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Decoupling 2D Topological CSS codes (translation invariant, on qubits)

(Yifei Wang, Zhongyi Ni, Mingxin He, Jinguo Liu, YG to appear)

- Motivation

  New codes $\longleftrightarrow$ Old codes (Toric codes)

  (Bivariate Bicycle codes)

- History
  - Bombin, Duclos-Cianci, Poulin 2012
  - Haah 2021
  - ~~[illegible]~~

  } break "too much" translation

- Language / Setup : CSS + Translation symmetry

  Chain Complex (Laurent) Polynomial Ring / Modules over R

  $R = \mathbb{F}_2[x^{\pm}, y^{\pm}]$

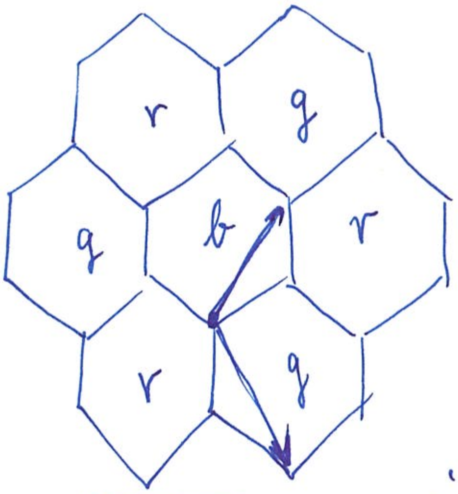
- Main Result : Proof the existence of decoupling unitary / circuit that preserves the maximal (Residual) translation symmetry

- Framework : Chain $\longrightarrow$ Chain map !

- Application? (decoder, logical gates ...)

- weak ~~b~~breaking of translation symmetry [Kitaev 2005]
  also known as <u>Anyon permutation</u>.

color code [Bombin, Martin-Delgado 2007]
[Kubica, Yoshida, Pastawski 2015]



X/Z - stabi = $\boxed{p}$

$= \prod_{j \in p} X_j$ or Z_j

$\boxed{\text{Break Translation Symmetry}}$ "e/m" Anyons : r, g, b $\mathbb{Z}_2$- type.

$r \times g \times b = 1$

- CSS [Calderbank - Shor - Steane 1996, <u>Kitaev 1997, Freedman Meyer 1998..</u>

homological algebra ..

<u>Finite code</u>

$$\mathbb{F}_2^{n_z} \xrightarrow{H_z^T} \mathbb{F}_2^{n_q} \xrightarrow{H_x} \mathbb{F}_2^{n_x}$$

Z-check qubits Syndrome.

$$\boxed{H_x H_z^T = 0 \Rightarrow \text{Chain Complex}}$$

Toric Code. $C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$

$\downarrow$ plaquette $\downarrow$ edge $\downarrow$ vertex

$$H_1 = \frac{\mathrm{Ker}(H_x)}{\mathrm{Im}(H_z^T)} = \text{logical operators}$$

$$\mathrm{coker}(H_x) = \frac{\mathbb{F}_2^{n_x}}{\mathrm{Im}(H_x)} = \text{anyons. (e-anyons).}$$

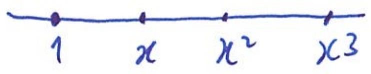
all can be dualized to

$$\mathbb{F}_2^{n_x} \xrightarrow{H_x^T} \mathbb{F}_2^{n_q} \xrightarrow{H_z} \mathbb{F}_2^{n_z}$$

...

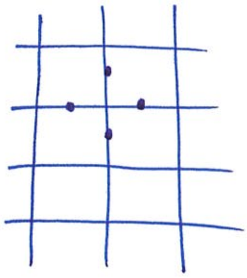
Polynomial Ring and Module Theoretical Language [Haah 2012]

$\mathbb{F}_2 \rightarrow \mathbb{F}_2[x, x^{-1}]$ Classical code.



$1+x^3 \Rightarrow X_0 X_3$.

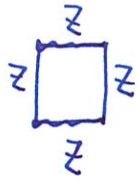
Quantum codes



$H_X = (1+x, 1+y)$ $\quad \bar{x}, \bar{y} = x^{-1}, y^{-1}$

$H_Z = (1+\bar{y}, 1+\bar{x})$

$H_Z^\dagger = \begin{pmatrix} 1+y \\ 1+x \end{pmatrix} =$



Toric code $\quad H_X = (\underbrace{1+x}_{f(x)}, \underbrace{1+y}_{g(y)})$: product code [2 classical repetition code 1D]

$H_Z = (\bar{g}(y), \bar{f}(x))$

Color code. $\quad H_X = (\underbrace{1+x+xy}_{f(x,y)}, 1+y+xy)$

Bivariate Bicycle: $\quad xy \rightarrow x^\alpha y^\beta$, more generally $\gcd(f(x,y), g(x,y)) = 1$

[Further more, need to compactify $R = \mathbb{F}_2[x, y, x^{-1}, y^{-1}] \Rightarrow R/I$
$I = \langle x^l - 1, x^m - 1 \rangle$ $\quad l \times m$ lattice]

Ex. Gross code: $\quad f(x,y) = 1 + x + x^{-1}y^3$
$g(x,y) = 1 + y + x^3 y^{-1}$. $\quad$ ~~coker~~

Free module: $\quad R \xrightarrow{H_Z^\dagger} R^2 \xrightarrow{H_X} R$

$\mathrm{coker}(H_X) = (\mathbb{F}_2)^8 \quad$ 8 toric code.

Compactify on 12×6 lattice (144 qubits) $\quad \otimes R/I$.

[[144, 12, 12]]

↳ loses 4 logicals, due to mismatch of lattice with T.I [12×12 lattice]

- Frame work

2D CSS T.I. code. Chain over $\mathbb{F}_2[x, y, x^{-1}, y^{-1}] = R$.

$$R^{n_z} \xrightarrow{H_z^{+}} R^{n_q} \xrightarrow{H_x} R^{n_x}$$

n_z, n_q, n_x : ~~qubit~~ per unit cell.

Now, to understand decoupling (and many other properties...) we need to consider not just chain itself, we need to consider chain map! (morphism of chain complexes)

$$\begin{array}{ccccc} R^{n_z} & \longrightarrow & R^{n_q} & \longrightarrow & R^{n_x} \\ \downarrow \varphi_2 & & \downarrow \varphi_1 & & \downarrow \varphi_0 \\ R^{n_z} & \longrightarrow & R^{n_q} & \longrightarrow & R^{n_x} \end{array}$$

$\boxed{\leftarrow \text{New code}}$

$\boxed{\text{Toric code}}$

copy = ~~$\dim_{\mathbb{F}_2}$~~ $\dim_{\mathbb{F}_2}(\mathrm{coker}\, H_x$

(diagram commutes)

Unitary decoupling = $\varphi_2, \varphi_1, \varphi_0$ isomorphism.

φ_2, φ_0 redefining stab.

$\underline{\underline{\varphi_1}}$ map between qubits

Standard chain (Toric code) ~~[illegible]~~ : $t^{\dim_{\mathbb{F}_2}(\mathrm{coker}\, H_x)}$ - copies of T.C.

$+ Z, X$, ancillas

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