Beyond the Spectrum: Non-Normality in Open Quantum Dynamics

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Motivation

- Lindblad master equation

$$\partial_t \hat{\rho}(t) = \mathcal{L}\left[\hat{\rho}(t)\right] = i\left[\hat{\rho}(t), \hat{H}\right] + \sum_k \gamma_k \left[\hat{L}_k \hat{\rho}(t) \hat{L}_k^\dagger - \frac{1}{2}\left\{\hat{L}_k^\dagger \hat{L}_k, \hat{\rho}(t)\right\}\right]$$

- The generator, the *Liouvillian* $\mathcal{L}$, is linear

$$\mathcal{L}\left[\hat{\rho}_n\right] = \mu_n \hat{\rho}_n, \qquad \mathcal{L}^\dagger\left[\hat{\varrho}_n\right] = \mu_n^* \hat{\varrho}_n$$

- *CPTP* dynamics constrains the spectrum $\{\mu_n\}$: $\mathrm{Re}(\mu_n) \leq 0$. Eigenvalues appear in conjugate pairs. The *steady state* $\hat{\rho}_{\mathrm{ss}}$, $\mu_{\mathrm{ss}} = 0$.
- *Biorthogonality*

$$\mathrm{Tr}\left[\hat{\varrho}_m^\dagger \hat{\rho}_n\right] = \delta_{mn}$$

Time evolution

$$\hat{\rho}(t) = \hat{\rho}_{\mathrm{ss}} + \sum_{n \neq \mathrm{ss}} d_n e^{\mu_n t} \hat{\rho}_n$$

- *Liouvillian gap* $\Delta_L = \min_{n \neq \mathrm{ss}} |\mathrm{Re}(\mu_n)| > 0$. For CPTP dynamics, $\mathrm{Tr}\left[\hat{\rho}_n\right] = 0$ for all non-steady modes.

Motivation

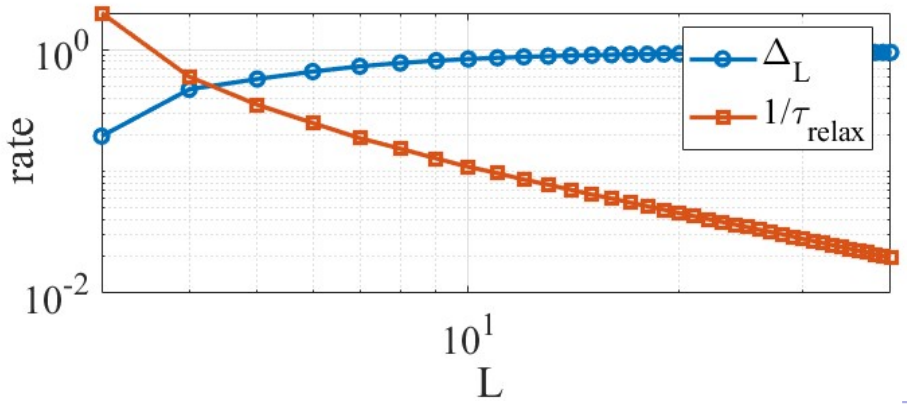
- Given $\mathcal{L}$, what is the relaxation time scale?
- Example: finite tight-binding chain with dissipative bias

$$\hat{H} = -J \sum_{i=1}^{N-1} \left(|i+1\rangle\langle i| + \text{h.c.} \right), \qquad \hat{L} = \sum_{i=1}^{N-1} |i+1\rangle\langle i|.$$

- We compare the Liouvillian gap Δ_L with the inverse relaxation time extracted from the *Uhlmann fidelity*

$$F(t) = \text{Tr}\left[\sqrt{\sqrt{\hat{\rho}(t)}\, \hat{\rho}_{\text{ss}}\, \sqrt{\hat{\rho}(t)}} \right],$$

starting from $\hat{\rho}(0) = |1\rangle\langle 1|$.



Motivation

- Recall

$$\hat{\rho}(t) = \hat{\rho}_{\rm ss} + \sum_{n \neq {\rm ss}} d_n e^{\mu_n t} \hat{\rho}_n$$

- The coefficients may satisfy $|d_n| \gg 1$, and in general

$$\mathrm{Tr}\left[\hat{\rho}_n^\dagger \hat{\rho}_m\right] \neq \delta_{nm}.$$

- If the eigenmodes are almost parallel, the Liouvillian is highly *non-normal*.
- Toy model

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\gamma & G \\ 0 & -\gamma \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} x_2(t) &= x_2(0)e^{-\gamma t}, \\ x_1(t) &= e^{-\gamma t}\big(x_1(0) + G x_2(0) t\big) \end{aligned}$$

- Initial growth of $x_1(t)$ before decay: *transient growth*.
- **Message:** For highly non-normal generators, the spectrum alone is only one piece of the puzzle.

Outline

1. Chaos
2. Non-normal mathematics
3. Open Quantum Systems
4. Intermission – chaos again
5. Results
6. Conclusions

Chaos – Background

A brief history of *quantum chaos*:

- 1890s: Poincaré and sensitivity to initial conditions — classical chaos.
- 1920s–1950s: The search for a quantum analogue.
- 1950s: Wigner's *random matrix theory* (RMT) — universal spectral features of quantum Hamiltonians.
- 1970s: *Berry–Tabor conjecture* — spectral properties of integrable systems.
- 1980s: *Bohigas–Giannoni–Schmit conjecture* — spectral properties of chaotic systems.
- 2000s–present: *Eigenstate thermalization hypothesis*, *quantum information scrambling*, and beyond.

Chaos – Berry–Tabor Conjecture & Level-Spacing Statistics

- Does the spectrum reveal information about chaotic versus regular motion?
- *For a generic quantum system whose classical limit is integrable, the spacings of energy levels (after unfolding to unit mean spacing) follow a Poisson distribution.*
- Spectrum: E_n; *level spacing:* $s_n = E_{n+1} - E_n$

$$P(s) = e^{-s}$$

- The energy levels are *uncorrelated*; they can overlap, leading to small s values. This reflects the presence of integrals of motion.

Chaos – Bohigas–Giannoni–Schmit Conjecture

- *If a quantum system has a fully chaotic classical counterpart, its energy-level statistics follow the predictions of random matrix theory (RMT).*

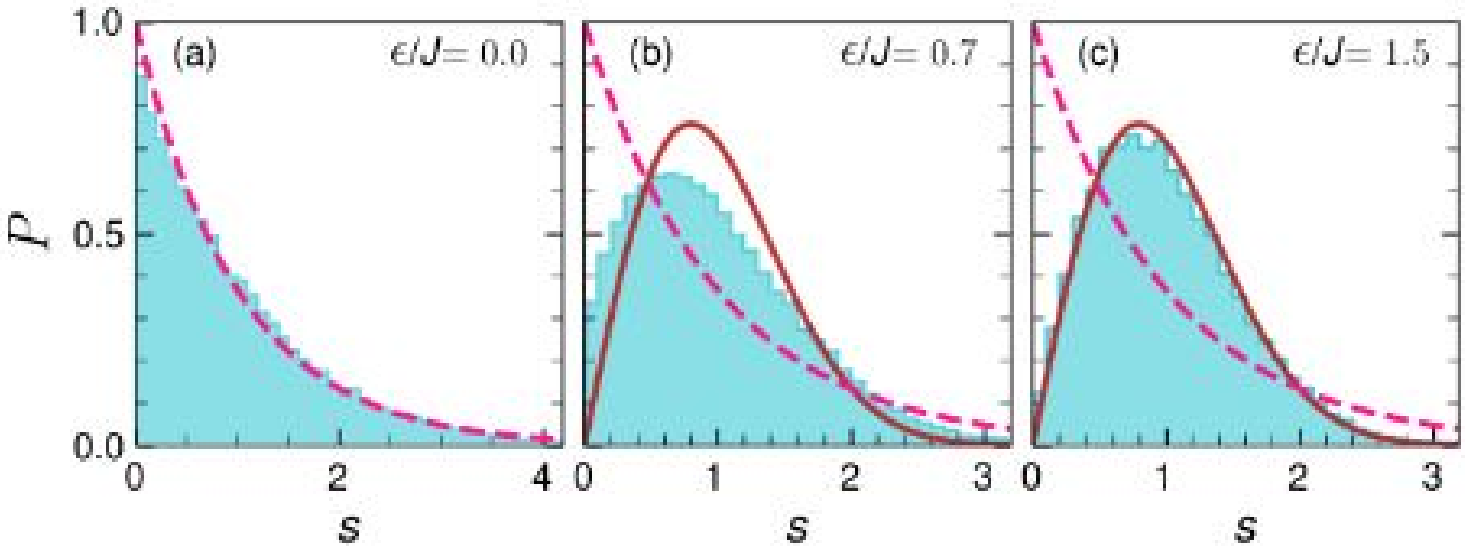
- Depending on symmetries:

$$P(s) \sim s^{\beta} e^{-\alpha_\beta s^2},$$

  - $\beta = 1$: *Gaussian Orthogonal Ensemble (GOE)* — time-reversal (TR) symmetry.
  - $\beta = 2$: *Gaussian Unitary Ensemble (GUE)* — broken TR symmetry.
  - $\beta = 4$: *Gaussian Symplectic Ensemble (GSE)* — TR symmetry with Kramers degeneracy.

Chaos – Characteristics

- Interacting bosons ($N = 270$) in a (tilted) triple-well potential; *Phys. Rev. E* **105**, 034204.



- As the tilt increases, the system undergoes a crossover from regular to chaotic behavior; *level-repulsion* $P(s) \to 0$ as $s \to 0$.

Non-Normal Mathematics – What's Not Normal

- A matrix is *non-normal* if

$$[M, M^\dagger] \neq 0.$$

  Non-normal $\Rightarrow$ non-Hermitian, but the reverse is not necessarily true.

- If diagonalizable:

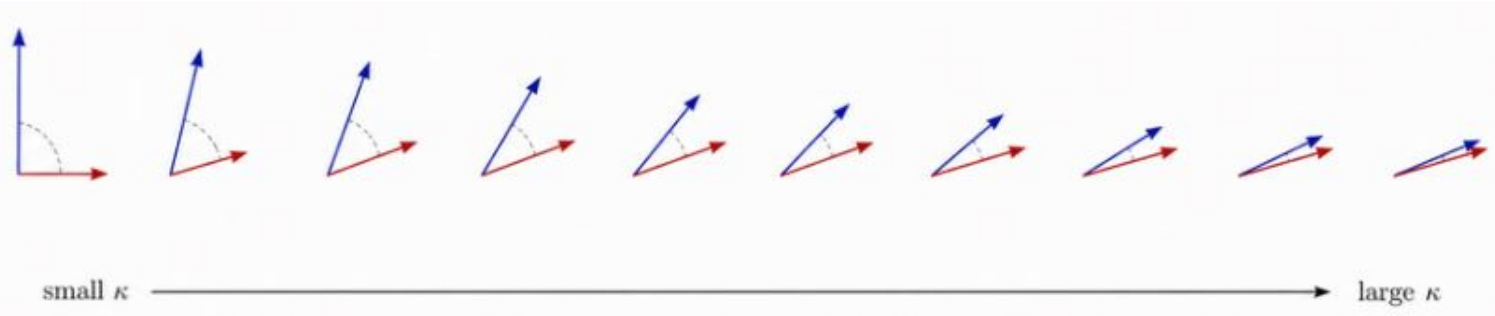
$$M|\phi_n^R\rangle = \lambda_n|\phi_n^R\rangle, \qquad M^\dagger|\phi_n^L\rangle = \lambda_n^*|\phi_n^L\rangle.$$

- *Left/right eigenvectors* are generally not orthogonal:

$$\langle\phi_m^R|\phi_n^R\rangle \not\propto \delta_{mn}, \quad \text{but} \quad \langle\phi_m^L|\phi_n^R\rangle \propto \delta_{mn}.$$

Non-Normal Mathematics – What's Not Normal

- The more non-normal a matrix is, the less orthogonal its eigenvectors tend to be:



- Assume M is diagonalizable, $M = VDV^{-1}$, the *condition number*

$$\kappa(V) = \|V\|_2 \, \|V^{-1}\|_2$$

quantifies the non-orthogonality of the eigenvectors.

- The induced 2-norm is

$$\|V\|_2 = \sup_{|x|=1} |Vx| = \sigma_{\max}(V),$$

where $\sigma_{\max}(V)$ is the largest singular value of V.

- If V is unitary, then $\kappa(V) = 1$. As M approaches a defective (non-diagonalizable) matrix, $\kappa(V) \to \infty$.

Non-Normal Mathematics – What's Not Normal

- Intuition:
  - $\|V\|_2$ measures the largest stretching induced by V.
  - $\|V^{-1}\|_2$ measures the largest amplification required to undo that stretching.
- If the eigenvectors are nearly parallel, V becomes ill-conditioned and $\|V^{-1}\|_2$ becomes very large $\Rightarrow \kappa(V) \gg 1$.
- Consequently, small perturbations ε can induce large changes in the eigenvalues and eigenvectors.
- In a finite system, there exists a basis $\{|\theta_n\rangle\}$ such that the right eigenvectors $\{|\phi_n^R\rangle\}$ are localized near one "boundary".
- The left eigenvectors $\{|\phi_n^L\rangle\}$ are localized near the opposite boundary:

$$|\langle \phi_n^L | \phi_n^R \rangle| \ll 1.$$

- **Skin effect.**

Non-Normal Mathematics – What's Not Normal

- Perturbation theory, $M \to M + \epsilon$:

$$\lambda_n \to \lambda_n + \frac{\langle \phi_n^L | \epsilon | \phi_n^R \rangle}{\langle \phi_n^L | \phi_n^R \rangle} \sim \lambda_n + \|\epsilon\| \, \kappa(V).$$

- Often:

$$\kappa(V) \sim e^{\alpha L},$$

where $\alpha \sim 1$ is a system-dependent parameter, and L is the system size.

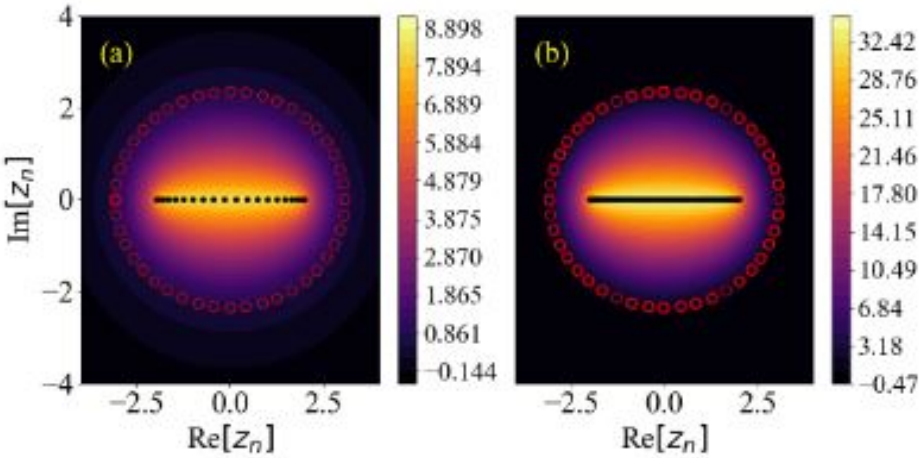
Example: if $\epsilon \sim 10^{-5}$, there is effectively no stability for $L \lesssim 10$.

Non-Normal Mathematics – What's Not Normal

- Mathematicians have long known that for non-normal matrices M, the spectrum alone is generally not a reliable descriptor of their behavior.
- *Pseudospectrum*:

$$\sigma_\varepsilon(M) = \left\{ z \in \mathbb{C} \;\middle|\; \|(zI - M)^{-1}\|_2 > \frac{1}{\varepsilon} \right\}.$$

- For $\varepsilon = 0$, σ_ε is concentrated at the eigenvalues λ_n. For $\varepsilon \neq 0$, it quantifies spectral sensitivity to perturbations, forming "blobs" around the eigenvalues (normal matrix, circle of radius ε around λ_n).



Pseudospectrum (logarithmic scale; $\log_{10} f(z)$, $f(z) = \|(zI - M)^{-1}\|_2$) for the Hatano–Nelson model.

Left: 20 sites. Right: 100 sites.

Open Quantum Systems – Vectorized Lindblad Equation

- Linear equation $\longrightarrow$ *vectorization*: stack the columns of ρ into a single vector,

$$\rho \longrightarrow |\rho\rangle\rangle \,,$$

yielding

$$\partial_t \, |\rho\rangle\rangle = \bar{\mathcal{L}} \, |\rho\rangle\rangle \,.$$

- Liouville space is the doubled Hilbert space,

$$\mathfrak{L} = \mathfrak{H} \otimes \mathfrak{H}^*,$$

and

$$\rho = \sum_{n,m} c_{nm} |\varphi_n\rangle\langle\varphi_m| \longrightarrow |\rho\rangle\rangle = \sum_{n,m} c_{nm} |\varphi_n\rangle \otimes |\varphi_m^*\rangle.$$

- The vectorized *Liouvillian* reads

$$\bar{\mathcal{L}} = -i\Big(I \otimes H^\dagger - H \otimes I\Big) + \sum_k \left(L_k \otimes L_k^* - \frac{1}{2} I \otimes L_k^\dagger L_k - \frac{1}{2} (L_k^\dagger L_k)^T \otimes I \right).$$

Chaos – Level-repuslion & the Grobe-Haake-Sommers Conjecture

- For open quantum systems, the earlier conjectures must be modified for Liouvillians:

  If the classical system is integrable, the Liouvillian eigenvalues exhibit Poisson-like statistics (no level repulsion).

  If the classical system is chaotic, the Liouvillian eigenvalues show cubic level repulsion, similar to the Ginibre random matrix ensemble.

- $\Rightarrow$ Spectral level statistics serve as a diagnostic of chaos. General beyond classical limit.

Models

- Integrable models: comparison of numerics with analytics.

1 Driven harmonic oscillator:

$$H_{\mathrm{dHO}} = \omega a^\dagger a + \eta(a + a^\dagger), \qquad L_1 = a, \qquad L_2 = a^\dagger.$$

The spectrum and steady-state properties are analytically known. The infinite Hilbert space requires truncation at boson number N_{tr}.

2 Open tight-binding model:

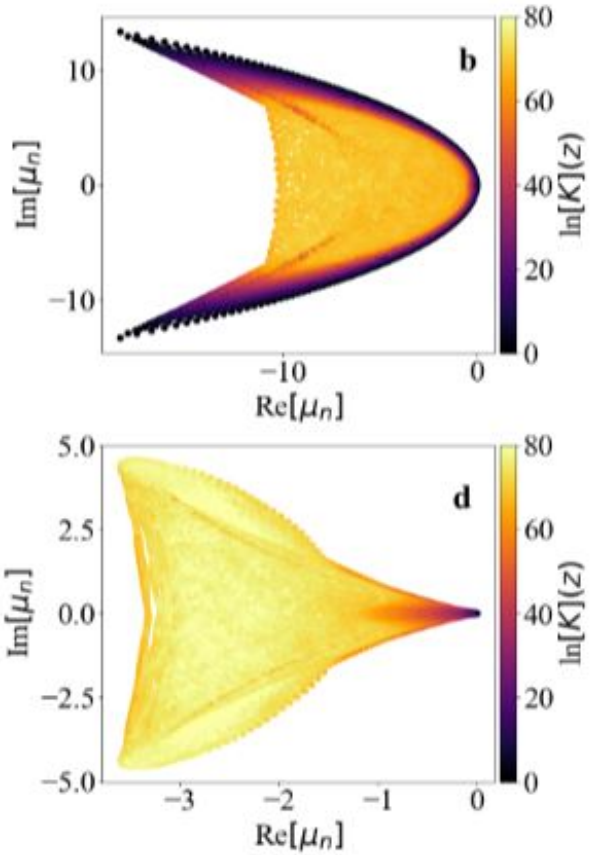
$$H_{\mathrm{TB}} = -J \sum_n \left(|n+1\rangle\langle n| + |n\rangle\langle n+1| \right), \quad L_1 = \sum_n |n\rangle\langle n+1|, \quad L_2 = L_1^\dagger.$$

For periodic boundary conditions (PBC), the spectrum and steady state are translationally invariant and analytically tractable. For open boundary conditions (OBC), boundary effects render the Liouvillian non-normal.

Spectral Instabilities

- Spectral structure:

$$\begin{aligned}\mu_{m,n} &= -(\gamma_1-\gamma_2)(m+n) - i\omega(m-n), \quad m,n\in\mathbb{N}_0\\ \mu(q,\tilde{q}) &= -(\gamma_1+\gamma_2)\left[1+\cos(q-\tilde{q})\right]\\ &\quad +i\Big[(\gamma_1-\gamma_2)\sin(q-\tilde{q})+2J(\cos q-\cos\tilde{q})\Big]\end{aligned}$$



Numerical spectrum, $L = N_{\rm tr} = 100$.

Colormap shows the *eigenvalue condition number*

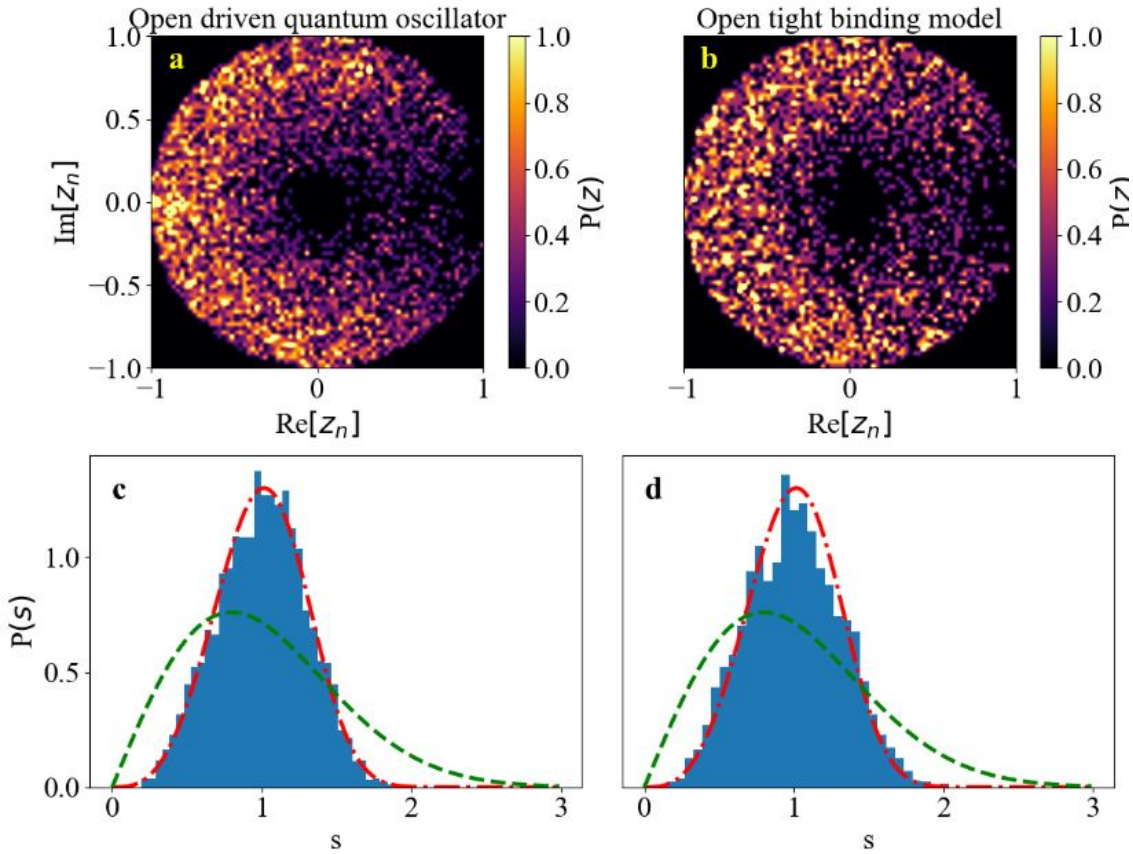
$$\kappa(\mu_n) = \frac{\|\rho_n^L\|_2\,\|\rho_n^R\|_2}{|\langle\rho_n^L|\rho_n^R\rangle|}.$$

Spectral Instabilities

- *Level spacing ratio*:

$$\mu_n = \frac{\lambda_n - \lambda_n^{\mathrm{NN}}}{\lambda_n - \lambda_n^{\mathrm{NNN}}}.$$

and spacing distribution $P(s)$



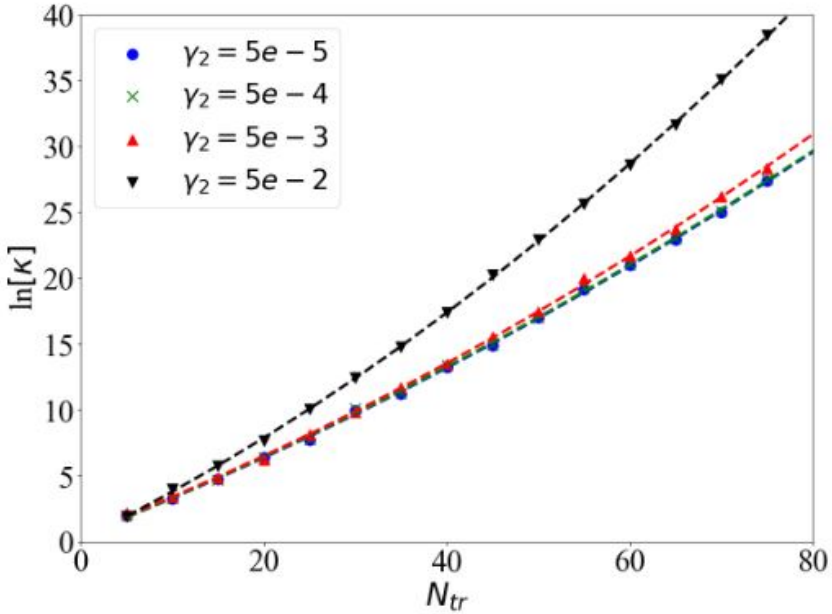
Spectral instabilities

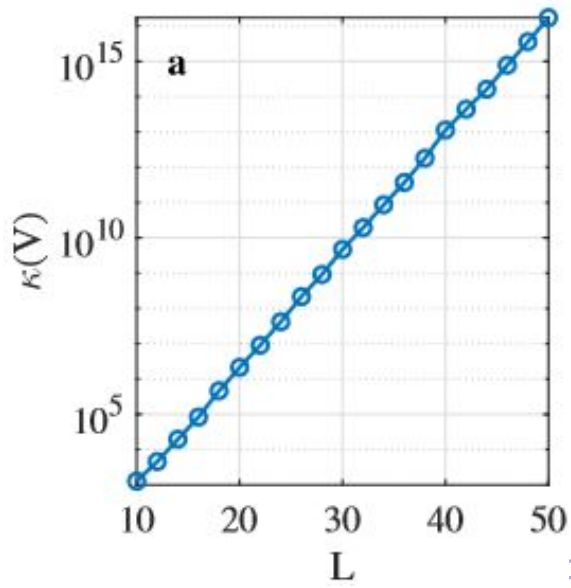
- Condition numbers

$$\kappa(\hat{V}) \sim e^{\alpha N_{\mathrm{tr}}}$$

  High non-normality $\longrightarrow$ spectral instability

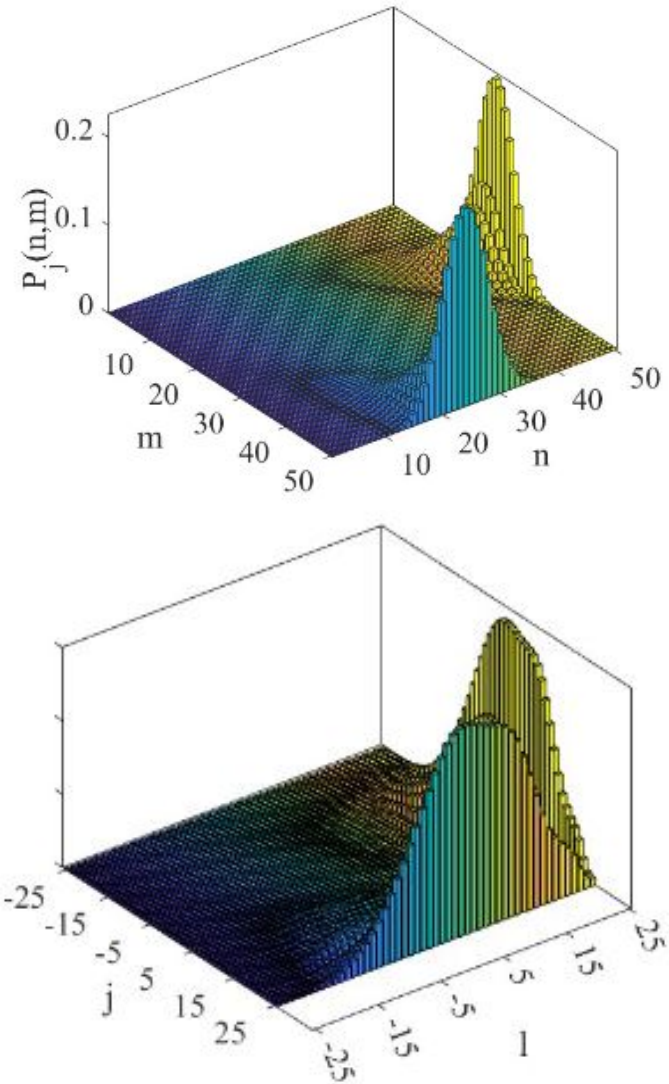
- Numerical round-off errors $\varepsilon \sim 10^{-16}$ cause level repulsion





Spectral instabilities

- A large number of right eigenvectors $|\rho_n^R\rangle\rangle$ become nearly parallel.
- As a result, there exists an orthonormal basis in which these states occupy only a small subset of basis directions.
- This is a manifestation of strong *non-normality* (Liouvillian skin effect).
- Figures show two representative eigenvectors in the number (Fock) basis.
- The condition number $\kappa(\hat{V})$ grows rapidly (often exponentially) with system size, indicating strongly localized skin modes.
- Open boundary conditions or numerical truncation N_{tr} introduce boundaries and can induce a skin effect.

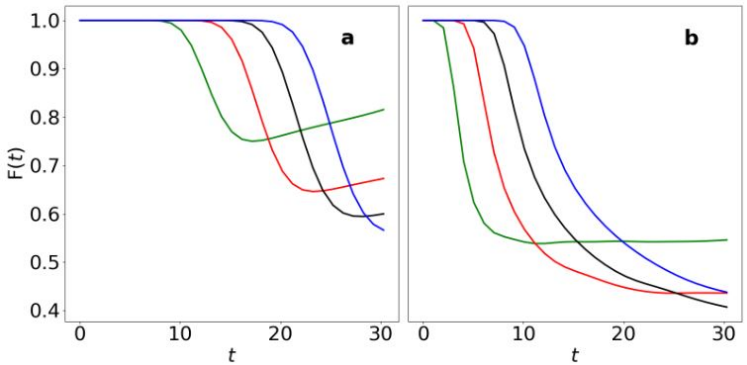


Level repulsion failure

- Are the models chaotic?
  - The analytically solvable models studied here are not chaotic.
  - The Liouvillian $\mathcal{L}$ has a local structure in Fock space, so dynamics from localized initial states is insensitive to distant truncation boundaries.
  - Relaxation is governed by approach to a steady state $\hat{\rho}_{\rm ss}$ rather than chaotic mixing.
- Uhlmann fidelity $F(t) \approx 1$ until the dynamics feels the truncation boundary, suggesting that

$$\lim_{L,\, N_{\rm tr}\to\infty} F(t) \approx 1.$$

- A comparison of relaxation time scales $\tau_{\rm relax}$ and level-repulsion scales $\tau_{\rm lr} \sim 1/\varepsilon$ shows: $\tau_{\rm lr} \gg \tau_{\rm relax} \Rightarrow$ No dynamical chaos!



Spectrum and dynamics of open quantum systems

- Recall

$$\hat{\rho}(t) = \hat{\rho}_{\mathrm{SS}} + \sum_{n \neq \mathrm{SS}} d_n e^{\mu_n t} \hat{\rho}_n$$

  This separates dynamics into three contributions:

  1. **Initial-state coefficients** d_n: determined by projection onto eigenmodes; can be large ($|d_n| \gg 1$), leading to transient amplification.
  2. **Spectrum** μ_n: governs decay rates and oscillations, but its dynamical relevance depends on non-normality and the eigenvector structure.
  3. **Eigenvectors** $\hat{\rho}_n$: biorthogonal structure; sensitivity captured by matrix/eigenvector condition numbers and Petermann-like factors.

- The full state $\hat{\rho}(t)$ and propagator $\hat{W}(t) = e^{\mathcal{L}t}$ are well-defined and stable.

  However, spectral and modal diagnostics should be based on robust quantities that account for non-normality.

Spectrum and dynamics of open quantum systems

- Diagnostics and dynamical characteristics are best defined from robust physical quantities.
- *Non-Hermitian sensors* rely on spectral sensitivity. It is not fully clear how instabilities of the spectrum of an effective (generally non-Hermitian) generator manifest in physical observables.
- Many works numerically plot the Liouvillian spectrum; the question is how these spectral features should be interpreted physically.
- Gap closing $\Delta_L \to 0$ may still be well-behaved dynamically if Petermann factors remain small near the steady state.
- Adiabaticity criteria in open quantum systems:

$$\Lambda_{nm} = \left| \frac{\mathrm{Tr}\left[\hat{\varrho}_m^\dagger \dot{\mathcal{L}}\, \hat{\rho}_n\right]}{(\mu_n - \mu_m)^2} \right| \ll 1.$$

The quantity Λ_{nm} may itself be sensitive to non-normality and ill-conditioning.

Conclusion

- Open quantum systems are described by non-normal generators. Boundary effects can induce strong non-normality, with $\kappa(\hat{V}) \sim e^{\alpha L}$.
- Mathematics: for non-normal matrices, the spectrum alone is generally not sufficient to characterize stability. Small perturbations (e.g. numerical noise) can strongly distort spectral features.
- Outlook:
  1. What is a suitable notion/diagnostic of chaos in open quantum systems?
  2. The full system (system + environment) is typically well-behaved and robust. How do instabilities emerge in effective (reduced) descriptions?
  3. How do non-analytic structures, such as exceptional points, arise in effective theories? What do our results imply for non-Hermitian sensors?
  4. Beyond Markovianity:

$$\hat{\rho}(t) = \hat{\Phi}_t[\hat{\rho}(0)]\,.$$