Analytical Fluxes from Generic Schwarzschild Geodesics

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Based on 2604.XX w/ C. Kavanagh, O. Telem

INTRODUCTION

Context of this work

Non-relativistic Keplerian/ EM 0PA inspirals

2310.03798

w/ O. Telem

Bound-unbound universal relativistic Keplerian generalization

2503.23317

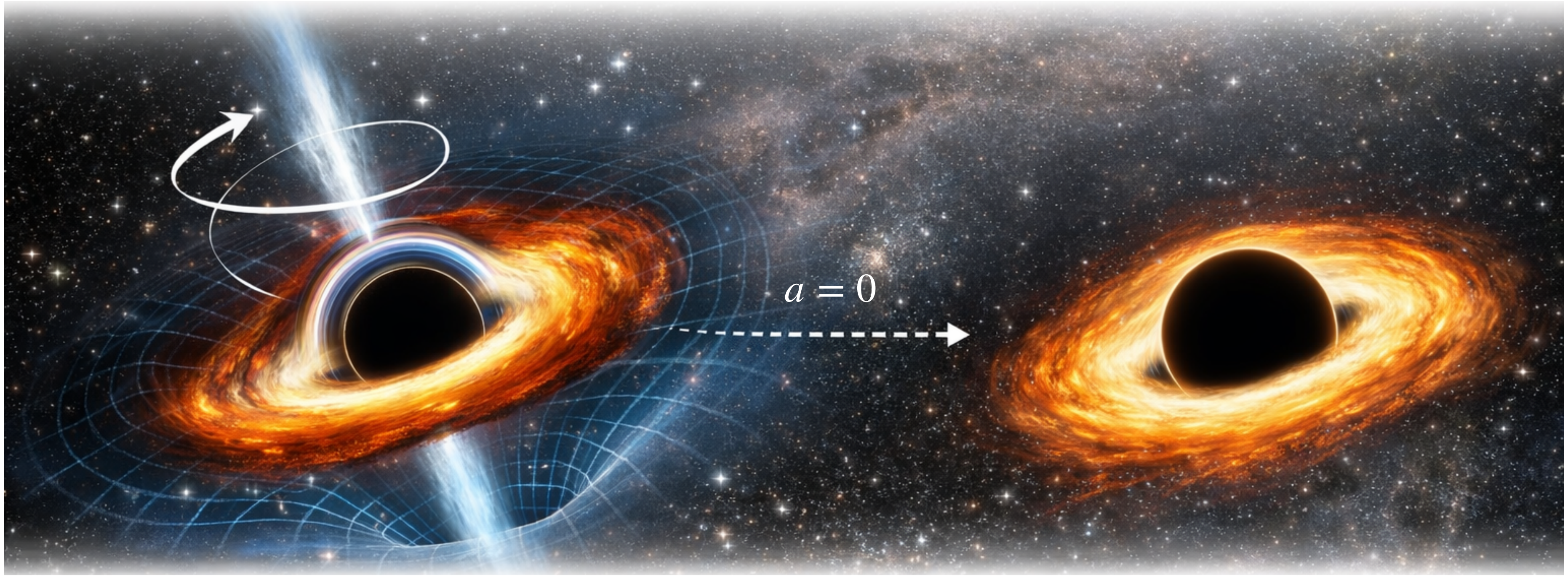
w/ C-H. Shen, O. Telem

This talk

Schwarzschild generalization

2604.XX

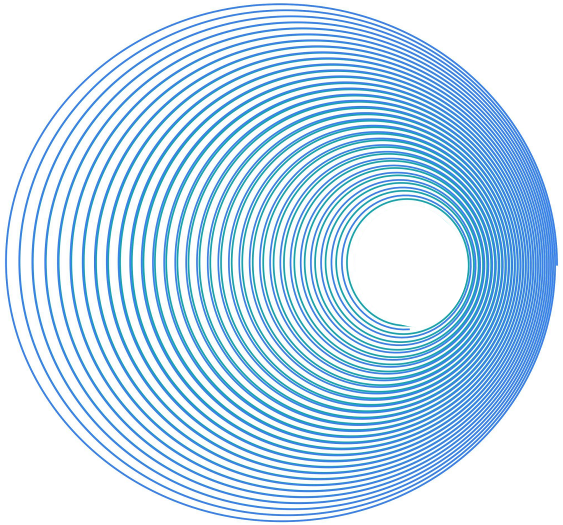
w/ C. Kavanagh, O. Telem



INTRODUCTION

Adiabatic fluxes

- Need 0PA energy/angular momentum fluxes to obtain 0PA inspiral.
- For concreteness, consider fluxes radiated to ∞.



$$\left(\dot{E}_{\text{tot}}\right)^{\infty}=\sum_{nlm}\frac{\left|Z^{\infty}_{nlm}\right|^{2}}{4\pi\omega_{nm}^{2}},\quad \left(\dot{L}_{\text{tot}}\right)^{\infty}=\sum_{nlm}m\,\frac{\left|Z^{\infty}_{nlm}\right|^{2}}{4\pi\omega_{nm}^{3}}$$

$$Z^{\infty}_{nlm}=\left\{\hat{Z}^{\infty}_{nlm}\right\}^{nm}_{\mathrm{S}},\quad \hat{Z}^{\infty}_{nlm}\equiv\mathscr{D}R^{\text{in}}_{l}(r;\omega_{nm})$$

$$\omega_{nm}\equiv\Omega^{r}n+\Omega^{\varphi}m$$

Outline

- Refresher: Schwarzschild and Keplerian conservative motion
- Fourier elements
- Schwarzschild-Kepler matching and analytical reduction
- PN—Chebyshev expansion
- Application: analytical fluxes and benchmarks
- Conclusions

PROBLEM SETUP

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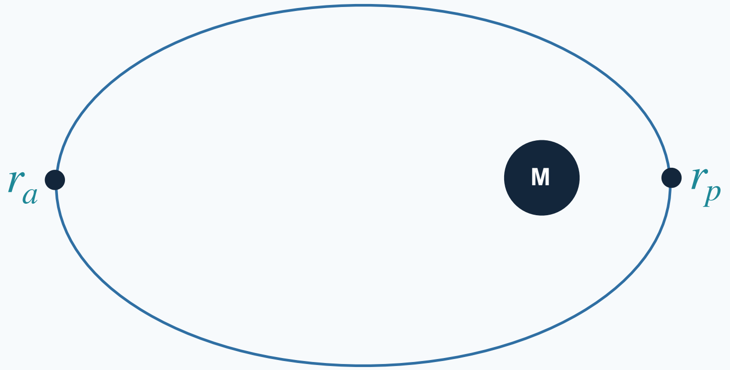
Bound Schwarzschild geodesics

We work with bound motion parameterised by semi-latus rectum p and eccentricity e.

- Schwarzschild coordinates: $t_S(r)$ and $\varphi_S(r)$.
- Let r_p and r_a be the periapsis and apoapsis.

$$\frac{dt_S}{dr} = \frac{Er^4}{\Delta(r)\sqrt{U_S^r(r)}} \qquad \frac{d\varphi_S}{dr} = \frac{L_S}{\sqrt{U_S^r(r)}}$$

$$\Delta(r) = r\,(r - 2GM\,) \qquad U_S^r(r) = E^2r^4 - \Delta(r)r^2\left(\frac{L_S^2}{r^2} + \mu^2\right)$$



bound orbital interval: $r_p \leq r \leq r_a$

$$r_p = \frac{GMp}{1+e} \qquad r_a = \frac{GMp}{1-e}$$

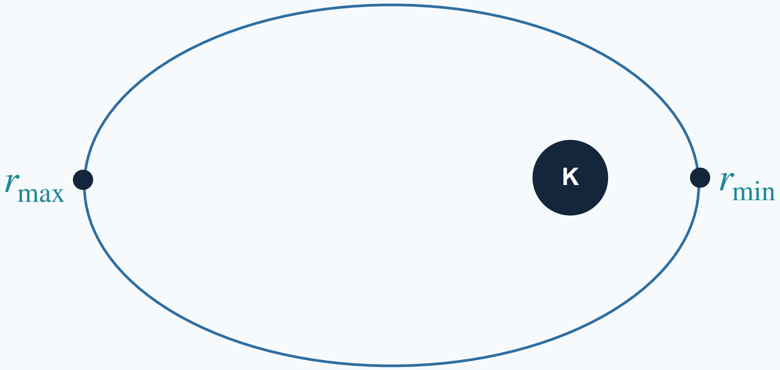
PROBLEM SETUP

Relativistic Keplerian Orbits

- Phase and time coordinates: $t_{\rm K}(r)$ and $\varphi_{\rm K}(r)$.
- Let $r_{\min}$ and $r_{\max}$ be the periapsis and apoapsis.

$$\frac{dt_{\rm K}}{dr} = \frac{E + \frac{\alpha}{r}}{\sqrt{U^r_{\rm K}(r)}} \qquad \frac{d\varphi_{\rm K}}{dr} = \frac{L_{\rm K}}{r^2\sqrt{U^r_{\rm K}(r)}}$$

$$U^r_{\rm K}(r) = E^2 - \mu^2 + \frac{2E\alpha}{r} - \frac{L_{\rm K}^2 - \alpha^2}{r^2}$$



PROBLEM SETUP

Fourier elements as the central objects

Consider an arbitrary function $f(r)$.

$$\{f\}_i^{nm} \equiv \frac{1}{\pi}\int_{r_p}^{r_a} f(r)\,\cos\left(A_{nm}^i(r)\right)\;t_i'(r)\,dr, \quad i = \mathrm{S},\mathrm{K}$$

$$A_{nm}^i(r) = \omega_{nm}^i t_i(r) - m\varphi_i(r),\ \omega_{nm}^i \equiv \Omega_i^r n + \Omega_i^\varphi m$$

Some relations:

$$\Omega_i^r = \frac{2\pi}{T_i^r}, \qquad \Omega_i^\varphi = \Omega_i^r\,\Delta\varphi_i/\pi$$

$$T_i^r = 2\int_{r_p}^{r_a} t_i'(r)\,dr, \qquad \Delta\varphi_i = \int_{r_p}^{r_a} \varphi_i'(r)\,dr$$

How can we compute these Fourier coefficients analytically for Schwarzschild?

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CORE IDEA

Schwarzschild-Kepler Matching

Choose Keplerian parameters so that the radial turning points coincide with the Schwarzschild ones.

Schwarzschild side

Direct evaluation is analytically cumbersome.

match

$$r_p = r_{\min}$$
$$r_a = r_{\max}$$

well-behaved reduction

Keplerian side

- Easier to handle analytically.
- Already has analytical expressions for Fourier coefficients.

$$\frac{\alpha}{GM\mu} = (p-4)\sqrt{\frac{p}{\left((p-2)^2-4e^2\right)\left(p-e^2-3\right)}}$$

$$\frac{L_{\mathrm{K}}}{\alpha} = \sqrt{\frac{p(p-3)-4e^2}{p-4}}$$

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CORE IDEA

Main analytical reduction

Schwarzschild Fourier coefficients are reduced to Keplerian Fourier coefficients.

$$\{f\}_{\mathrm{S}}^{nm} = \left\{ \mathscr{K}_{nm}[f] \Big/ \left(1 + \frac{\alpha}{Er}\right) \right\}_{K}^{nm}$$

where

$$\mathscr{K}_{nm}[f] \equiv \mathscr{A}_{nm}(r) f(r) + \mathscr{B}_{nm}(r) \int_{r_p}^{r} \mathscr{C}_{nm}(r') f(r')\, dr', \quad \mathscr{A}_{nm}(r) = \frac{r^2 \cos\left[\Delta A_{nm}(r)\right]}{\Delta(r)\sqrt{1 - \frac{r_h}{r}}}$$

$$\mathscr{B}_{nm}(r) = \frac{E}{\sqrt{\mu^2 - E^2}} \left[\omega_{nm}^{\mathrm{K}} \left(1 + \frac{\alpha}{Er}\right) - \frac{mL_{\mathrm{K}}}{Er^2} \right], \quad \mathscr{C}_{nm}(r) = \frac{r^3 \sin\left(\Delta A_{nm}(r)\right)}{\Delta(r)\sqrt{1 - \frac{r_h}{r}}\sqrt{(r - r_p)(r_a - r)}},$$

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CORE IDEA

Main analytical reduction

Schwarzschild Fourier coefficients are reduced to Keplerian Fourier coefficients.

$$\{f\}_{\mathrm{S}}^{nm} = \left\{ \mathcal{K}_{nm}[f] \Big/ \left(1 + \frac{\alpha}{Er}\right) \right\}_{K}^{nm}$$

where

$$\mathcal{K}_{nm}[f] \equiv \mathcal{A}_{nm}(r)\, f(r) + \mathcal{B}_{nm}(r) \int_{r_p}^{r} \mathcal{C}_{nm}(r')\, f(r')\, dr', \quad \mathcal{A}_{nm}(r) = \frac{r^2 \cos\left[\Delta A_{nm}(r)\right]}{\Delta(r)\sqrt{1-\frac{r_h}{r}}}$$

$$A^{S}_{nm} - A^{K}_{nm}$$

$$\frac{2GMp}{p-4}$$

$$\mathcal{B}_{nm}(r) = \frac{E}{\sqrt{\mu^2 - E^2}} \left[\omega^{\mathrm{K}}_{nm} \left(1 + \frac{\alpha}{Er}\right) - \frac{mL_{\mathrm{K}}}{Er^2} \right], \quad \mathcal{C}_{nm}(r) = \frac{r^3 \sin\left(\Delta A_{nm}(r)\right)}{\Delta(r)\sqrt{1-\frac{r_h}{r}}\sqrt{(r-r_p)(r_a-r)}},$$

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CORE IDEA

Why the reduction is useful

Once the mapped integrand is expanded into r^j, known Keplerian coefficients do the rest.

Schwarzschild $f(r)$ → $\mathscr{K}_{nm}[f]$ → **Laurent-like series** $\mathscr{K}_{nm}[f] \simeq \sum_{j=-N_d}^{N_u} a_j(N_u, N_d)\, r^j$ → **Use explicit analytic Keplerian coefficients for** r^j

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CORE IDEA

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$$\left\{\frac{r^j}{1+\frac{\alpha}{Er}}\right\}^{nm}_{\mathrm{K}} = \rho(-1)^{n+1}\left(\frac{GMpe}{2(1-e^2)}\right)^j \left(\frac{1+\sqrt{1-e^2}}{e}\right)^{j-n}(j-\tilde{m}-n+2)_n$$

$$\times\sum_{k=0}^{\infty} \frac{\left(\frac{1-\sqrt{1-e^2}}{e}\right)^{2k}(-j-\tilde{m}-1)_k(-j+\tilde{m}+n-1)_k}{k!\,\Gamma(k+n+1)}\, {}_1F_1(-k;\, j-k+\tilde{m}+2;\, -(n+\tilde{m})\rho)$$

$$\times\, {}_1F_1(-k-n;\, j-k-\tilde{m}-n+2;\, (n+\tilde{m})\rho)\,.$$

No eccentricity expansion!

where

$$\rho(p,e) \equiv \frac{(1+\sqrt{1-e^2})\left((p-2)^2-4e^2\right)}{2p(p-e^2-3)}, \quad \tilde{m} \equiv m\left[1+\frac{p-4}{4e^2+3p-p^2}\right]^{-1/2}$$

Interpretation

The hard Schwarzschild calculation is traded for a structured expansion problem plus a tractable analytic basis.

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METHOD

What must be expanded

The practical task is to turn $\mathscr{K}_{nm}[f]$ into a series that is accurate over the full orbital interval.

Target form

- Assume $f(r)$ can be preprocessed into a Laurent-like expansion.
- Seek an analogous expansion for the full mapped object $\mathscr{K}_{nm}[f]$.

Why it is nontrivial

$\mathscr{K}_{nm}[f]$ contains oscillatory cosine and sine factors, rational prefactors, and an integral term. Those structures must all be made compatible with the Laurent-like basis.

Guiding principle

Use a hybrid expansion strategy that respects both post-Newtonian structure and the need for good behavior all the way from r_p to r_a.

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METHOD

Hybrid PN + Chebyshev expansion strategy

Chebyshev pieces are introduced to maintain control on the whole interval $r_p \leq r \leq r_a$.

PN layer

Expand smooth non-oscillatory radial functions in inverse powers of r. For example,

$$\Delta(r)^{-1},\ (1-\frac{r_h}{r})^{-1/2}$$

Chebyshev layer

Replace $\cos,\ \sin,\ \log$ and $\arctan$ with their Chebyshev/Gegenbauer expansions.

PN organizes the algebra; Chebyshev protects convergence over the orbit.

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METHOD

Why Chebyshev?

Chebyshev pieces are introduced to maintain control on the whole interval $r_p \leq r \leq r_a$.

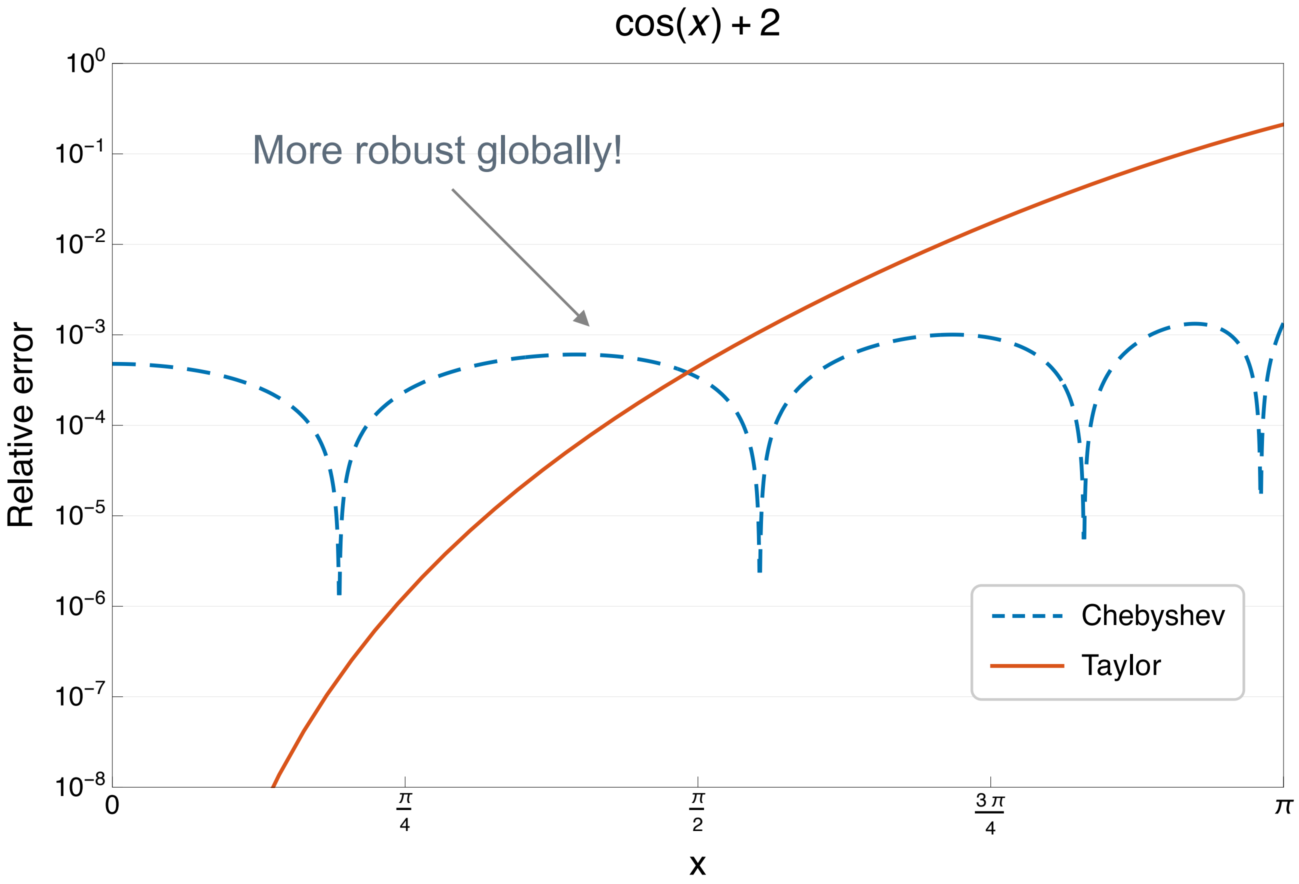
Taylor expansion up to 6th order:

$$3-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$$

Chebyshev expansion up to 6th order:

$$2+J_0(\pi)-2J_2(\pi)\,T_2\left(\frac{x}{\pi}\right)$$
$$+2J_4(\pi)\,T_4\left(\frac{x}{\pi}\right)-2J_6(\pi)\,T_6\left(\frac{x}{\pi}\right)$$

$$T_n(x)=\cos\big(n\arccos x\big)$$



More robust globally!

METHOD

Structure of the phase difference $\Delta A_{nm}(r)$

$$\Delta A_{nm}(r) = \sqrt{(r-r_p)(r_a-r)}\left[\chi_1^{(1)}(r)\,n + \chi_1^{(2)}(r)\,m + \left(\chi_2^{(1)}n + \chi_2^{(2)}m\right)g(r)\right]$$

where

$$g(r) \equiv \arctan\left[\frac{\sqrt{(r-r_p)(r_a-r)}}{r+\sqrt{r_p r_a}}\right]\Big/\sqrt{(r-r_p)(r_a-r)}$$

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METHOD

Structure of the phase difference $\Delta A_{nm}(r)$

Readily PN-expanded   Exact and independent of r

$$\Delta A_{nm}(r) = \sqrt{(r-r_p)(r_a-r)}\left[\chi_1^{(1)}(r)\,n + \chi_1^{(2)}(r)\,m + \left(\chi_2^{(1)}n + \chi_2^{(2)}m\right)g(r)\right]$$

where

$$g(r) \equiv \arctan\left[\frac{\sqrt{(r-r_p)(r_a-r)}}{r+\sqrt{r_p r_a}}\right]\Big/\sqrt{(r-r_p)(r_a-r)}$$

METHOD

Laurent-like expansion of $\mathscr{A}_{nm}(r)$

Reminder:

$$\mathscr{A}_{nm}(r) = \frac{r^2 \cos\left[\Delta A_{nm}(r)\right]}{\Delta(r)\sqrt{1-\frac{r_h}{r}}}$$

METHOD

Laurent-like expansion of $\mathscr{A}_{nm}(r)$

Reminder:

$$\mathscr{A}_{nm}(r) = \frac{r^2 \cos\left[\Delta A_{nm}(r)\right]}{\Delta(r)\sqrt{1-\frac{r_h}{r}}}$$

Use 'cos' Chebyshev

PN expand

Chebyshev of 'cos':

$$\cos(x) = J_0(a) + 2\sum_{i=1}^{\infty} (-1)^i J_{2i}(a)\, T_{2i}(\frac{x}{a})$$

Note that this introduces factors of $g\,(r)^k$.

METHOD

Chebyshev expansion of $g(r)^k$

To cast $\mathscr{A}_{nm}(r)$ into a Laurent-like form, need to also find good expansions for $g(r)^k$

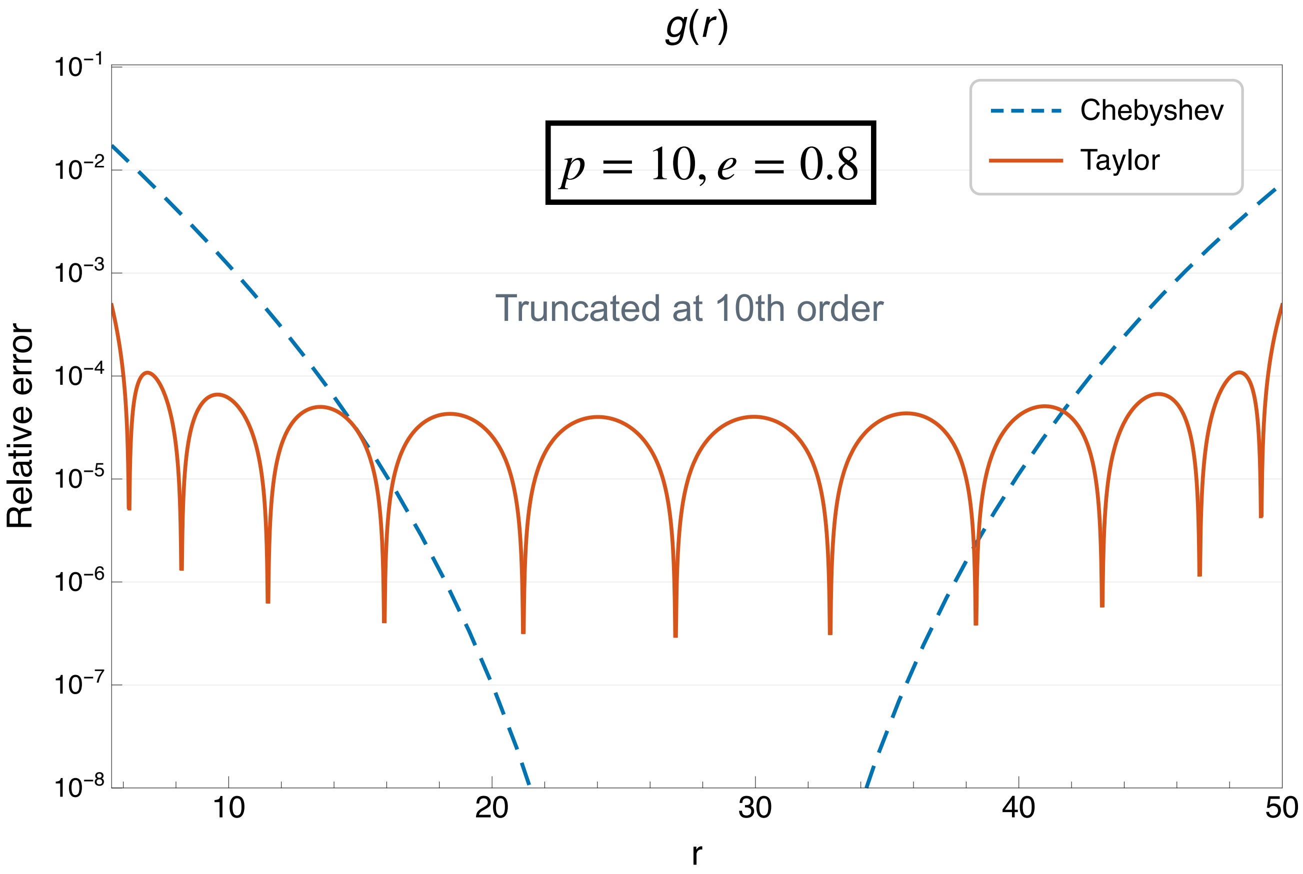
Chebyshev expansion:

$$g(r) = \frac{2}{r_a - r_p} \sum_{i=0}^{\infty} \frac{(-1)^i q^{i+1}}{i+1} U_i(x),$$

$$x = \frac{2r - r_a - r_p}{r_a - r_p}, \qquad q = \frac{\sqrt{r_a} - \sqrt{r_p}}{\sqrt{r_a} + \sqrt{r_p}},$$

$$U_n(x) = \frac{\sin\big((n+1)\arccos x\big)}{\sin(\arccos x)}$$

Expansions for $k > 1$ are obtained from this expansion through convolution of coefficients.



METHOD

Laurent-like expansion of $\mathscr{C}_{nm}(r)$

Reminder:

$$\mathscr{C}_{nm}(r) = \frac{r^3 \sin\big(\Delta A_{nm}(r)\big)}{\Delta(r)\sqrt{1-\frac{r_h}{r}}\sqrt{(r'-r_p)(r_a-r')}}$$

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METHOD

Laurent-like expansion of $\mathscr{C}_{nm}(r)$

Reminder:

$$\mathscr{C}_{nm}(r) = \frac{r^3 \sin\big(\Delta A_{nm}(r)\big)}{\Delta(r)\sqrt{1-\frac{r_h}{r}}\sqrt{(r'-r_p)(r_a-r')}}$$

Use ‘sin’ Chebyshev

PN expand

Chebyshev of ‘sin’:

$$\sin(x) = 2\sum_{i=0}^{\infty}(-1)^i J_{2i+1}(a)\, T_{2i+1}(\frac{x}{a})$$

The $g\,(r)^k$ are replaced with their Chebyshev expansions.

METHOD

Laurent-like expansion of $\mathscr{K}_{nm}[f]$

Reminder:

$$\mathscr{K}_{nm}[f] = \mathscr{A}_{nm}(r)f(r) + \mathscr{B}_{nm}(r)\int_{r_p}^{r} \mathscr{C}_{nm}(r')f(r')\,dr'$$

1. Use expansions for $f(r)$, $\mathscr{A}_{nm}$ and $\mathscr{C}_{nm}$.
2. Do the now-trivial integral. This can generate a $\log(r)$ term.
3. Use the known Chebyshev expansion of $\log(r)$.

The Laurent-like expansion of $\mathscr{K}_{nm}[f]$ is relatively quick (few mins) and easily scalable to arbitrary orders.

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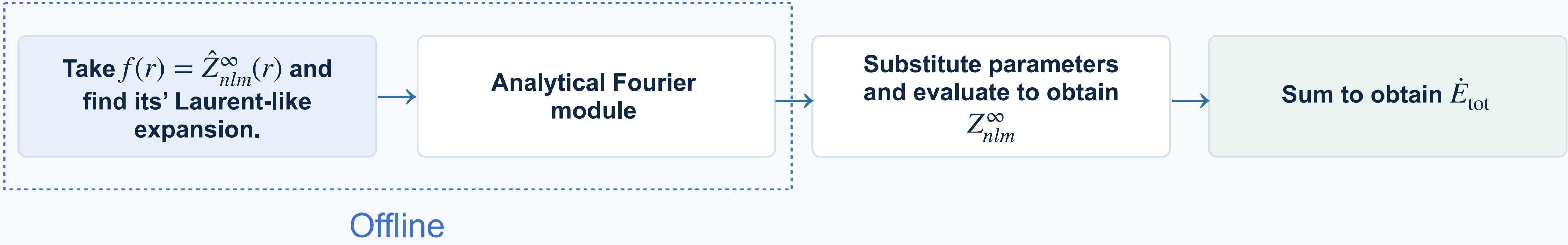
APPLICATION

From Fourier coefficients to radiated fluxes

The key application is adiabatic (0PA) energy and angular-momentum loss. We focus here on the energy loss without loss of generality.



Take $f(r) = \hat{Z}^{\infty}_{nlm}(r)$ and find its' Laurent-like expansion. → **Analytical Fourier module** → **Substitute parameters and evaluate to obtain Z^{∞}_{nlm}** → **Sum to obtain $\dot{E}_{\mathrm{tot}}$**

Offline

Need a Laurent-like expansion for $f(r)$

APPLICATION

Laurent-like expansion of $\hat{Z}^{\infty}_{nlm}(r)$

The expansion is analogous to that of $\mathscr{K}_{nm}[\,\cdot\,]$.

1. $R_l^{\text{in}}(r;\omega_{nm})$ admits MST+PN expansions, providing the starting point for analytic manipulation.
2. Trade the u^r piece in $\hat{Z}^{\infty}_{nlm}$ with an integral using IBP.
3. Replace the $\log^k$ terms with their truncated Gegenbauer expansions (generalizations of Chebyshev).
4. PN-expand everything else.
5. Do the now-trivial integral in the u^r piece.

VALIDATION

Definitions

Define the energy flux modes as

$$\dot{E}^{\infty}_{nlm} \equiv \frac{\left|Z^{\infty}_{nlm}\right|^2}{4\pi\omega^2_{nm}}$$

such that

$$\left(\dot{E}_{\text{tot}}\right)^{\infty} = \sum_{nlm} \dot{E}^{\infty}_{nlm}$$

Define the weighted relative error (WRE) as

$$\text{WRE} = \frac{\dot{E}^{\text{BHPT}}_{nlm}}{\dot{E}^{\text{BHPT}}_{\text{tot}}} \left| \frac{\dot{E}^{\text{analytical}}_{nlm} - \dot{E}^{\text{BHPT}}_{nlm}}{\dot{E}^{\text{BHPT}}_{nlm}} \right|$$

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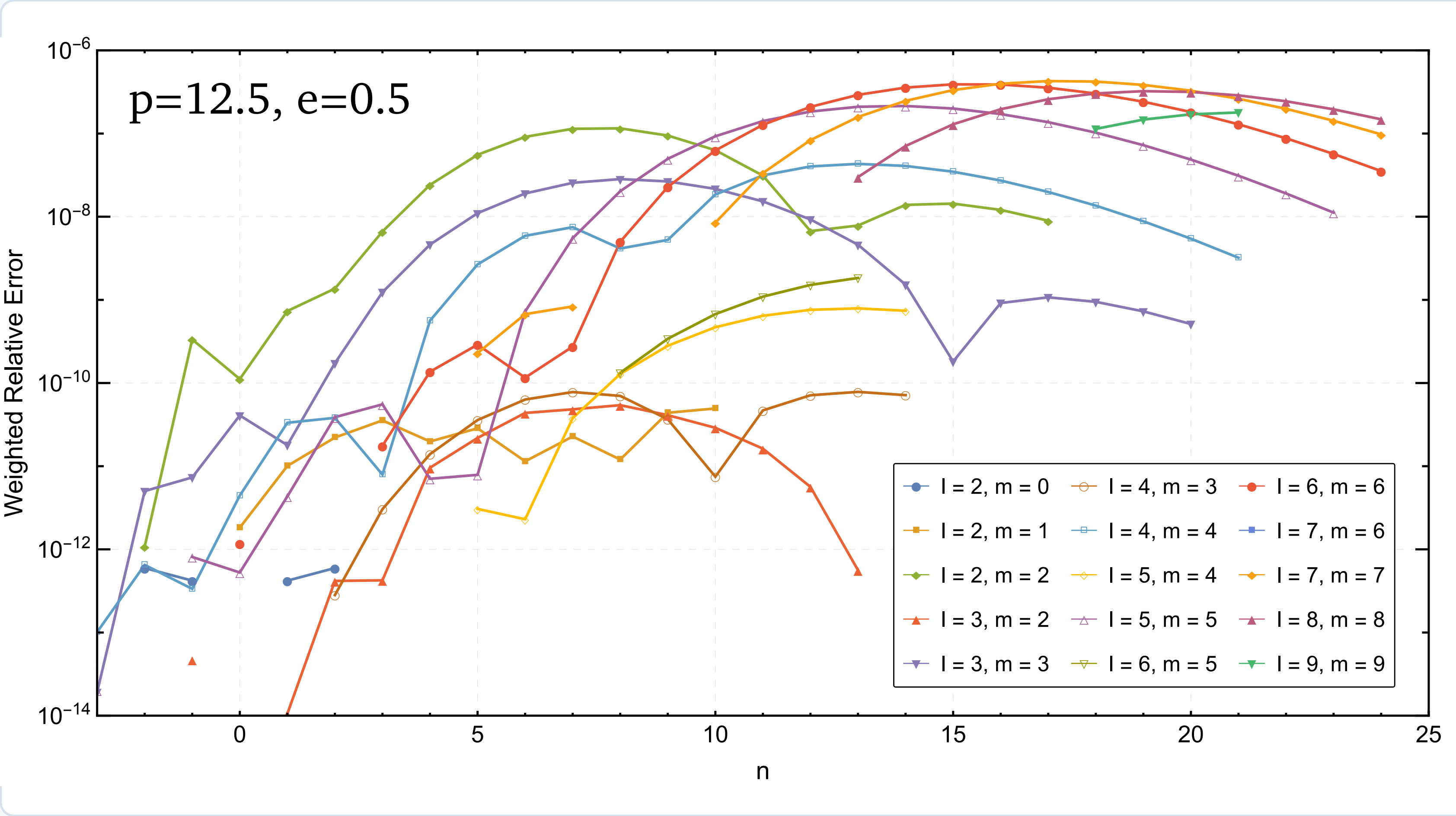
VALIDATION

Demonstration for $p = 12.5$ and $e = 0.5$

The flux here is fully converged with an accuracy of 10^{-5}.



PN order of $\hat{Z}$: **15**

Internal PN order: **15**

$\cos$ Chebyshev: **5**

$\sin$ Chebyshev: **6**

$g(r)^k$ Chebyshev: **16**

$\log^k$ Gegenbauer: **15**

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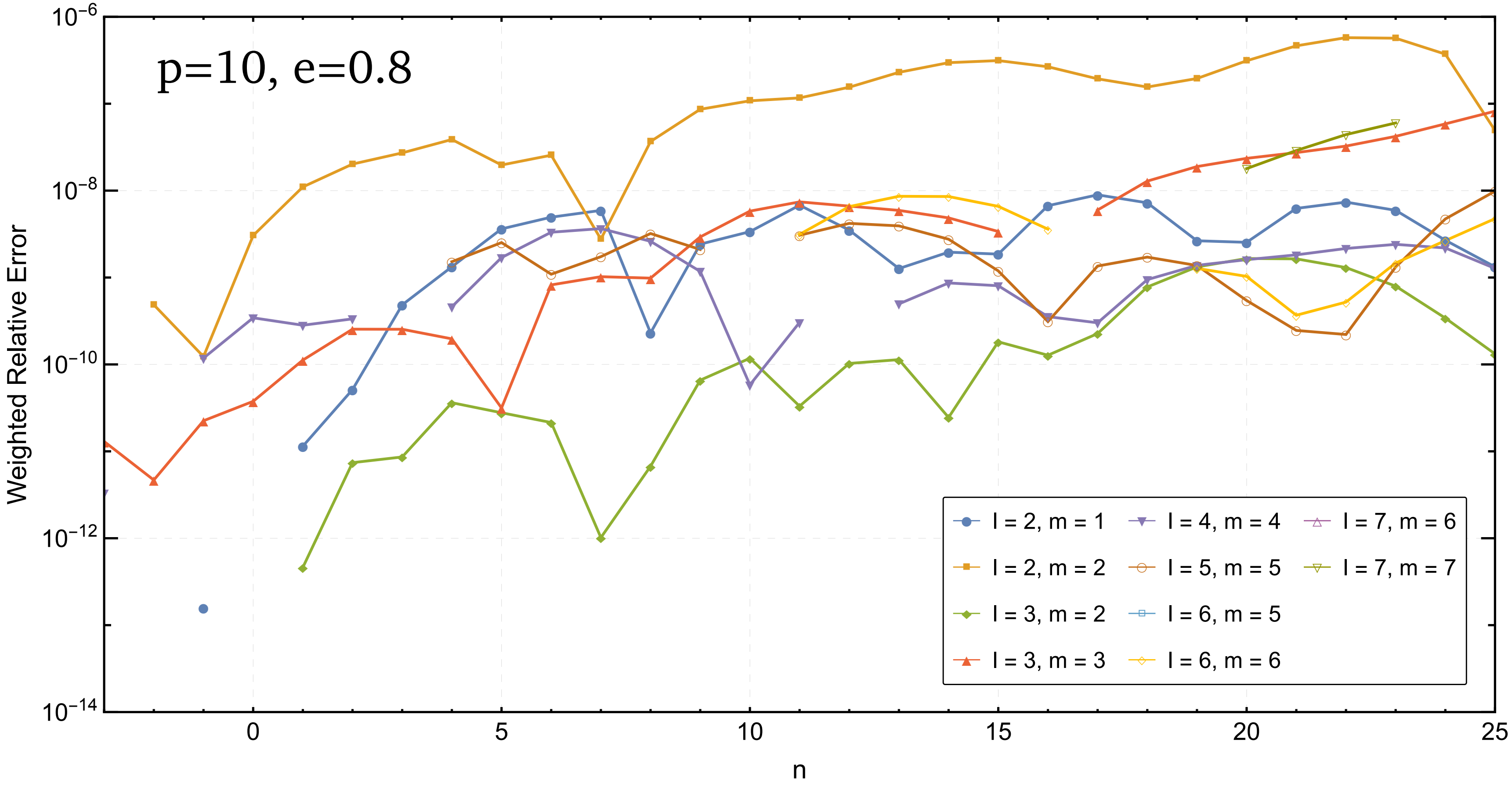
VALIDATION

Demonstration for $p = 10$ and $e = 0.8$



PN order of $\hat{Z}$: **15**

Internal PN order: **20**

$\cos$ Chebyshev: **5**

$\sin$ Chebyshev: **6**

$g(r)^k$ Chebyshev: **25**

$\log^k$ Gegenbauer: **15**

CONCLUSION

Summary and outlook

We saw:

- An approach/algorithm to analytically compute Fourier elements in Schwarzschild, with no eccentricity expansions.
- Reduce Schwarzschild Fourier coefficients to Keplerian coefficients.
- Chebyshev/Gegenbauer expansions control error globally.
- Analytical computation of adiabatic flux modes.

Possible directions:

- Eliminating PN expansions entirely in favor of Chebyshev expansions. Main challenge is doing that for the Teukolsky radial modes.
- Generalization to Kerr.
- Going beyond 0PA order.