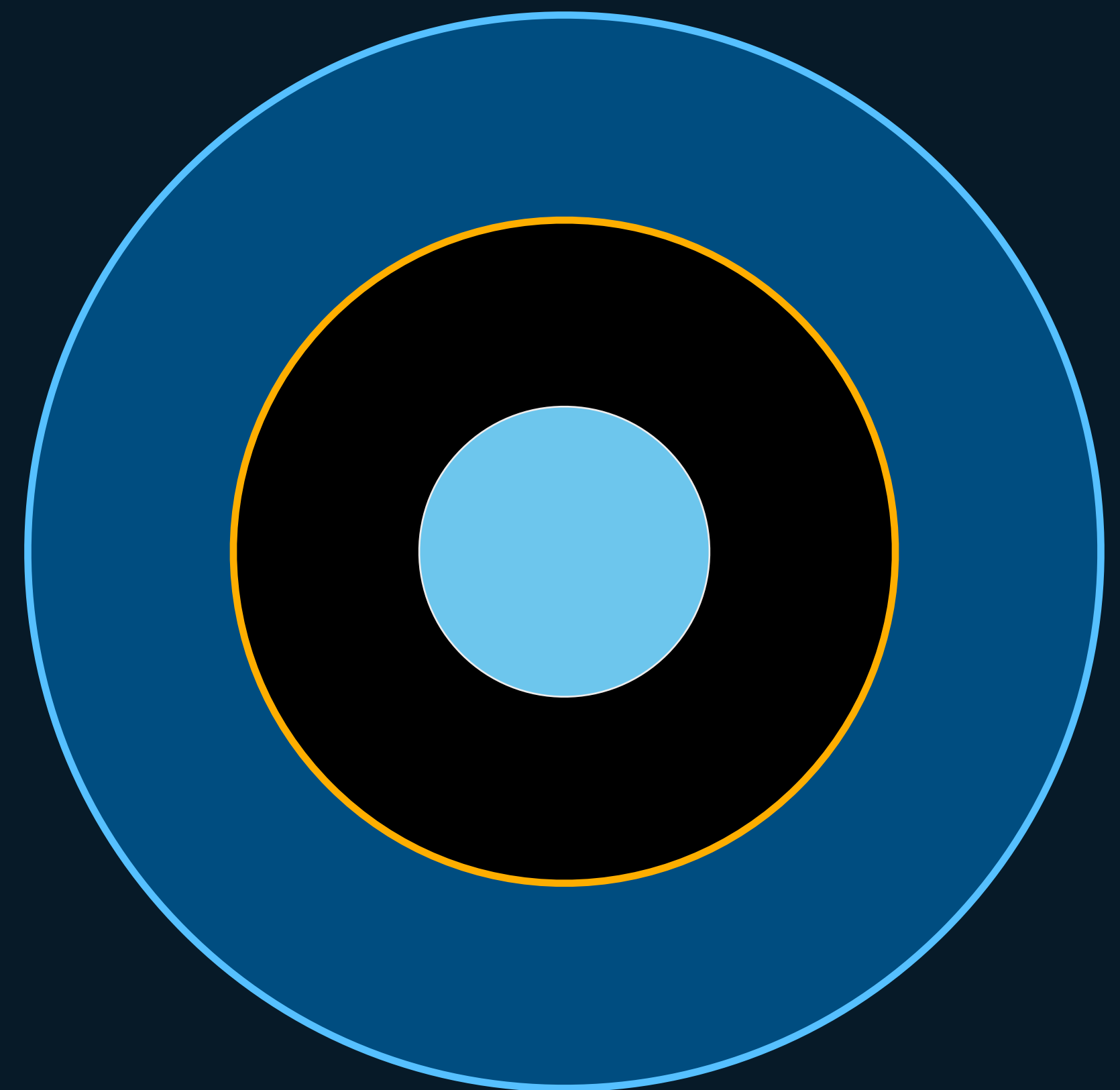
A recursive multipolar post-Minkowskian Formalism

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based on work in collaboration with Poul Henrik Damgaard,
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 (2603.16493, 2604.XXXX)

***Amplitude, Strong-Field Gravity and Resumption,
 Nordita, 17/04/2026***



1. Motivation & Setup
2. The multipolar post-Minkowskian (MPM) Formalism
3. Recursive post-Minkowskian Construction
4. Kerr Matching as special case
5. Summary & Outlook

The Problem

- ▶ In Newtonian gravity, the external field of a star is *completely and cleanly* determined by its multipole moments — one-to-one correspondence, no mixing, solved by Poisson's equation exactly
- ▶ In GR, non-linearity *contaminates* this: the metric at order $1/r^2$ gets contributions from G^2 , the metric at order $1/r^3$ gets contributions from G^2 mixing monopole with quadrupole, and so on

The question is: *can we still organize this systematically?*

Momentum space

$$g^{\mu\nu}(x^\mu) \equiv \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} - \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu}$$

Convolutions in x - space become products in k - space. Momentum space currents becomes polynomial in k .

$$\square h_{(1PM)}^{\mu\nu}(x) = -16\pi T^{\mu\nu}(x)$$



$$-k^2 \mathcal{J}_{(1PM)}^{\mu\nu}(k) = -16\pi \tilde{T}^{\mu\nu}(k)$$

Non-linearities at higher PM orders becomes recursive momentum space integrals.

The Formalism

Momentum space

$$g^{\mu\nu}(x^\mu) \equiv \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} - \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu}$$

Convolutions in x - space become products in k - space. Momentum space currents becomes polynomial in k .

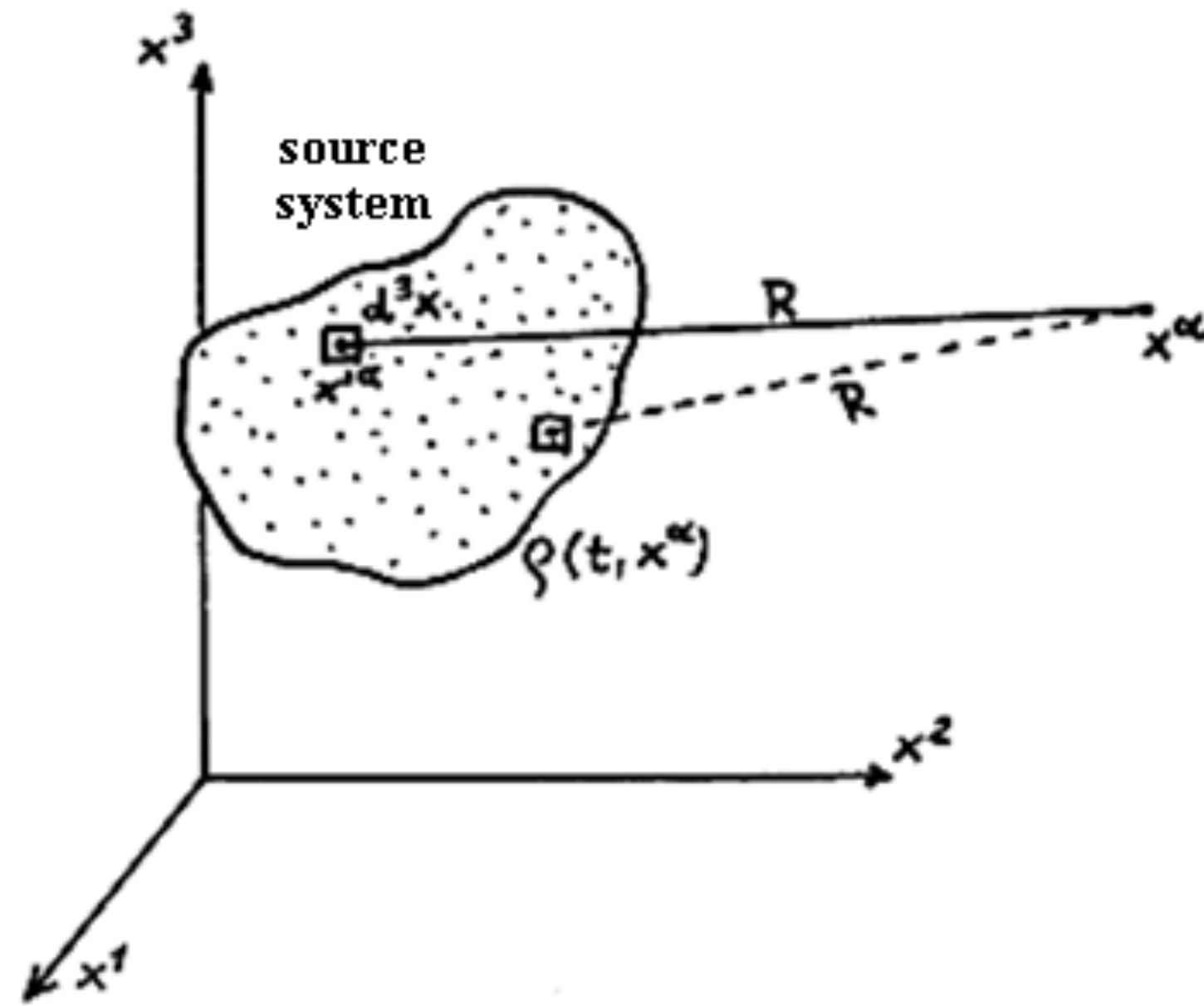
$$\square h_{(1PM)}^{\mu\nu}(x) = -16\pi T^{\mu\nu}(x)$$

NO Amplitude techniques, NO worldline diagrams!

$$-k^2 \mathcal{J}_{(1PM)}^{\mu\nu}(k) = -16\pi \tilde{T}^{\mu\nu}(k)$$

Non-linearities at higher PM orders becomes recursive momentum space integrals.

The MPM Setup



All gravitational waves, no matter how strong or weak, can be completely characterized by a set of multipole moments of the source.

[Thorne (1980)]

The Double Expansion

[Hansen (1974), Thorne (1980), Damour, Iyer, Blanchet (80's - 90's)]

Post-Minkowskian Expansion:

$$g^{\mu\nu}(x^\mu) \equiv \sqrt{-g} g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}, \quad |h_{\mu\nu}| \sim \frac{GM}{rc^2} \ll 1$$

$$\eta^{\alpha\beta} = \text{diag}(-1, +1, +1, +1)$$

Weak Field, any velocity expansion around Minkowski : $h^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu}$

Multipolar Post-Minkowskian (MPM) Expansion:

$$g_{\mu\nu}^{\text{ext}} \longleftrightarrow \{M_L, S_L\}_{\ell=0}^{\infty}$$

Each $h_{(n)}^{\mu\nu}$ admits finite multipolar decomposition: $h^{\mu\nu} = \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu} [M_L, S_L]$,

$$M^L(t) = \int d^3x T^{00}(t, \mathbf{x}) \hat{x}^L, \quad S^L(t) = \int d^3x x^{L-1} \epsilon_{jk}^i x^j T^{0k}(t, \mathbf{x})$$

$$L = i_1 \cdots i_\ell$$

Exterior of a Source

General time-dependent source

e.g. Binary inspirals

- **Near zone** ($R < r \ll \lambda$)
PN expansion valid
 $PN : v \ll c$
- **Exterior** ($R < r_e < r$)
MPM construction lives here
 $PM : \frac{GM}{rc^2} \sim \frac{v^2}{c^2}$
- **Wave zone** ($r \gg \lambda, r \rightarrow \infty$)
Radiation field, Far Zone

$M_L(t), S_L(t)$ - time dependent
Give real emitted radiations



Stationary Kerr / Star

Damgaard, Lee, Lee, TR (2603.16493)

- **Exterior Zone only** ($r > R$)

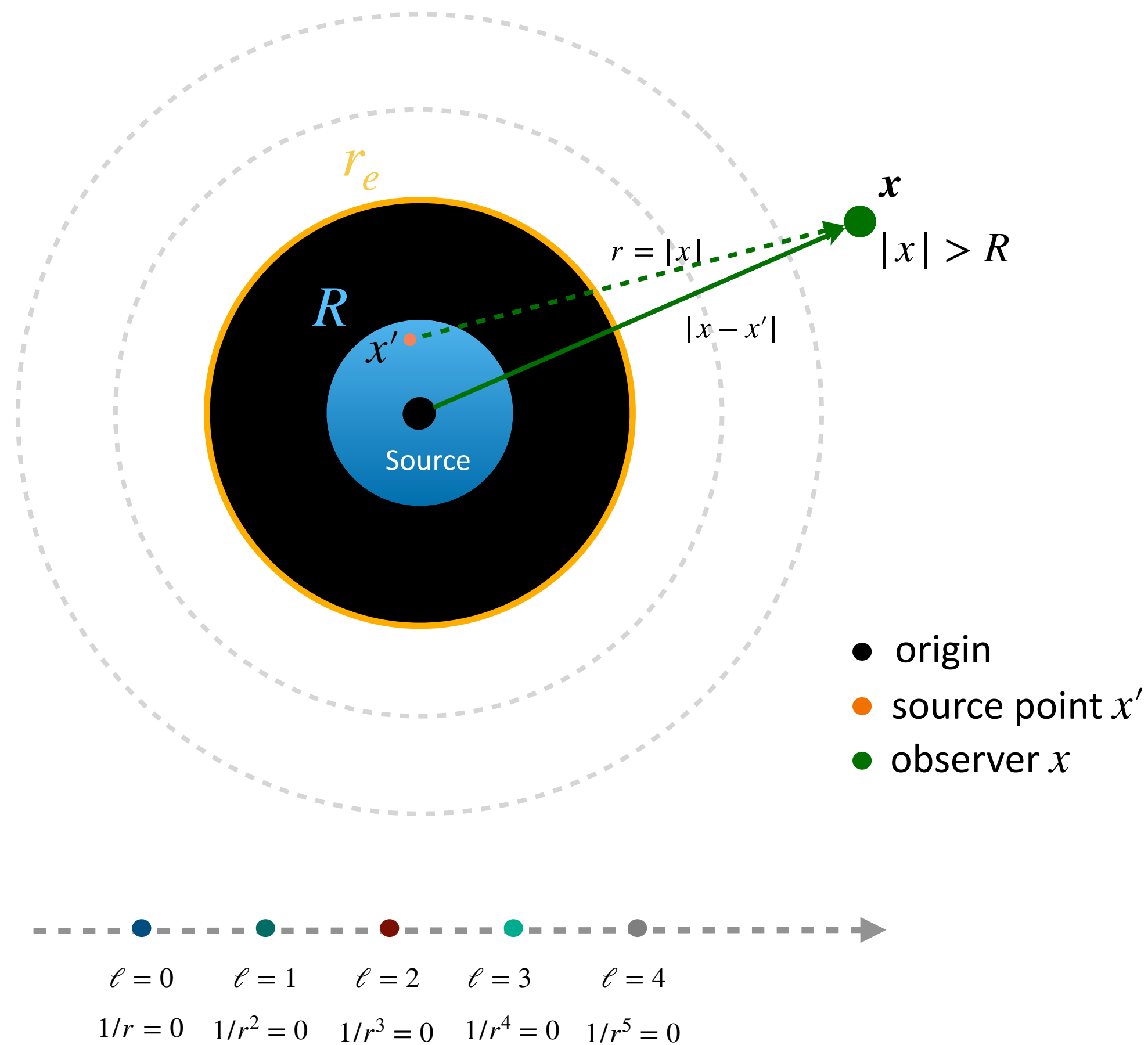
MPM valid everywhere outside
No near zone, no wave zone
No radiation
 $\lambda \rightarrow \infty$
(No physical scale)

M_L, S_L - constant
Fixed by M and J/M

VS

The Setup - 1PM in position space

Geometry



Taylor Expansion $r = |x| > |x'|$

$$\square h_{(1PM)}^{\mu\nu}(x) = -16\pi T^{\mu\nu}(x)$$

$$h_{(1PM)}^{\mu\nu}(x) = 4 \int d^4x' \frac{T^{\mu\nu}(x')}{|\mathbf{x} - \mathbf{x}'|}$$

$$\frac{1}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'_L \partial_L \left(\frac{1}{r} \right)$$

The Recursion

Recursive EOM

Field Equations:

$$\square h^{\mu\nu} = -16\pi G (\tau^{\mu\nu} + t^{\mu\nu})$$

$$\tau^{\mu\nu} = T^{\mu\nu} + \sum_{n=1}^{\infty} G^n \tau_{(nPM)}^{\mu\nu},$$

$$t^{\mu\nu} = \sum_{n=1}^{\infty} G^n t_{(nPM)}^{\mu\nu}$$

Recursive hierarchy:

$$\square h_{(1PM)}^{\alpha\beta} = -16\pi T^{\alpha\beta},$$

$$\square h_{(2PM)}^{\alpha\beta} = -16\pi \left(\tau_{(1PM)}^{\alpha\beta} + t_{(1PM)}^{\alpha\beta} \right),$$

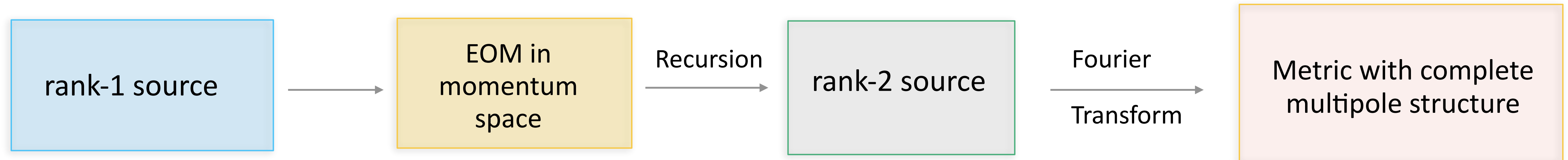
\vdots

$$\square h_{(nPM)}^{\alpha\beta} = -16\pi \left(\tau_{((n-1)PM)}^{\alpha\beta} + t_{((n-1)PM)}^{\alpha\beta} \right).$$

Solve these!

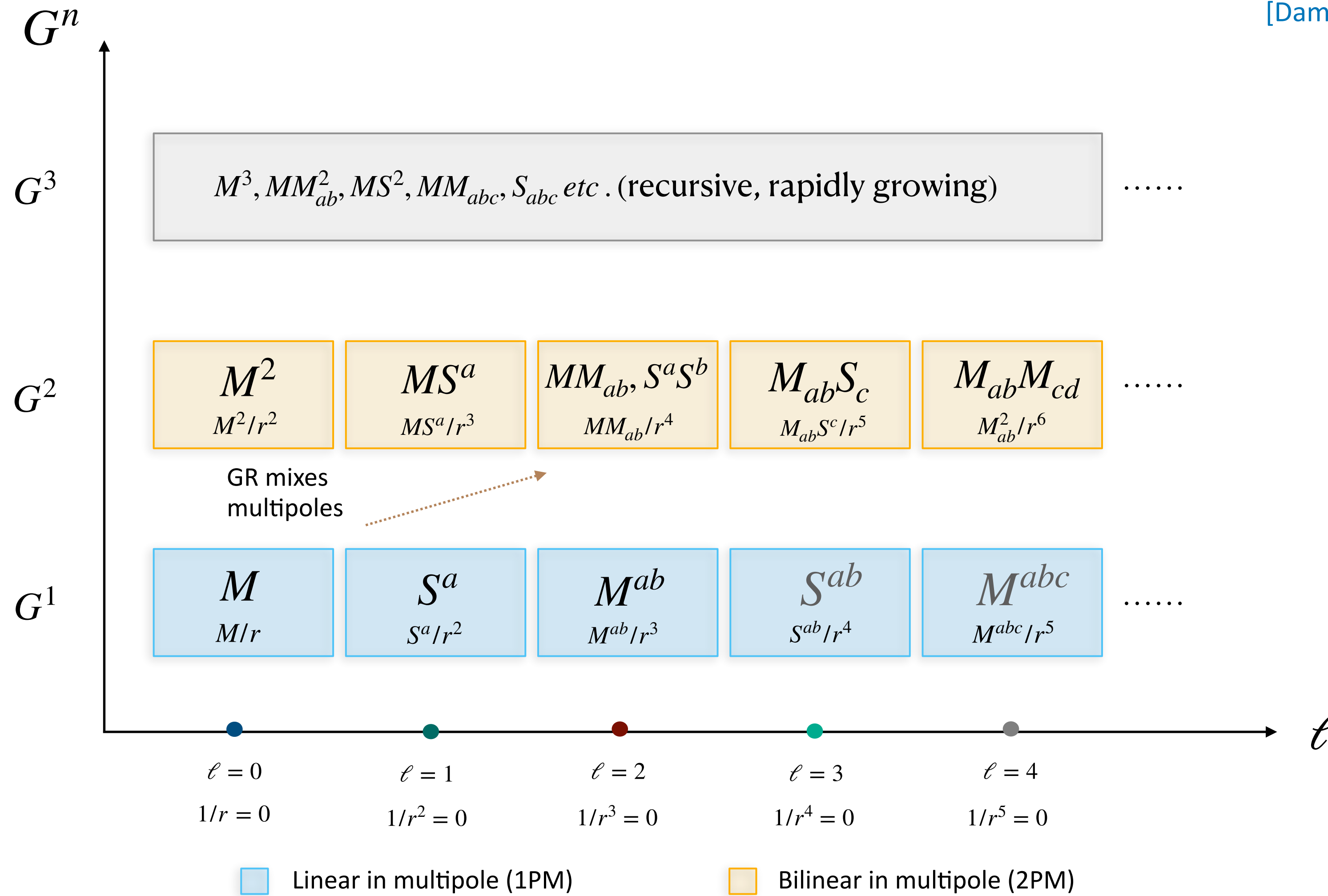
Recursion Pipeline

[Damgaard, Lee, Lee, TR (2026)]



MPM mixing

[Damgaard, Lee, Lee, TR (2026)]



$$M^L(t) = r^\ell \int d^3x T^{00}(t, \mathbf{x}) \hat{n}^L,$$

$$S^L(t) = \int d^3x x^{L-1} \epsilon_{jk}^i x^j T^{0k}(t, \mathbf{x})$$

Initial Conditions

[Damgaard, Lee, Lee, TR (2026)]

$$h^{\mu\nu}(\mathbf{x}) = \sum_{n=1}^{\infty} G^n h_{(n)}^{\mu\nu}(\mathbf{x}) \implies \mathcal{J}^{\mu\nu}(\mathbf{k}) = \sum_{n=1}^{\infty} G^n \mathcal{J}_{(n)}^{\mu\nu}(\mathbf{k})$$

► Fourier transform:

$$h_{(n)}^{\mu\nu} [M_L, S_L] \equiv \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \underbrace{\mathcal{J}_{(n)|\mathbf{k}}^{\mu\nu} [M_L, S_L]}_{\text{Rank-}n \text{ Graviton Current}}$$

- At **1PM** : The initial condition for the recursion is given by **rank-1 current**:

$$\mathcal{J}_{(1)|\mathbf{k}}^{\mu\nu} [M_L, S_L] = \int_{\mathbf{x}} e^{-i\mathbf{k}\cdot\mathbf{x}} h_{(1)}^{\mu\nu} [M_L, S_L]$$

$$M^L(t) = r^\ell \int d^3x T^{00}(t, \mathbf{x}) \hat{n}^L,$$

$$S^L(t) = \int d^3x x^{L-1} \epsilon_{jk}^i x^j T^{0k}(t, \mathbf{x})$$

Stationary Metric at 2PM

[Damgaard, Lee, Lee, TR (2026)]

Stationary Case

$$\nabla^2 h_{(1PM)}^{\mu\nu}(\mathbf{x}) = -16\pi T^{\mu\nu}(\mathbf{x})$$

- For instance, expansion to mass quadrupole term $\ell = 0, 2$ and current monopole terms ($\ell = 1$)

$$\begin{aligned} h_{(1)}^{00} &= \frac{4M}{r} + \frac{6M_{ab}}{r^3} \hat{n}_{ab} \\ h_{(1)}^{ij} &= 0 \\ h_{(1)}^{0i} &= -2\epsilon_{iab} n^a \frac{S^b}{r^2} \end{aligned}$$

$$\int_{\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}} (\cdot)$$

$$\begin{aligned} \mathcal{F}_{(1)|k}^{00} &= 16\pi G \left(\frac{M}{|\mathbf{k}|^2} - \frac{M_{ab} \hat{\mathbf{k}}^a \hat{\mathbf{k}}^b}{2|\mathbf{k}|^2} \right) \\ \mathcal{F}_{(1)|k}^{ij} &= 0 \\ \mathcal{F}_{(1)|k}^{0i} &= i\epsilon_{iab} \frac{8\pi G S^b k^a}{|\mathbf{k}|^2} \end{aligned}$$

Seeds for recursion

Stationary Metric at 2PM

[Damgaard, Lee, Lee, TR (2026)]

► 2PM EOM: $\square h_{(2PM)}^{\alpha\beta} = -16\pi \left(\tau_{(1PM)}^{\alpha\beta} + t_{(1PM)}^{\alpha\beta} \right)$

$$\partial^2 h_{(2)}^{00} = \frac{7}{8} \partial_i h_{(1)}^{00} \partial_i h_{(1)}^{00} + \frac{5}{4} h_{(1)}^{00} \partial^2 h_{(1)}^{00} - \frac{1}{2} \partial_i h_{(1)}^{0j} \partial_i h_{(1)}^{0j} - \frac{3}{2} \partial_i h_{(1)}^{0j} \partial_j h_{(1)}^{0i} - h_{(1)}^{0i} \partial^2 h_{(1)}^{0i}$$

Algebraic equation in Fourier space

$$\mathcal{F}_{(2)|-k_1}^{00} = \frac{1}{|k_1|^2} \int_{k_2} \left[-\frac{7}{8} k_{12} \cdot k_2 + \frac{5}{4} |k_2|^2 \right] \mathcal{F}_{(1)|-k_{12}}^{00} \mathcal{F}_{(1)|k_2}^{00} + \left[\left(\frac{1}{2} k_{12} \cdot k_2 - |k_2|^2 \right) \mathcal{F}_{(1)|-k_{12}}^{0i} \mathcal{F}_{(1)|k_2}^{0i} + \frac{3}{2} k_2^i k_{12}^j \mathcal{F}_{(1)|-k_{12}}^{0i} \mathcal{F}_{(1)|k_2}^{0j} \right]$$

$$\mathcal{F}_{(1)|k}^{00} = 16\pi G \left(\frac{M}{|k|^2} - \frac{M_{ab} \hat{k}^a \hat{k}^b}{2|k|^2} \right)$$

$$\mathcal{F}_{(1)|-k}^{0i} = -i\epsilon_{iab} \frac{8\pi G S^b k^a}{|k|^2}$$

Stationary Metric at 2PM

[Damgaard, Lee, Lee, TR (2026)]

► 2PM EOM: $\square h_{(2PM)}^{\alpha\beta} = -16\pi \left(\tau_{(1PM)}^{\alpha\beta} + t_{(1PM)}^{\alpha\beta} \right)$

$$\partial^2 h_{(2)}^{00} = \frac{7}{8} \partial_i h_{(1)}^{00} \partial_i h_{(1)}^{00} + \frac{5}{4} h_{(1)}^{00} \partial^2 h_{(1)}^{00} - \frac{1}{2} \partial_i h_{(1)}^{0j} \partial_i h_{(1)}^{0j} - \frac{3}{2} \partial_i h_{(1)}^{0j} \partial_j h_{(1)}^{0i} - h_{(1)}^{0i} \partial^2 h_{(1)}^{0i}$$

Algebraic equation in Fourier space

$$\mathcal{F}_{(2)|-k_1}^{00} = \frac{1}{|k_1|^2} \int_{k_2} \left[-\frac{7}{8} k_{12} \cdot k_2 + \frac{5}{4} |k_2|^2 \right] \mathcal{F}_{(1)|-k_{12}}^{00} \mathcal{F}_{(1)|k_2}^{00} + \left[\left(\frac{1}{2} k_{12} \cdot k_2 - |k_2|^2 \right) \mathcal{F}_{(1)|-k_{12}}^{0i} \mathcal{F}_{(1)|k_2}^{0i} + \frac{3}{2} k_2^i k_{12}^j \mathcal{F}_{(1)|-k_{12}}^{0i} \mathcal{F}_{(1)|k_2}^{0j} \right]$$

$$\mathcal{F}_{(2)|-k_1}^{00} = \frac{(16\pi G)^2}{|k_1|^2} \int_{k_2} \left[-\frac{7}{8} k_{12} \cdot k_2 + \frac{5}{4} |k_2|^2 \right] \times \left[\left(\frac{M^2}{|k_2|^2 |k_{12}|^2} - \frac{MM_{ab} k_2^a k_2^b}{2 |k_{12}|^2 |k_2|^2} + \dots \right) \right]$$

$$\frac{(16\pi G)^2}{|k_1|^2} \int_{k_2} -\frac{7}{8} \left(\frac{M^2 k_{12} \cdot k_2}{|k_2|^2 |k_{12}|^2} \right) = (16\pi G)^2 \left(-\frac{7}{8} \left(-\frac{M^2}{16 |k_1|} \right) \right)$$

Nothing but Bubbles!

The Results

2PM Quadrupole orders

[Damgaard, Lee, Lee, TR (2026)]



$$h_{(2)}^{00} = \frac{7M^2}{r^2} + \frac{21MM_{ab}}{r^4}\hat{n}_{ab} - \frac{5|\mathbf{S}|^2}{3r^4} + \frac{7S^aS^b}{r^4}\hat{n}^{ab} + \frac{63M_{ab}M_{cd}\hat{n}^{abcd}}{4r^6} + \frac{9M_{ab}^2\hat{n}^{ab}}{r^6} + \frac{21M_{ab}M_{ab}}{10r^6},$$

$$h_{(2)}^{0i} = \frac{2M(\mathbf{S} \times \mathbf{n})^i}{r^3} + \frac{3\tilde{S}_{ia}M_{bc}\hat{n}^{abc}}{2r^5} - \frac{19\tilde{S}_{ia}(M \cdot \mathbf{n})_a}{10r^5} + \frac{M_{ij}(\mathbf{S} \times \mathbf{n})^j}{2r^5},$$

$M_{ab}^2 = M_{ac}M_{cb}, \tilde{S}_{ia} = \epsilon_{iab}S^b,$

$$h_{(2)}^{ij} = \frac{1}{r^2}H_2^{ij} + \frac{1}{r^4}H_4^{ij} + \frac{1}{r^6}H_6^{ij}$$

Long expressions of Mass quadrupole terms
[Blanchet (1998)]

$$H_2^{ij} = M^2 \left[\hat{n}^{ij} + \frac{1}{3}\delta^{ij} \right],$$

$$H_4^{ij} = MM_{ab} \left[\frac{15}{2}\hat{n}^{ijab} + \frac{11}{7}\delta^{ij}\hat{n}^{ab} \right] - \frac{12}{7}MM^{a(i}\hat{n}_a^{j)} + \left[\frac{9}{2}\hat{n}^{ijab} - \frac{13}{7}\delta^{ij}\hat{n}^{ab} \right] S_a S_b - 2\hat{n}^{ab}\tilde{S}_{ia}\tilde{S}_{jb} + \frac{32}{7}S^a\hat{n}_a^{(i}S^{j)} - \frac{S^i S^j}{15} - |\mathbf{S}|^2 \left[\frac{20}{7}\hat{n}^{ij} - \frac{2}{15}\delta^{ij} \right].$$

2PM All Multipole orders

[Damgaard, Lee, Lee, TR (2026)]

Initial conditions for recursion:

$$\mathcal{F}_{(1)|\pm k}^{00} = 16\pi \sum_{l=0}^{\infty} \frac{(\pm i)^l M_L \hat{k}^L}{l! |\mathbf{k}|^2}$$

$$\mathcal{F}_{(1)|\pm k}^{0i} = 16\pi \sum_{l=0}^{\infty} \frac{(\pm i)^l l}{(l+1)!} \epsilon_{iab} \frac{S_{bL-1} \hat{k}^{aL-1}}{|\mathbf{k}|^2}$$

$$\mathcal{F}_{(1)|\pm k}^{ij} = 0$$

$$T_{r|e}^{i_1 \dots i_n} = \frac{n! \delta^{(i_1 i_2 \dots \delta^{i_{2r-1} i_{2r}} \ell^{i_{2r+1} \dots \ell^{i_n})}}{(2r)!(n-2r)!}$$

$$C_{n,r}^{[b]} = (-1)^r \frac{\Gamma\left(n-r+\frac{1}{2}\right)}{2^{r+3} \sqrt{\pi} (2r)!(n-2r)!}$$

$$\mathcal{F}_{(2)|-k}^{00} = \frac{(16\pi)^2}{|\mathbf{k}|^2} \sum_{l,l'=0}^{\infty} \frac{7(-1)^{l+1} i^{l+l'}}{8 l! l'!} \sum_{q=0}^l \binom{l}{q} \sum_{s=0}^{\min(l',q)} C_{n,s}^{[a]} |\mathbf{k}|^{2s+1} M_{SA} k^A M_{SB} k^B$$

$$+ \frac{(16\pi)^2}{2 |\mathbf{k}|^2} \sum_{l,l'=1}^{\infty} \frac{(-1)^l i^{l+l'} l l'}{(l+1)!(l'+1)!} \sum_{p=0}^{l-1} \binom{l-1}{p} \left[\sum_{r=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{C_{n+1,r}^{[b]}}{|\mathbf{k}|^{1-2r}} (\text{Tr}^{rf_{(1)}})_A k^A + \sum_{r'=0}^{\lfloor \frac{n}{2} \rfloor + 1} \frac{C_{n+2,r'}^{[b]}}{|\mathbf{k}|^{1-2r'}} (\text{Tr}^{r'f_{(2)}})_B k^B \right]$$

Closed form to all multipole orders @2PM!

Recursive Integrals

[Damgaard, Lee, Lee, TR (2026)]

The problem merges to recursive bubble integrals

$$\int \frac{d^3 k}{(2\pi)^3} \frac{k^{i_1} \dots k^{i_n}}{|k|^2 |\ell - k|^2} = \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(-1)^r \Gamma(n - r + \frac{1}{2})}{2^{r+3} \sqrt{\pi n}!} T_{r|\ell}^{i_1 i_2 \dots i_n} |\ell|^{2r-1}$$

FT

$$\int \frac{d^3 \ell}{(2\pi)^3} \frac{\ell^{i_1} \dots \ell^{i_m}}{|\ell|^n} e^{i\ell \cdot x} = \sum_{r=0}^{\lfloor m/2 \rfloor} i^{m-2r} \frac{\Gamma(\frac{3-n+2m-2r}{2})}{2^{n-m+r} \pi^{\frac{3}{2}} \Gamma(\frac{n}{2})} \frac{T_{r|x}^{i_1 \dots i_m}}{|\mathbf{x}|^{3-n+2m-2r}},$$

$$T_{r|\ell}^{i_1 \dots i_n} = \frac{n! \delta^{(i_1 i_2 \dots \delta^{i_{2r-1} i_{2r}} \ell^{i_{2r+1}} \dots \ell^{i_n})}}{(2r)!(n-2r)!}$$

The Kerr Check & Implications

The Kerr Corner

- ▶ The Kerr BH occupies a very special, highly degenerate corner:

Mass moments: $M_L = (-1)^l M a^{2l} \hat{s}^L$

Current moments: $S_L = (-1)^l M a^{2l+1} \hat{s}^L$

$$s^i = \delta^{i3}, i = 1,2,3$$

An infinite tower of moments, *entirely* determined by mass (M) and spin ($a = J/M$).



- ▶ But more importantly: *nearby* this corner live compact stars that are Kerr-like but are not black holes - no horizons, yet arbitrarily similar multiple structure from far away.

Consistency Check

[Damgaard, Lee, Lee, TR (2026)]

On matching Procedure

- ▶ Kerr in harmonic coordinate [\[Jiang, Lin \(2014\)\]](#)
- ▶ Expansion $g_{Kerr}^{\mu\nu}$ in G and $1/r$ at 1PM order
- ▶ Read off M_L and S_L multipole coefficients at each order in $1/r$
- ▶ Compare coefficient-by-coefficient against our MPM recursive expansion

FULL AGREEMENT: all Kerr moments including infinite tower reproduced!

Identifying Kerr Multipoles

[Damgaard, Lee, Lee, TR (2026)]

Tag each 1PM term with indicator $Q[n_1]$ and run the recursion. Each product $Q[n_1]Q[n_2]$ in 2PM shows the multipole interactions

$$Q[\ell] = a^\ell$$

Initial conditions: $M^{-2}h_{(1)}^{00} \Big|_{\mathcal{O}(a^2)} = \frac{4M}{r}Q[0] - \frac{6Ma^2}{r^3} \left(\frac{z^2}{r^2} - \frac{1}{3} \right) Q[2], \quad M^{-1}h_{(1)}^{0i} = \frac{-2May}{r^2}Q[1]$

Kerr Recursion results*: $M^{-2}h_{(2)}^{00} \Big|_{\mathcal{O}(a^4)} = \frac{7Q[0]^2}{r^2} + \frac{7(r^2 - 3z^2)}{r^6}Q[0]Q[2] - \frac{(4r^2 - 7z^2)}{r^6}Q[1]^2 + \frac{7(r^2 - 3z^2)^2}{4r^{10}}Q[2]^2 - \frac{(4r^4 - 33r^2z^2 + 35z^4)}{r^{10}}Q[1]Q[3] + \frac{7(3r^4 - 30r^2z^2 + 35z^4)}{4r^{10}}Q[0]Q[4].$

MATCHING

$$\hat{n} \cdot \hat{s} = \frac{z}{r}$$

$$S^a = Ma \delta^{a3} = (0, 0, Ma),$$

$$M_{ab} = -Ma^2 \hat{s}_{ab} = -Ma^2 \left(s_a s_b - \frac{1}{3} \delta_{ab} \right).$$

Generic Recursion: $h_{(2)}^{00} \Big|_{quad} = \frac{7M^2}{r^2} + \frac{21MM_{ab}}{r^4} \hat{n}_{ab} - \frac{5|S|^2}{3r^4} + \frac{7S^a S^b}{r^4} \hat{n}^{ab} + \frac{63M_{ab}M_{cd} \hat{n}^{abcd}}{4r^6} + \frac{9M_{ab}^2 \hat{n}^{ab}}{r^6} + \frac{21M_{ab}M_{ab}}{10r^6}$

*work in progress

$$M_{ab}^2 = M_{ac}M_{cb}, \quad \tilde{S}_{ia} = \epsilon_{iab}S^b$$

Identifying Kerr Multipoles

[Damgaard, Lee, Lee, TR (2026)]

Tag each 1PM term with indicator $Q[n_1]$ and run the recursion. Each product $Q[n_1]Q[n_2]$ in 2PM shows the multipole interactions

$$Q[\ell] = a^\ell$$

	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$
<div style="display: flex; align-items: center;"> <div style="width: 15px; height: 15px; background-color: blue; margin-right: 5px;"></div> Even n <div style="width: 15px; height: 15px; background-color: green; margin-right: 5px; margin-left: 10px;"></div> Odd n </div>					
$h^{00} \Big _{Kerr}$	✓ ■ $Q[0]^2$ ■ Schwarzschild term		✓ ■ $Q[0]Q[2]$ ■ MM^{ab} ✓ ■ $Q[1]^2$ ■ $S^a S^b$		✓ ■ $Q[0]Q[4]$ ■ ✓ ■ $Q[2]^2$ ■ ■ $Q[1]Q[3]$ ■
$h_{(2)}^{0i} \Big _{Kerr}$		✓ ■ $Q[0]Q[1]$ ■ MS^a		✓ ■ $Q[1]Q[2]$ ■ $S^a M^{bc}$	■ $Q[0]Q[3]$ ■ MS^{abc}
$h_{(2)}^{ij} \Big _{Kerr}$	✓ ■ $Q[0]^2$ ■ Schwarzschild term		✓ ■ $Q[0]Q[2]$ ■ MM^{ab} ✓ ■ $Q[1]^2$ ■ $S^a S^b$		✓ ■ $Q[0]Q[4]$ ■ ✓ ■ $Q[2]^2$ ■ ■ $Q[1]Q[3]$ ■

Coefficients matched to general multipoles!

Caveat in Kerr-Recursion

Gauge redundancies

- ▶ Standard form of Kerr metric in harmonic gauge: [\[Jiang, Lin \(2014\)\]](#)

$$h^{00} = 1 + \frac{R^2 [a^2(3\bar{M}^2 + 4\bar{M}R + R^2) + (\bar{M} + R)^4]}{a^2 - \bar{M}^2 + R^2} + \frac{a^2 z^2}{a^2 z^2 + R^4},$$

$$h^{0i} = - \frac{2a\bar{M}R^2(\bar{M} + R)\epsilon_{ij3}x^j}{(a^2 z^2 + R^4)(a^2 - \bar{M}^2 + R^2)},$$

$$h^{ij} = \frac{\bar{M}^2 R^2}{R^4 + a^2 z^2} \zeta^i \zeta^j$$

$$R^2 = r^2 - a^2 \left(1 - \frac{z^2}{R^2}\right)$$

$$r^2 = x^2 + y^2 + z^2$$

$$\bar{M} = GM$$

$$\zeta^i = \left(\frac{Rx + ay}{R^2 + a^2}, \frac{-ax + Ry}{R^2 + a^2}, \frac{z}{R} \right)$$

Caveat in Kerr-Recursion

[Damgaard, Lee, Lee, TR (2026)]

Gauge redundancies

- Standard form of Kerr metric in harmonic gauge:

$$h^{00} = 1 + \frac{R^2 [a^2(3\bar{M}^2 + 4\bar{M}R + R^2) + (\bar{M} + R)^4]}{a^2 - \bar{M}^2 + R^2} + \frac{a^2 z^2}{a^2 z^2 + R^4},$$

$$h^{0i} = - \frac{2a\bar{M}R^2(\bar{M} + R)\epsilon_{ij3}x^j}{(a^2 z^2 + R^4)(a^2 - \bar{M}^2 + R^2)},$$

Expand in powers of $1/r$ $\left\{ \begin{array}{l} h^{ij} = \frac{\bar{M}^2 R^2}{R^4 + a^2 z^2} \zeta^i \zeta^j \xrightarrow{\mathcal{O}(a)} h_{(2)}^{12} = \frac{\bar{M}^2 a}{r^5} (y^2 - x^2) \end{array} \right.$

Under $x^i \rightarrow x^i + \xi^i$: $h_{(2)}^{12} \rightarrow h_{(2)}^{12} - \frac{\bar{M}^2 a}{r^5} (y^2 - x^2)$ Where, $\xi^i = -M^2 a \epsilon_{ij3} \frac{x^j}{3r^3}$

- 2PM Gauge vector to all orders in a : $\xi^i = \frac{M^2}{2a^2} \left[\frac{Ra}{R^2 + a^2} + \tan^{-1} \left(\frac{R}{a} \right) - \frac{\pi}{2} \right] \epsilon^{ij3} x^j$

$$R^2 = r^2 - a^2 \left(1 - \frac{z^2}{R^2} \right)$$

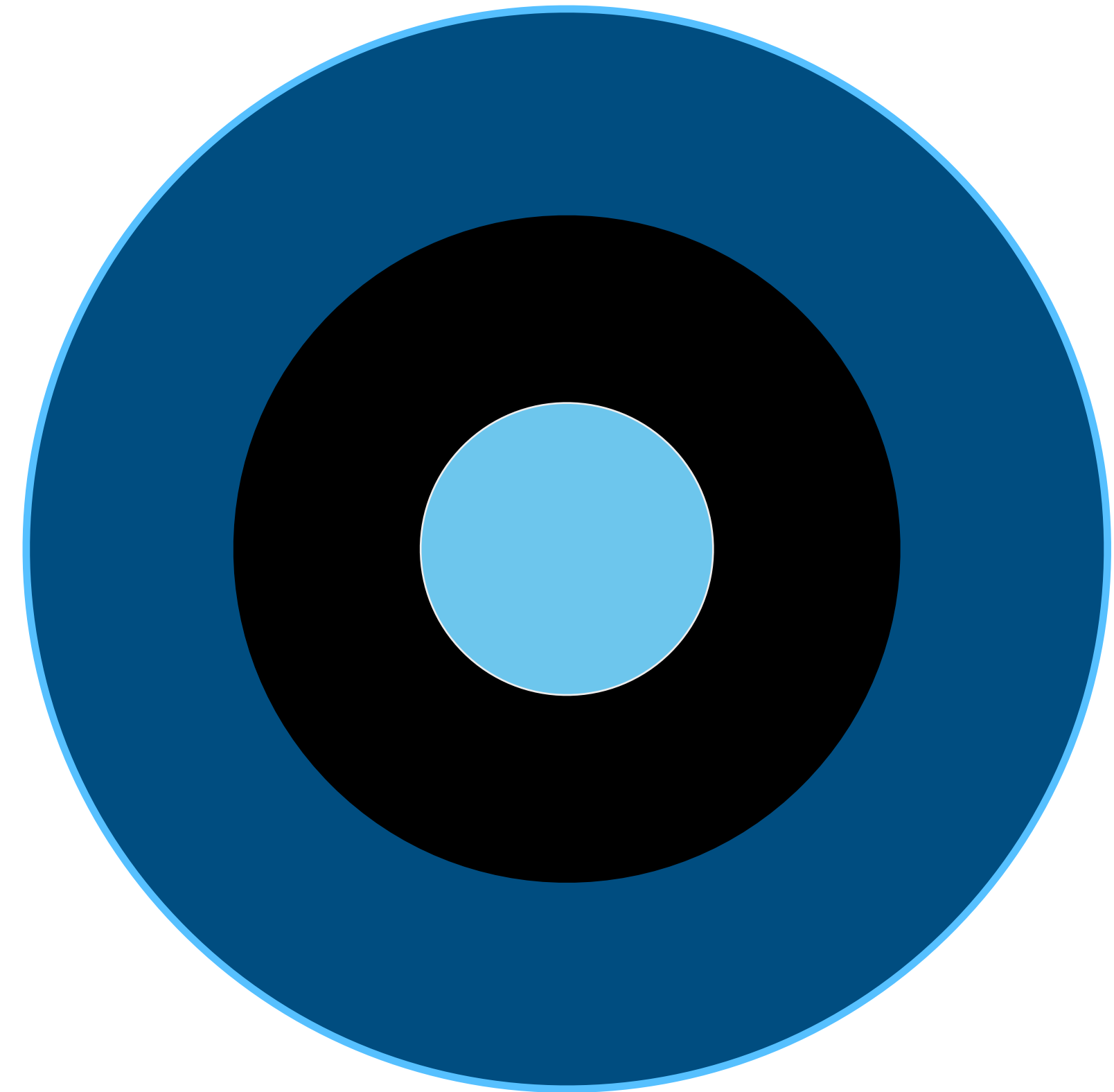
$$r^2 = x^2 + y^2 + z^2$$

$$\bar{M} = GM$$

$$\zeta^i = \left(\frac{Rx + ay}{R^2 + a^2}, \frac{-ax + Ry}{R^2 + a^2}, \frac{z}{R} \right)$$

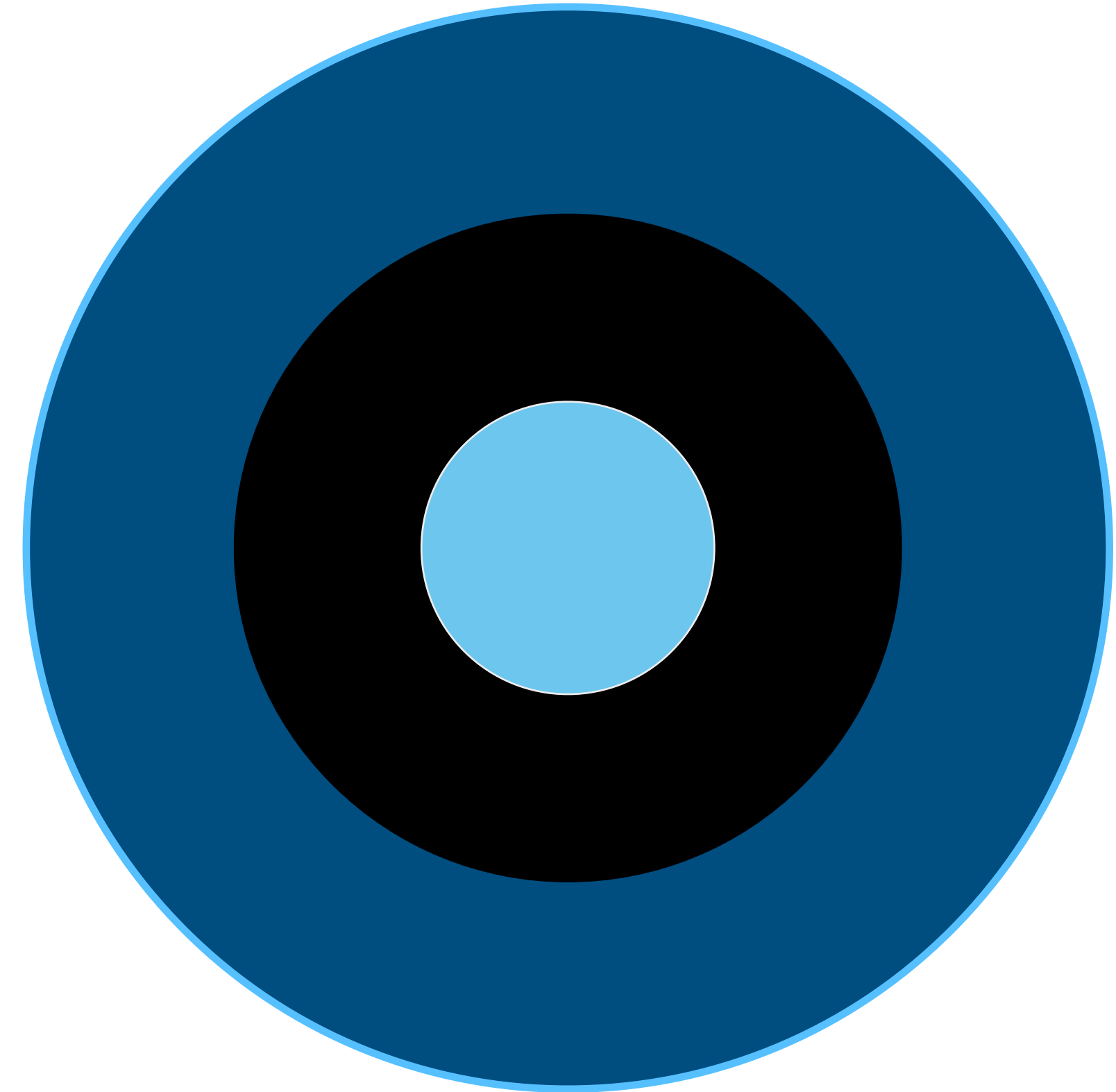
What we have done?

- ▶ First complete 2PM metric of a general stationary star with complete multipole structure.
- ▶ Closed-form all-order multipole expression in terms of momentum space bubble integrals.
- ▶ Kerr black hole recovered as a special case, including infinite towers of Kerr multipoles.
- ▶ Gauge redundancies has been identified in the harmonic Kerr metric and resolved explicitly.



Open Directions

- ▶ Extension to third and higher PM orders using same pipeline



Outlook

Open Directions

- ▶ Extension to third and higher PM orders using same pipeline
- ▶ Extension to time-dependent (non-stationary) sources - the binary inspiral case

General time-dependent source

e.g. Binary inspirals

- **Near zone** ($R < r \ll \lambda$)

PN expansion valid

$$PN : v \ll c$$

- **Exterior** ($R < r_e < r$)

MPM construction lives here

$$PM : \frac{GM}{rc^2} \sim \frac{v^2}{c^2}$$

- **Wave zone** ($r \gg \lambda, r \rightarrow \infty$)

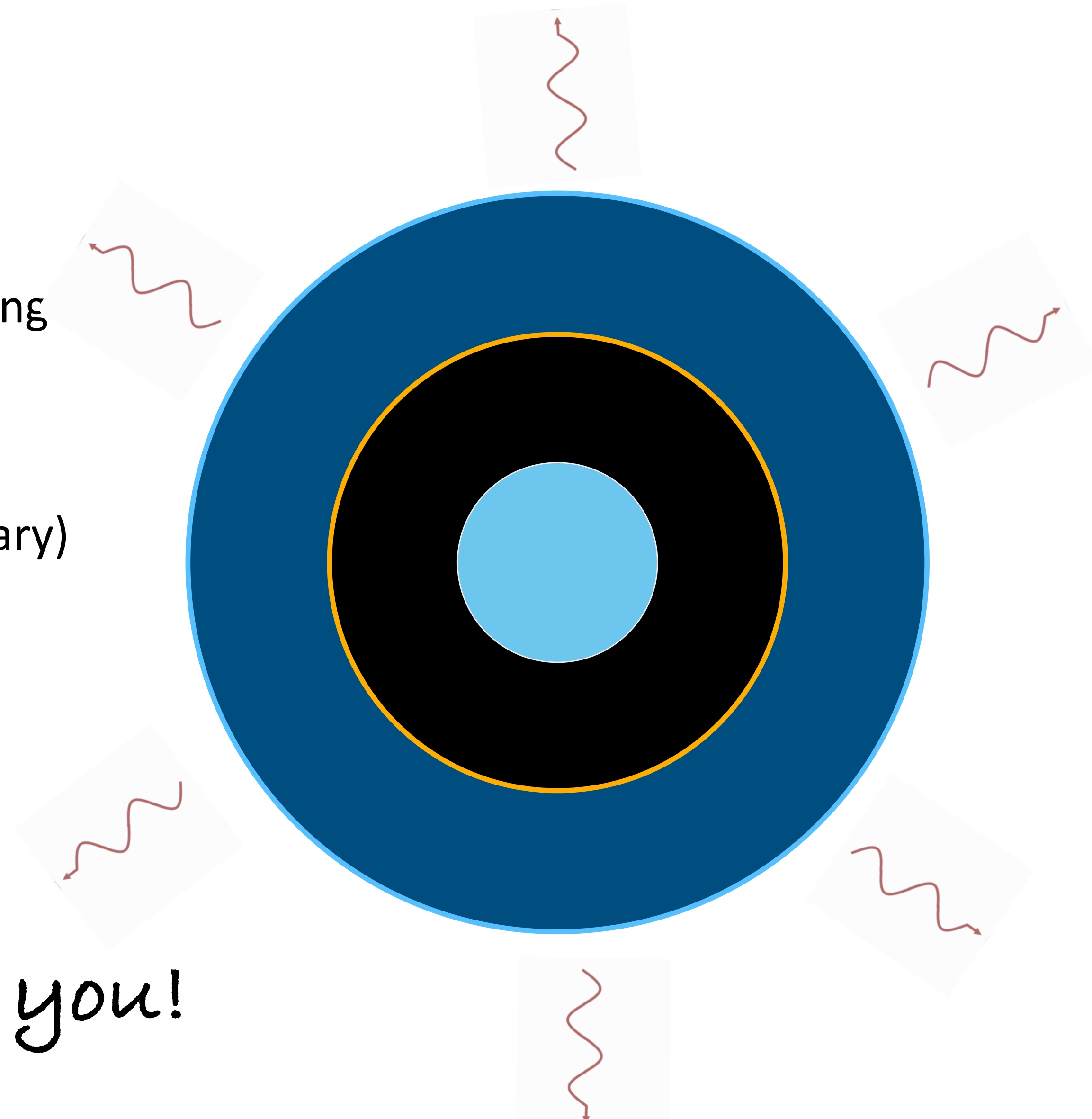
Radiation field, Far Zone

$M_L(t), S_L(t)$ - time dependent
 Give real emitted radiations

Outlook

Open Directions

- ▶ Extension to third and higher PM orders using same pipeline
- ▶ Extension to time-dependent (non-stationary) sources - the binary inspiral case
- ▶ Tidal deformations and Love numbers from the multipole expansion



Thank you!