Topological Defects and Non-Invertible Symmetries

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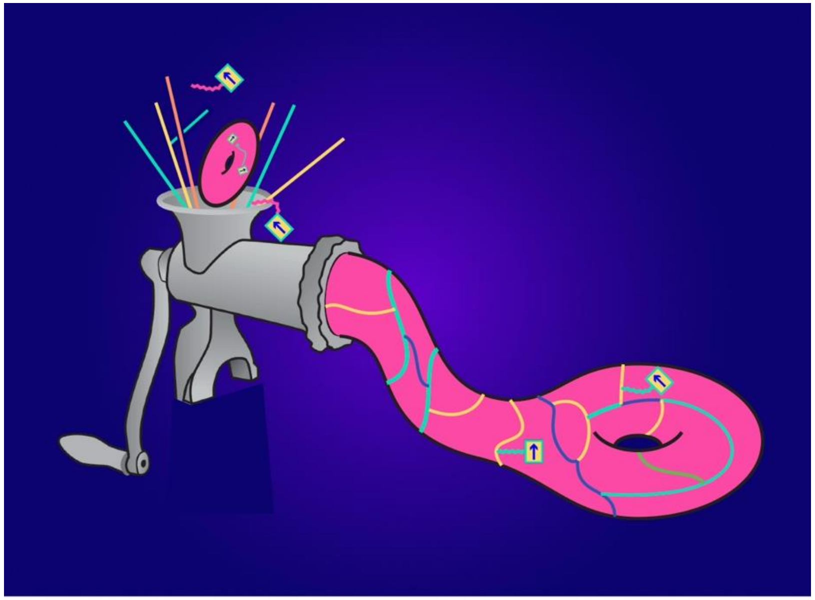


Based on work with David Aasen and Roger Mong and with Luisa Eck

Entanglement Dynamics: Open Quantum Systems, Monitored Circuits, and Topological Order

Canonical example is Kramers-Wannier duality

AUGUST 1, 1941 PHYSICAL REVIEW VOLUME 60

Statistics of the Two-Dimensional Ferromagnet. Part I

H. A. KRAMERS, *University of Leiden, Leiden, Holland*

AND

G. H. WANNIER, *University of Texas, Austin, Texas*[1]

(Received April 7, 1941)

In an effort to make statistical methods available for the treatment of cooperational phenomena, the Ising model of ferromagnetism is treated by rigorous Boltzmann statistics. A method is developed which yields the partition function as the largest eigenvalue of some finite matrix, as It is shown that these matrices possess a symmetry property which permits location of the Curie temperature if it exists and is unique. It lies at

$$J/kT_c = 0.8814$$

K-W showed that the energy spectra of the ordered and disordered phases of the Ising quantum spin chain are identical, i.e. found a "duality".

But there are two ground states in ordered phase, and one in the disordered.

Map cannot be invertible!

We now understand how to generalize

The key is to write lattice models in terms of a fusion category.

One then can construct topological defects that implement such maps.

Yields exact results without integrability, significantly generalizing Kramers-Wannier duality.

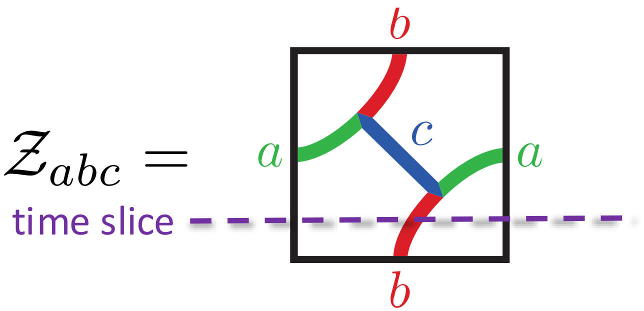
Much recent work has been done on such “symmetries” in QFT, but I will stick to lattice.

Topological defects generate categorical/topological/higher/non-invertible "symmetries" and "dualities"

The partition function in the presence of topological defects is independent of local deformations of the defects, both on the lattice and in the continuum.

Just need to keep track of how they branch/fuse, and how they wrap around cycles. Schematically:

$$\mathcal{Z}_{abc} =$$



With defects in spacetime, can still work in a Hilbert-space formalism. The defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix.

Symmetries and dualities are in quotes because these operators are in general non-unitary. Sometimes they are not even invertible.

Finding lattice topological defects

People found the duality defects in Ising by brute force, and that's not easy.

Grimm-Schutz 93; Oshikawa-Affleck 98

Beyond a few other simple examples, it's very difficult to get much further.

Our work exploits the fact that many interesting statistical mechanical models are conveniently described in terms of graphs and the corresponding knot and link invariants.

Temperley-Lieb 71; Fortuin-Kasteleyn 71; Baxter-Kelland-Wu 76; Jones '80s

The more systematic version is to define these models using fusion categories. The payoff is that exact topological defects follow automatically.

Feiguin et al 2006; Aasen-Fendley-Mong 2020

There is such a thing as a free lunch!

Outline

Statistical mechanics from fusion categories

1) What is a fusion category?

2) Topological defects occur automatically in lattice models defined using the category

3) Properties (e.g. consequences of gluing them together) follow directly

The uses of topological defects

Use #1: Generalized Kramers-Wannier duality

Use #2: Exact degeneracies for ground states and for kinks

Use #3: Non-obvious non-invertible mappings
(from XXZ to the Rydberg-blockade ladder)

Use #4: Exact lattice computation of g factor ratios and conformal spins

Fusion categories generalize knot polynomials

and define interesting lattice statistical mechanical models.

BULLETIN (New Series) OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 1, January 1985

A POLYNOMIAL INVARIANT FOR KNOTS VIA VON NEUMANN ALGEBRAS[1]

BY VAUGHAN F. R. JONES[2]

For real t, D. Evans pointed out that an explicit representation of A_n on $\mathbf{C}^{2n+2}$ was discovered by H. Temperley and E. Lieb [**23**], who used it to show the equivalence of the Potts and ice-type models of statistical mechanics. A readable account of this can be found in R. Baxter's book [**2**]. This represen-

Parallel with great discoveries in knot theory, closely related results came from integrable statistical mechanics, including quantum-group algebras and conformal field theory. Such interplay allowed the development of fusion categories.

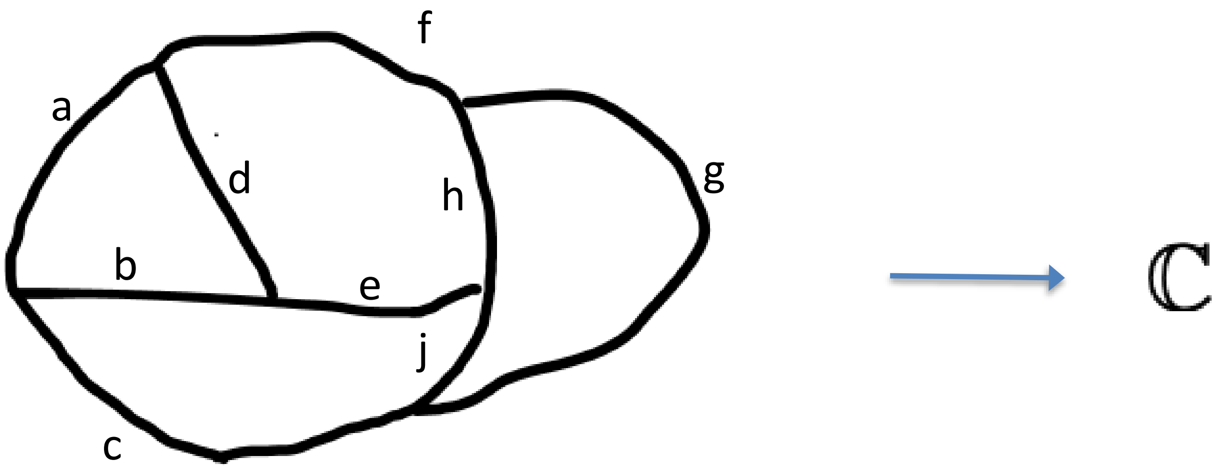
Physics reviews: Moore and Seiberg, Kitaev, Bonderson…

Stat-mech review: Wadati, Deguchi and Akutsu

A fusion category gives a set of rules to turn a fusion diagram into a number, an isotopy invariant

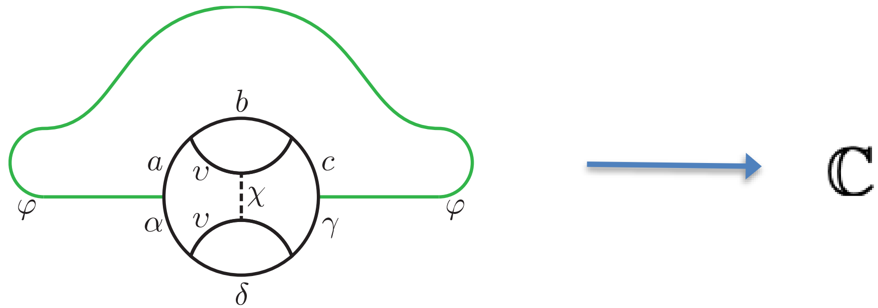
The Jones polynomial turns out to be the simplest example of an enormous class of topological invariants. Can't just use loops, but instead use a fusion diagram, i.e. a labelled trivalent graph.

The evaluation of such a graph is an isotopy invariant:



Isotopy invariance means that the evaluation is independent of any local deformations of the diagram.

Expressing diagram using a category



Category is a finite list of objects: a,b,c,… obeying fusion rules

$$a \otimes b = \sum_c N_{ab}^{c}\ c$$

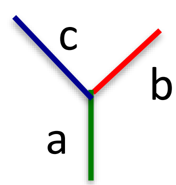
non-negative integer

Think tensor products of reps of Lie algebras, e.g. $\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$

The $SU(2)_k$ fusion rules are a truncated version of *SU*(2), with objects $0, \frac{1}{2}, 1, \ldots, \frac{k}{2}$

Crucial connection:

Vertex for each $N_{bc}^{a} = 1:$



0 is the identity, no line

Manipulating the diagram

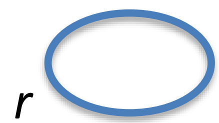
Data:

Non-negative integers N_{ab}^{c}

quantum dimensions d_a

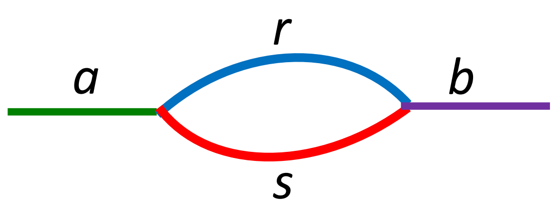
F symbols: $F_{tt'}\begin{bmatrix} r & s \\ a & b \end{bmatrix}$

Isolated loop:



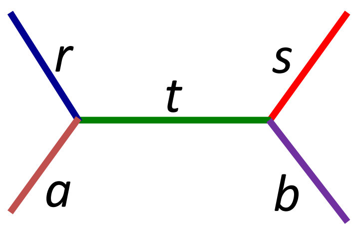
$= d_r$

bubble removal:

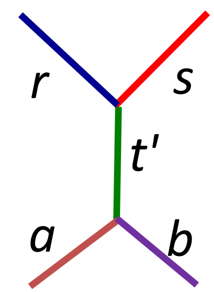


$= \delta_{ab}\sqrt{\frac{d_r d_s}{d_a}}$

F move:



$= \sum_{t'\in\mathcal{C}} F_{tt'}\begin{bmatrix} r & s \\ a & b \end{bmatrix}$

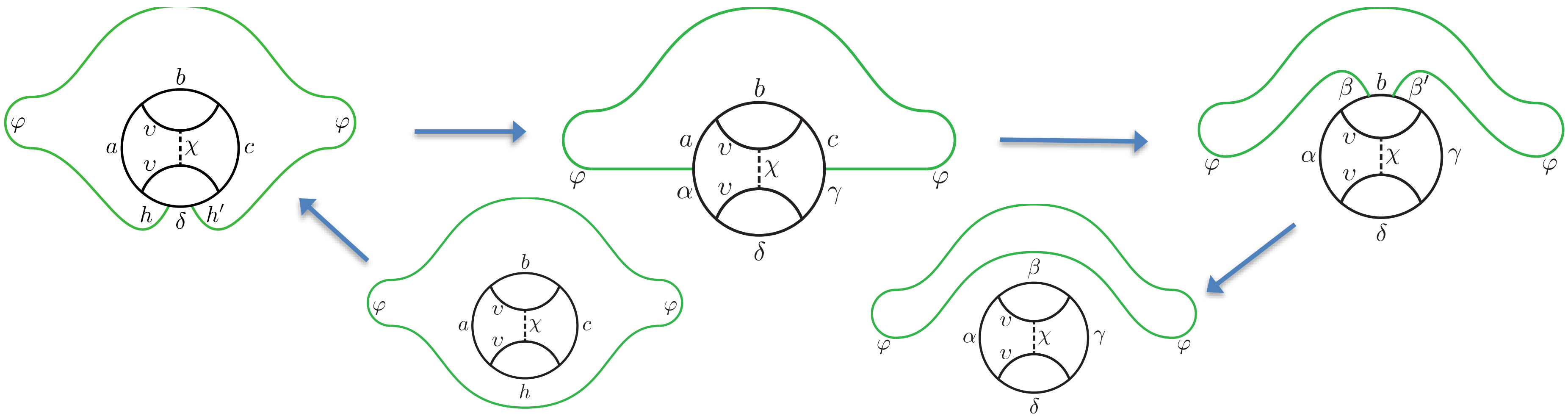


F moves preserve the evaluations. Note they result in a sum over diagrams.

Evaluating the diagram

F moves preserve the evaluations. Doing them and bubble removal repeatedly express a fusion diagram as a sum over closed loops, and hence a number.

Also very useful to relate different diagrams:

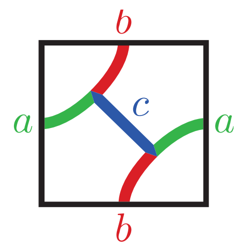


not writing sums and F symbols

Consistency guaranteed by the F symbols satisfying pentagon, hexagon equations.

Fusion categories allow one to construct topological defects both on the lattice and in the continuum

The partition function in the presence of topological defects is independent of local deformations of the defects. Just need to keep track of how they branch/fuse, and how they wrap around cycles. Schematically:



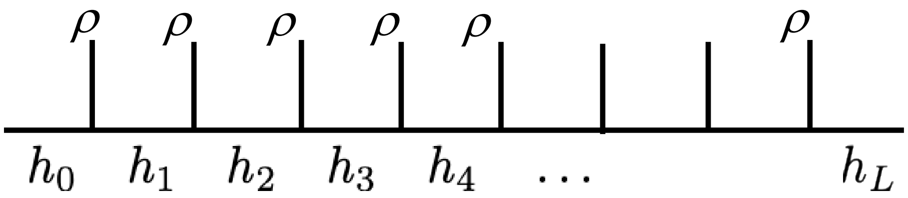
Working in a Hilbert-space formalism, the defect-line creation operators necessarily commute with the Hamiltonian/transfer matrix.

Topological defects generate non-invertible/categorical/ generalized ``dualities'' and symmetries.

The Hilbert space

To define model, first pick a category C, and a particular object $\rho \in C$.

Each basis element of Hilbert space is then given by a fusion tree:



Degrees of freedom are "heights" $h_0, h_1, h_2, \dots, h_L \in C$ obeying $N_{h_j \rho}^{h_{j+1}} = 1.$

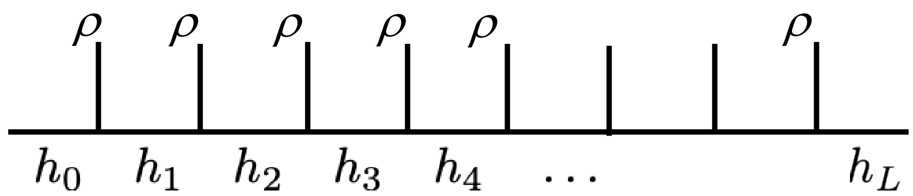
(recall $a \otimes b = \sum_c N_{ab}^c \, c$)

To get periodic boundary conditions, set $h_0 = h_L$.

One nice interpretation of such a Hilbert space is that of an anyon chain.

Feiguin et al 2006

Examples of Hilbert spaces

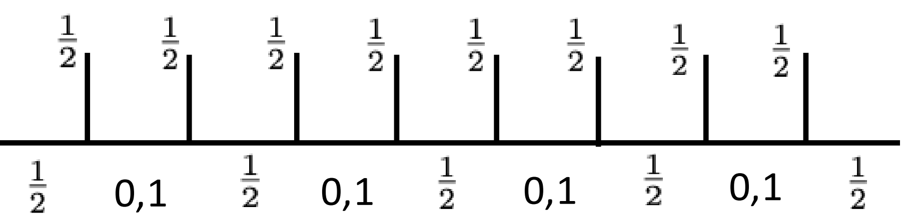


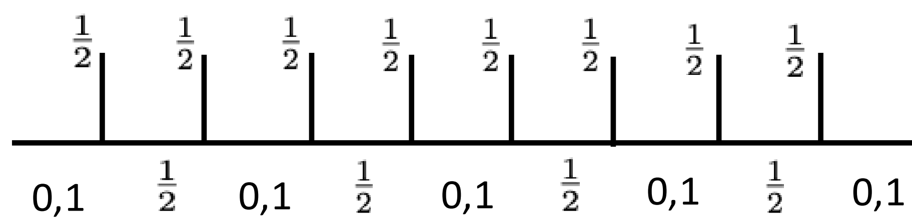
$$N_{h_j \rho}^{h_{j+1}} = 1$$

Ising/$SU(2)_2$ $\quad \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1\,, \quad 1 \otimes \frac{1}{2} = \frac{1}{2}\,, \quad \mathbf{1 \otimes 1 = 0} \quad \boldsymbol{a \otimes 0 = a}$

Take $\rho = \frac{1}{2}$.

Hilbert space then splits into two sectors:



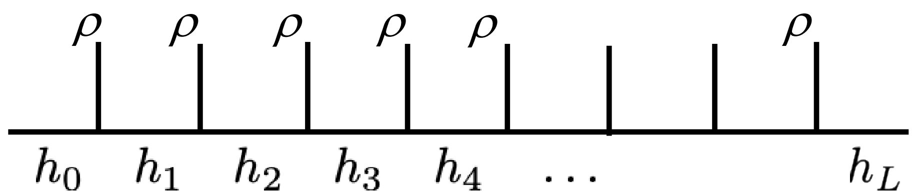


Within each sector, each such Hilbert space is simply a tensor-product one!

2d classical model follows same rules for nearest neighbours:

$$\begin{array}{ccccccccc}
0 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\
1 & \frac{1}{2} & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{2} & 1 \\
\frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2} & 1 & \frac{1}{2}
\end{array}$$

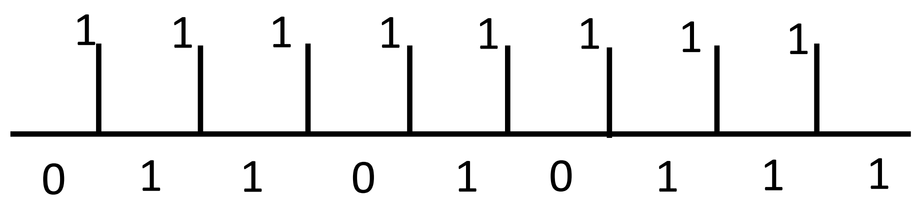
In general, not a tensor product



$$N_{h_j \rho}^{h_{j+1}} = 1$$

FIbonacci/hard squares/golden chain: $1 \otimes 1 = 0 \oplus 1$ with $\rho = 1$

e.g.



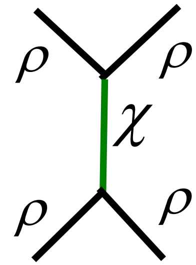
Cannot have two zeroes in a row, i.e. can't have $h_j = 0,\ h_{j+1} = 0$

If you think of 1 as being empty, and 0 as being a particle, get Rydberg blockade!

The Hamiltonian/transfer matrix

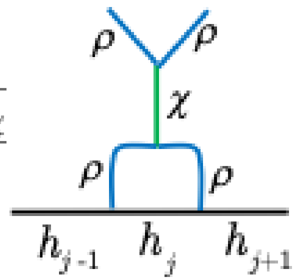
Hamiltonian or transfer matrix is any combination of projection operators

$$P^{(\chi)} = \frac{\sqrt{d_\chi}}{d_\rho}$$

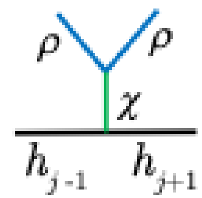


Each projector gives a linear operation on Hilbert space by gluing and using F moves:

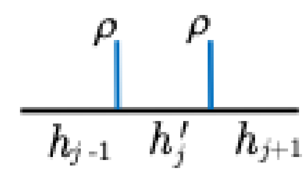
$$\frac{\sqrt{d_\chi}}{d_\rho}$$



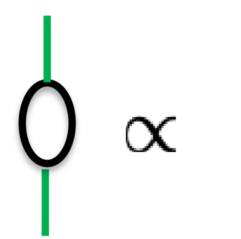
$$= F_{h_j \chi} \begin{bmatrix} \rho & \rho \\ h_{j-1} & h_{j+1} \end{bmatrix}$$



$$= \sum_{h'_j} F_{h_j \chi} \begin{bmatrix} \rho & \rho \\ h_{j-1} & h_{j+1} \end{bmatrix} F_{\chi h'_j} \begin{bmatrix} h_{j-1} & \rho \\ h_{j+1} & \rho \end{bmatrix}$$



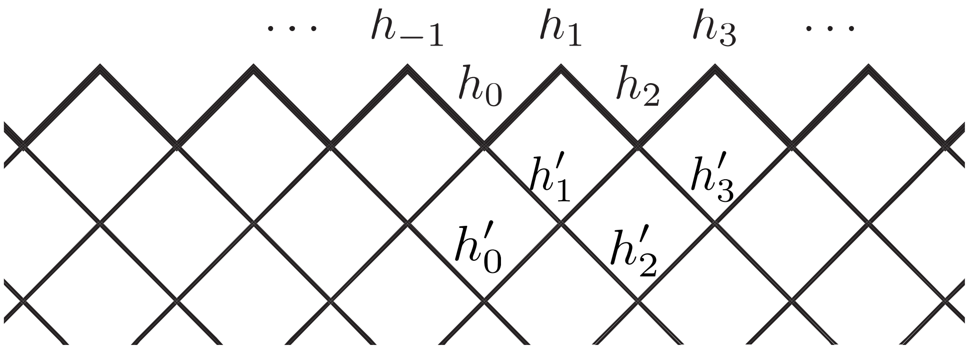
Use bubble removal to see that $P^{(\chi)} P^{(\chi')} = \delta_{\chi\chi'} P^{(\chi)}$

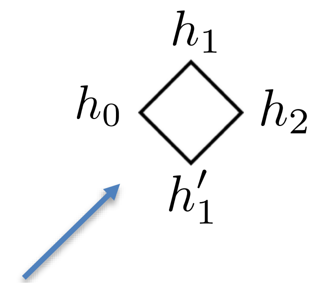


The beauty of the category!

Local statistical-mechanical models

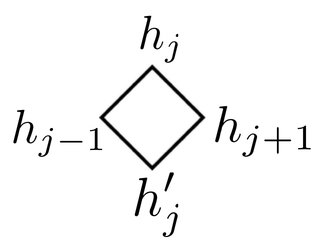
Discrete degrees of freedom called ``heights'' living on the sites of a square lattice:





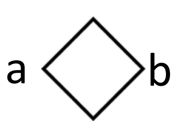
Boltzmann weight depends on interactions among the four heights ``round a face''

$$Z = \sum_{\text{heights}} \prod_{\text{faces}}$$

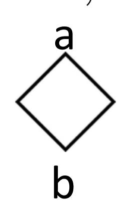


Like Hamiltonian, Boltzmann weights are built from projection operators $P^{(a)}$

For nearest-neighbor Ising, include degrees of freedom $a, b = 0, 1$ on half the sites:



$= e^{K_x \delta_{ab}}$



$= e^{K_y \delta_{ab}}$

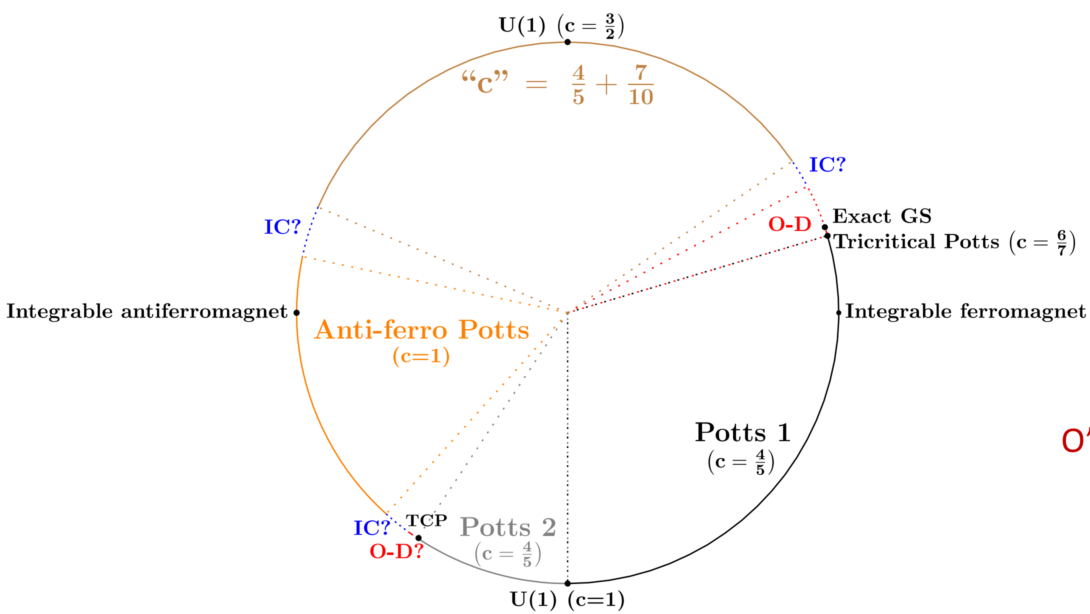
Many well-known models fit in this framework

Any ``categorical” lattice model has a family of topological defects!

Seems likely there exists at least at least one (often more than one) such lattice model for each RCFT.

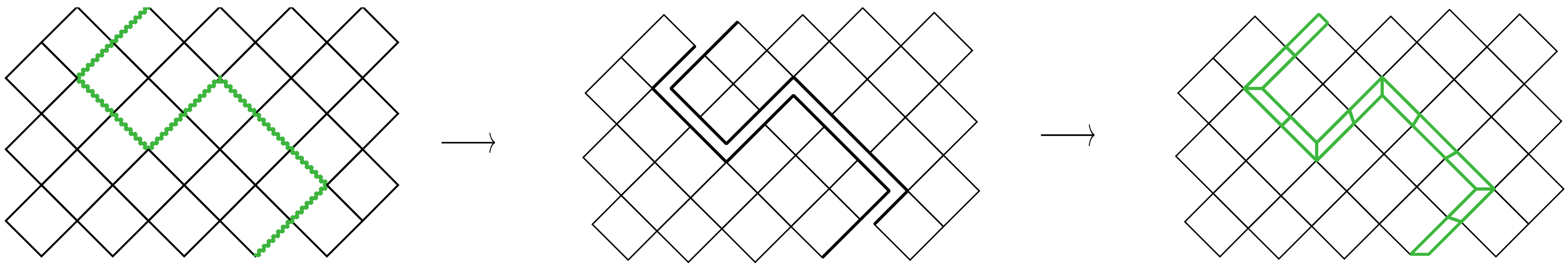
Category of lattice model is typically a subcategory of the category describing RCFT.

Important to emphasise that generically, these lattice models are neither critical, nor integrable, even with spatially uniform couplings. Very rich phase diagrams, e.g.

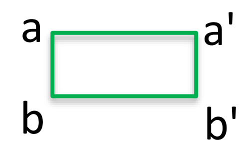


O’Brien & Fendley 2019

Inserting a defect



The defects have a weight depending on the adjacent heights:



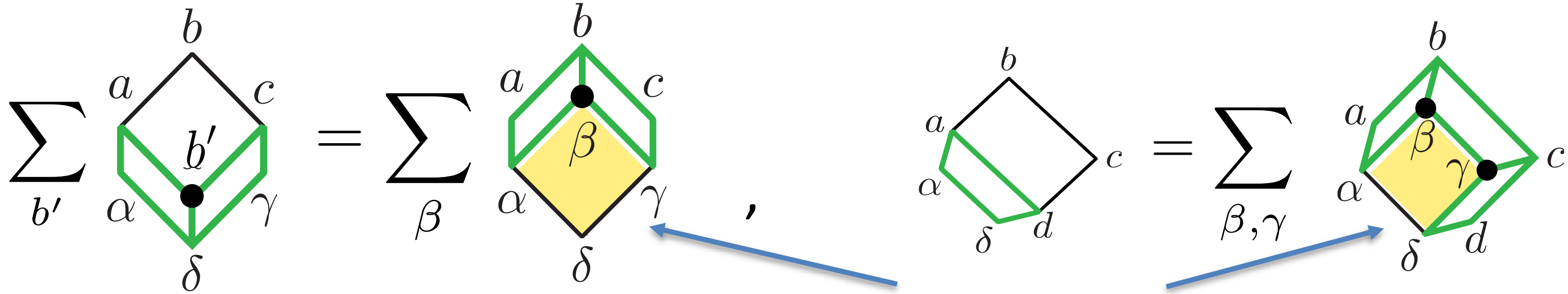
One type of defect for each object ϕ, with weight $\propto F_{\phi\rho}\begin{pmatrix} a & b' \\ b & a' \end{pmatrix}$

In the presence of the defect, the partition function is modified to

$$\mathcal{Z} = \sum_{\text{heights}} \prod_{\text{faces}} \Diamond \times \prod_{\text{along defect}} \square$$

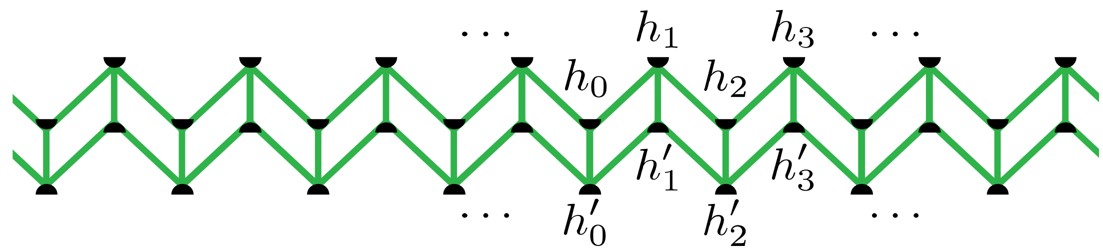
To make defects topological, they must satisfy defect commutation relations

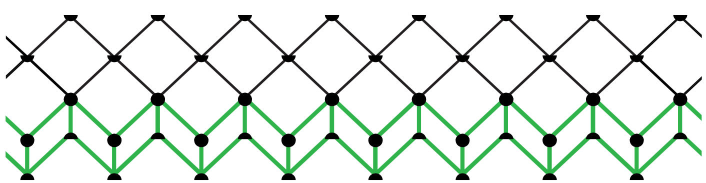
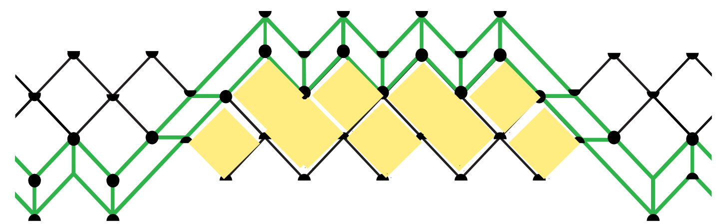
For $\mathcal{Z}$ to be invariant under deformations of the defect's path:

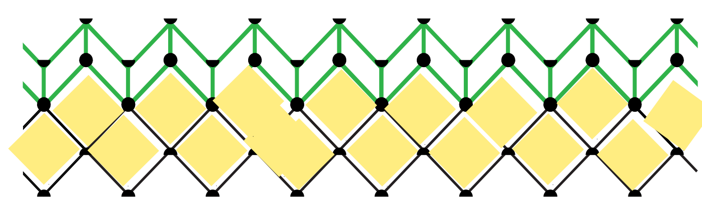


defects can separate “dual” models

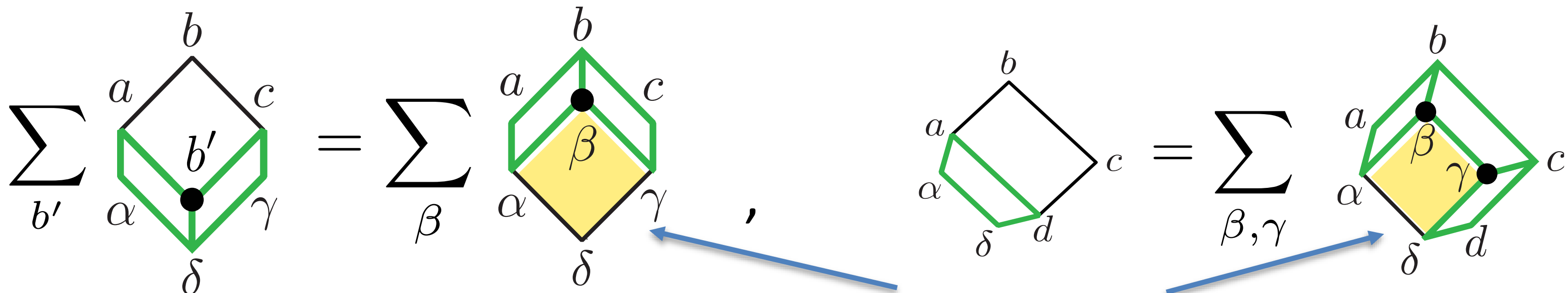
On Hilbert space, define
Defect-line creation operator:

$\mathcal{D} =$ 

$\mathcal{D}T_1 =$  = 

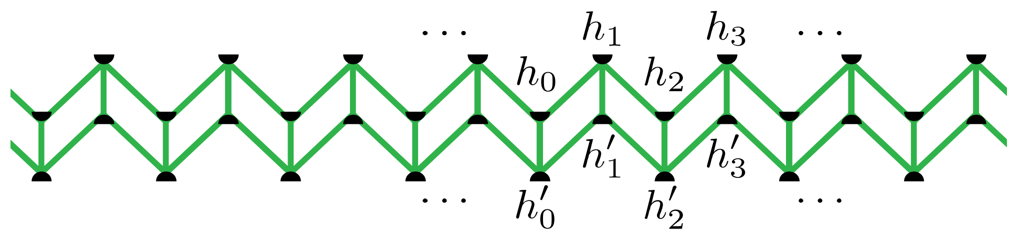
=  $= T_2\mathcal{D}$

To make defects topological, they must satisfy defect commutation relations

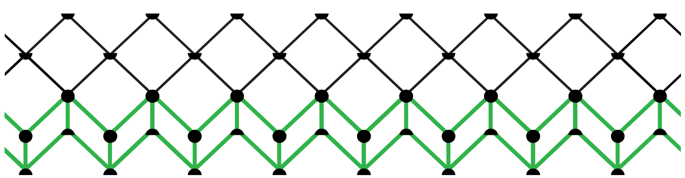
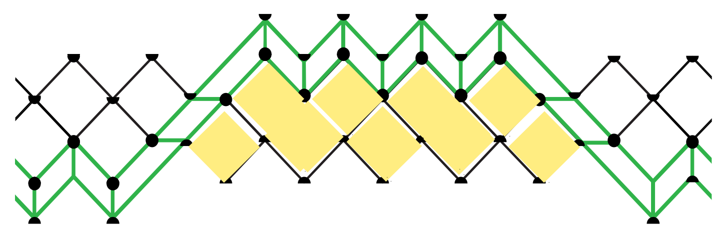


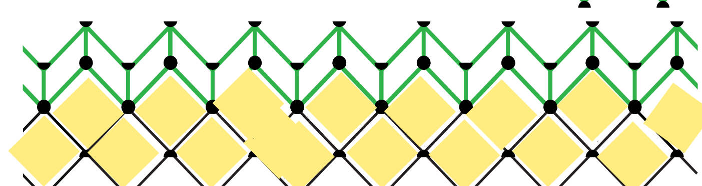
defects can separate "dual" models

On Hilbert space, define Defect-line creation operator: $\mathcal{D} =$



Commutes with each projector individually: $\mathcal{D}P_j^{(x)} = P_j^{(x)}\mathcal{D}$

$\mathcal{D}T_1 =$  = 

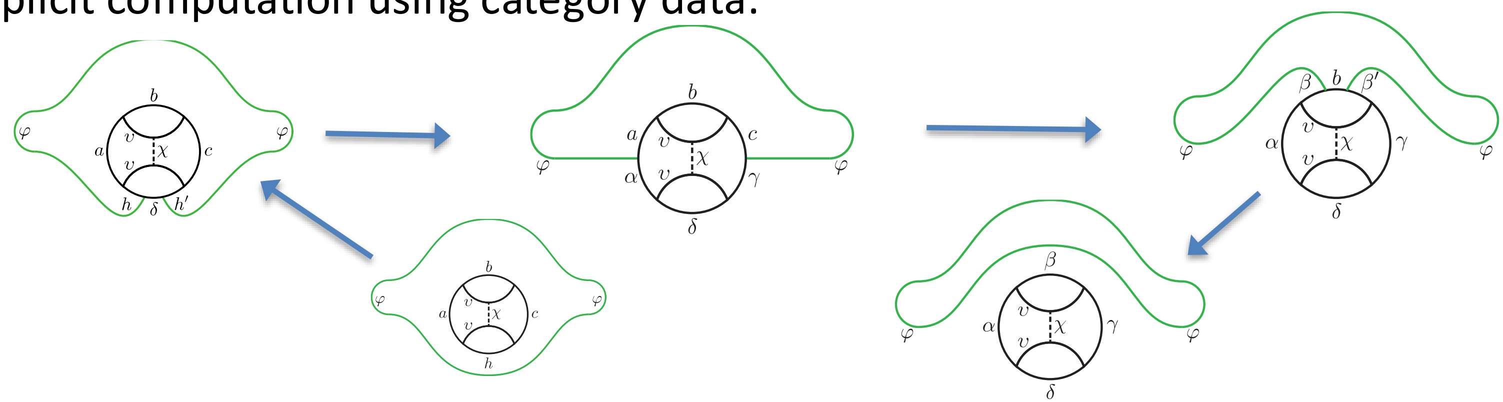
=  $= T_2\mathcal{D}$

Free lunch: define model using category, get a topological defect for each object

Methods of proof:

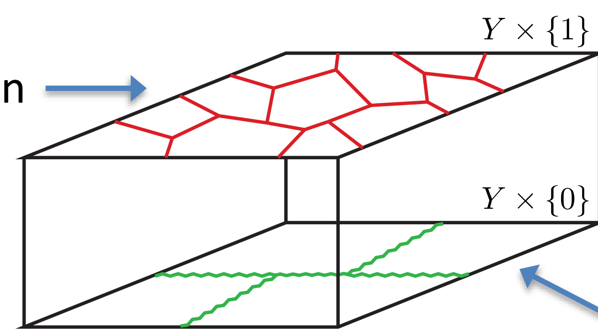
1) Defects are objects in the Drinfeld centre, so follows essentially by definition

2) Explicit computation using category data:



3) Using the Turaev-Viro-Barrett-Westbury ”state sum” approach to 3d TQFT (subsequently generalized and renamed ”SymTFT”)

Defect configuration



2d classical lattice model

Two types of topological defects in Ising

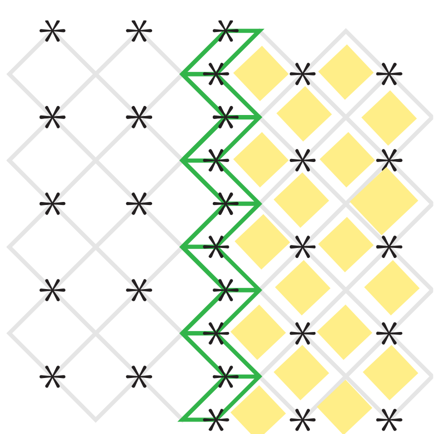
For Ising, fluctuating degrees of freedom on only half the sites:

$$a \diamond b = e^{K_x \delta_{ab}} \qquad \overset{b}{\underset{a}{\diamond}} = e^{K_y \delta_{ab}}$$

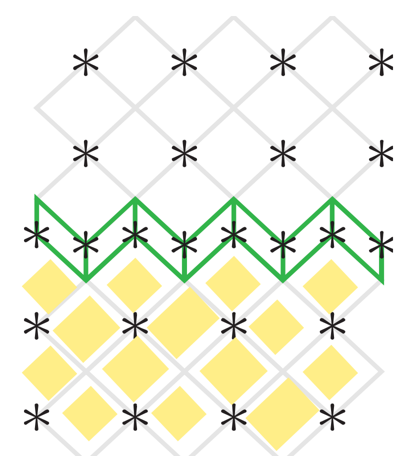
spin-flip defect: $\quad {}^{b}_{a}\square = \square^{b}_{a} = 1 - \delta_{ab}$

duality defect: $\quad {}_{a}\square^{b} = {}^{b}\square_{a} = \dfrac{1}{2^{1/4}}(-1)^{ab}$

Duality defect maps between Hilbert-space sectors!



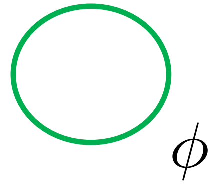
Couplings on one side of duality defect are dual to those on the other



Use #1: Generalized K-W duality

Automatically get identities between partition functions with different defect configuations

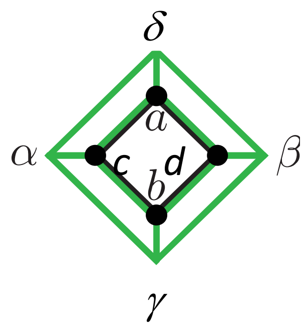
In a category,



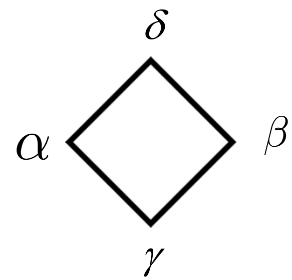
$= d_\phi$

By either the abstract manipulations or using the explicit defect weights, find

$\sum_{a,b,c,d}$

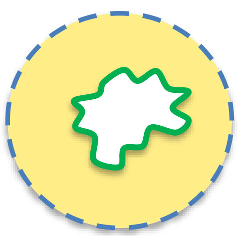


$= d_\phi$

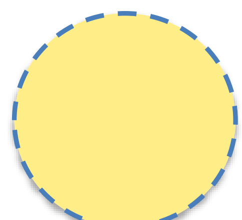


Use commutation relations to move defect around. For any path, partition functions on a disc are related as

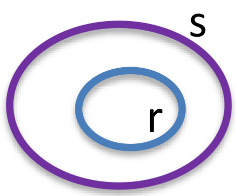
$$\mathcal{Z} = d_\phi Z$$

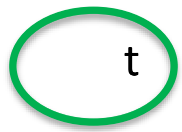


$= d_\phi$



Fusing

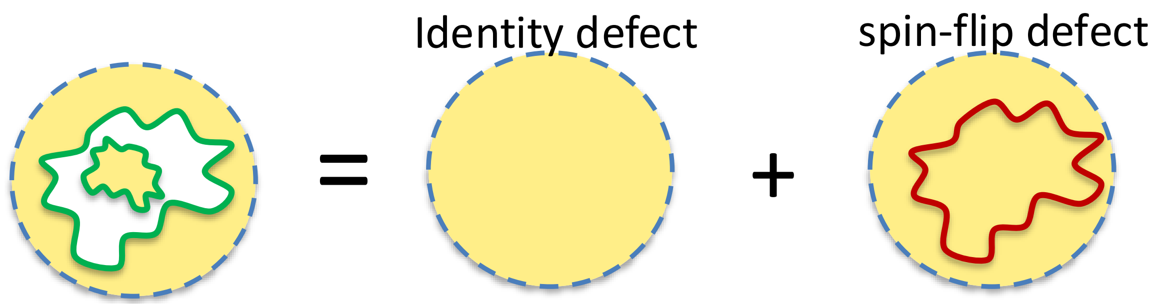


$= \sum_t N^t_{rs}$


so e.g. $\mathcal{D}_\alpha \mathcal{D}_\beta = \sum_\gamma N^\gamma_{\alpha\beta} \mathcal{D}_\gamma$

In Ising category (and CFT): $\mathcal{D}_{1/2}\mathcal{D}_{1/2} = 1 + \mathcal{D}_1$

Combine two duality defects:

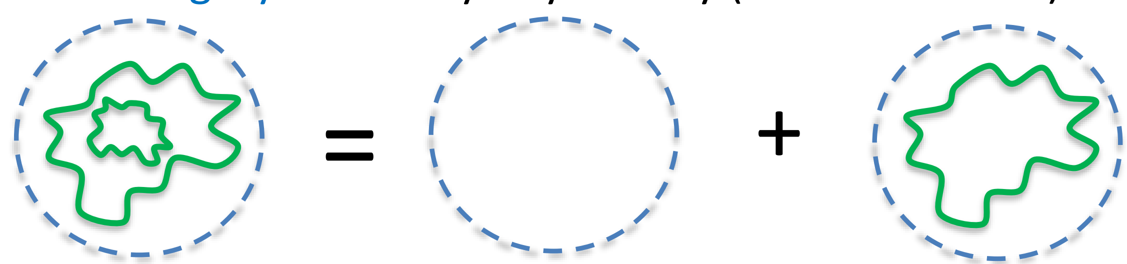


KW duality is not a symmetry – even at the critical point, it changes boundary conditions.

(At Ising critical point, in the Hamiltonian limit, with periodic b.c., you can mix KW duality with translation symmetry to get an actual symmetry.) Seiberg and Shao 2023

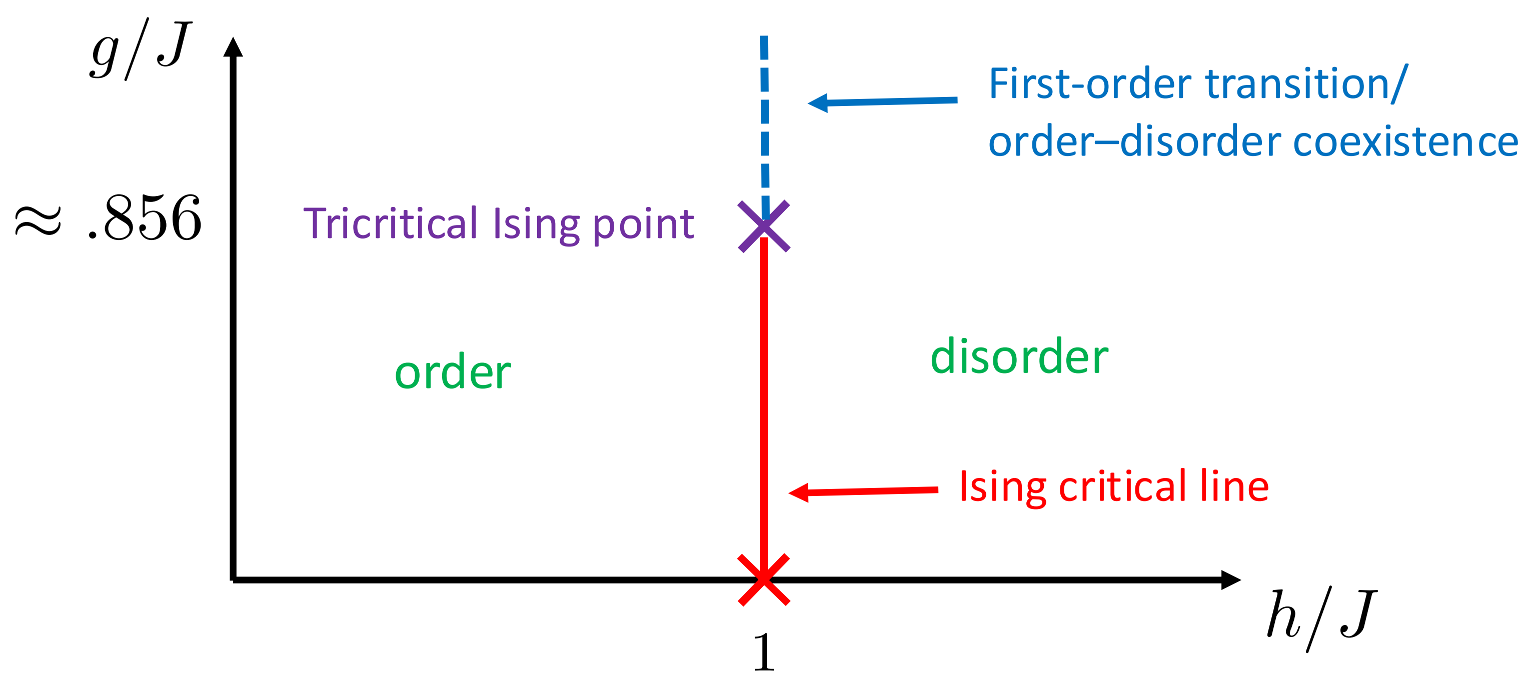
``Topological” symmetry for Fibonacci category is actually a symmetry (but is invertible)

$$\mathcal{D}_1\mathcal{D}_1 = 1 + \mathcal{D}_1$$



Useful in locating transition points

$$H = \sum_j \Big(X_j + Z_j Z_{j+1} + g\big(X_j Z_{j+1} Z_{j+2} + Z_j Z_{j+1} X_{j+2}\big)\Big)$$

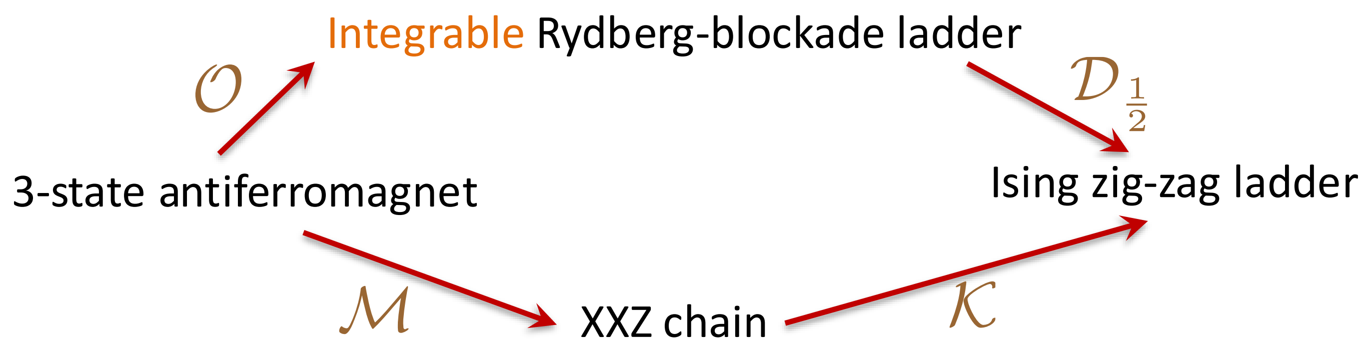


Self-duality means only one-parameter tuning needed to get to tricritical point

O'Brien and Fendley

Use #3: Non-invertible, non-obvious mappings, e.g. Rep(S_3)/ $SU(2)_4$

Eck & Fendley 2023; see also Lootens et al



The Hamiltonians obey e.g. $\mathcal{K}\, H_{\rm XXZ} = H_{\rm zig}\, \mathcal{K}$ $\qquad$ $\mathcal{K}^\dagger\, H_{\rm zig} = H_{\rm XXZ}\, \mathcal{K}^\dagger$

The maps obey e.g. $\mathcal{K}\,\mathcal{K}^\dagger = 1 + \mathcal{F}$ $\qquad$ $\mathcal{K}^\dagger\,\mathcal{K} = 1 + (-1)^L \mathcal{F}_{\rm zig}$

$$\mathcal{D}_{\frac{1}{2}}\,\mathcal{O} = \mathcal{M}\,\mathcal{K}$$

XXZ chain is well understood, so gives new results about physics of other models

Rydberg blockade

Rydberg-atom arrays provide a way of realizing all sorts of interesting phases in lattice models. The exceptional ability to tune interactions allows one to carefully examine these phases and the transitions between them.

In the Rydberg blockade, each site is effectively a two-state system that can be viewed as empty or as occupied by a hard-core boson. The blockade means that neighbouring (or more) sites cannot both be occupied.

Experiments on the chain have already revealed much interesting physics in strongly interacting systems. The golden chain has been realized!

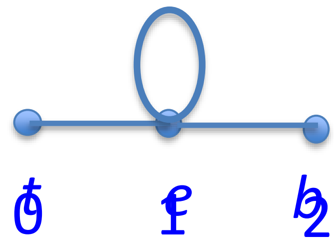
Experimental observation of conformal field theory spectra

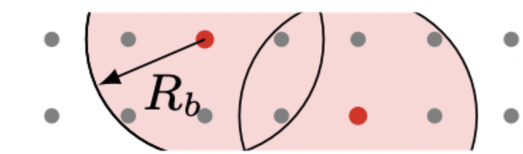
Xiangkai Sun,[1,*] Yuan Le,[1,*] Stephen Naus,[1,*] Richard Bing-Shiun Tsai,[1,*] Lewis R. B. Picard,[1] Sara Murciano,[2] Michael Knap,[3,4] Jason Alicea,[1] and Manuel Endres[1,†]

Here we directly observe the energy excitation spectra of emergent CFTs at quantum phase transitions—recovering universal energy ratios characteristic of the underlying field theories[13–15]. Specifically, we develop and implement a modulation technique to resolve a Rydberg chain's finite-size spectra, variably tuned to quantum phase transitions described by either Ising or tricritical Ising CFTs. We also employ lo-

Integrable Rydberg-blockade ladder

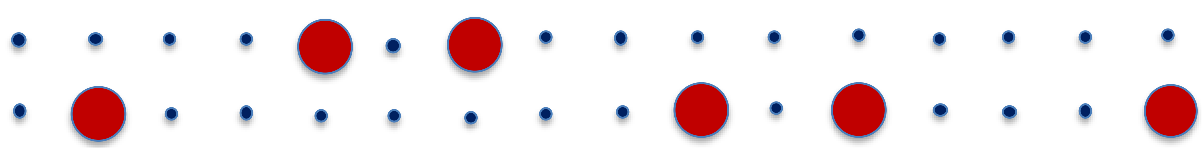
Using category, can construct ``height'' models with nice properties.
For $SU(2)_4$ - Rep(S_3) , degrees of freedom 0,1,2 obeying $1 \otimes 1 = 0 + 1 + 2$
adjacent heights on a chain must be adjacent on the diagram





Allowed configuration is 1211010112121112 -> ebeteteebebeeeb

Interpret as a ladder: 0 is particle on top rung, 2 on bottom, 1 is empty rung

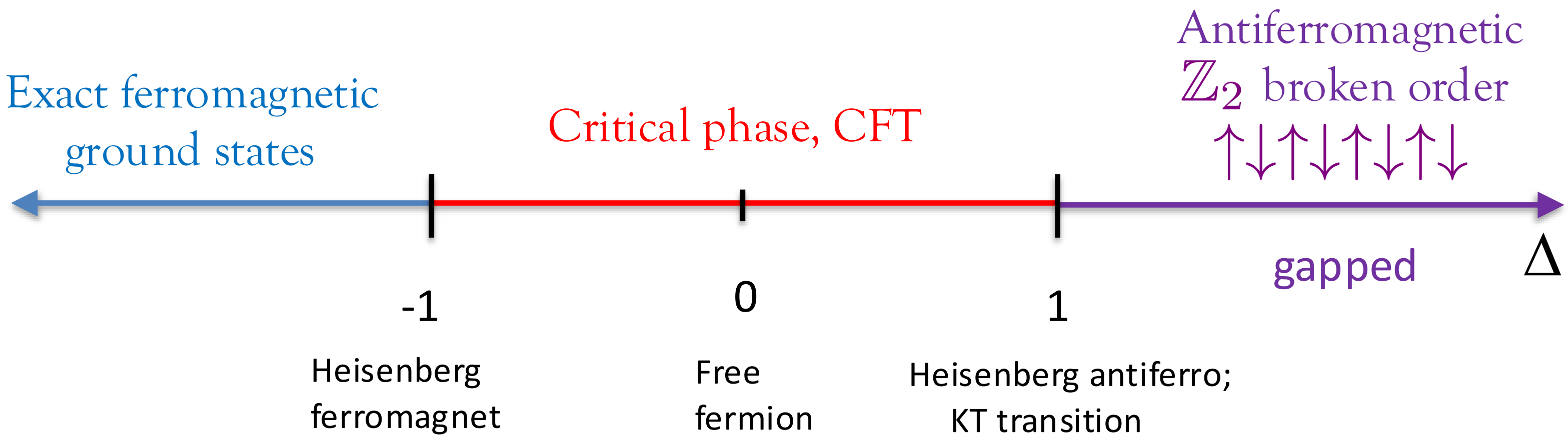


1 2 1 1 0 1 0 1 1 2 1 2 1 1 1 2

Rules mean at most one particle per square. The Rydberg blockade!

The physics of the integrable line

Phase diagram of XXZ



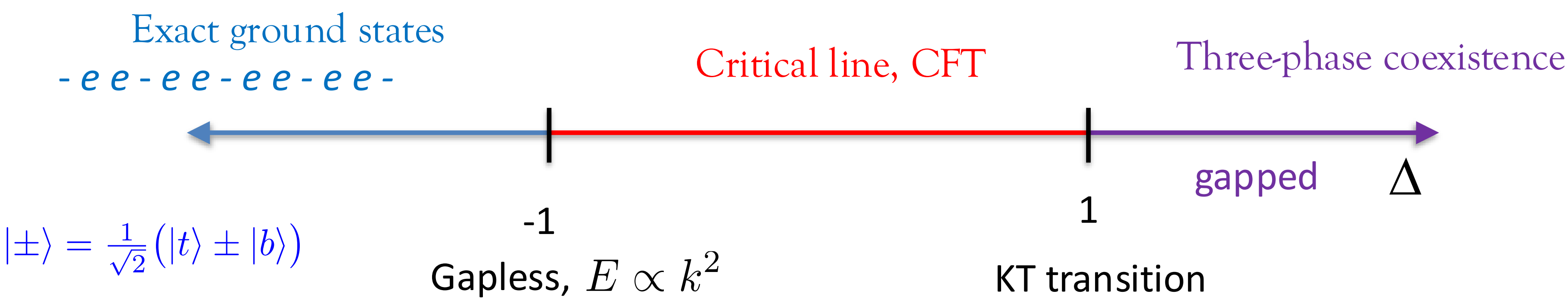
The effective field theory of XXZ is a free massless boson of radius R, where

$$\Delta = -\cos\frac{\pi}{2R^2} \qquad R \geq \frac{1}{\sqrt{2}}$$

Critical ground state survives all mappings: all models are critical for $-1 \leq \Delta \leq 1$

``Equivalent'' models are all c =1 CFTs, but not the same ones!

Phase diagram of integrable Rydberg-blockade ladder



We thus have located an exact critical line in the Rydberg-blockade ladder!

Its effective CFT is a c =1 orbifold with $\Delta = -\cos\frac{9\pi}{2R^2}$

As $\Delta \to \infty$, three ground states are

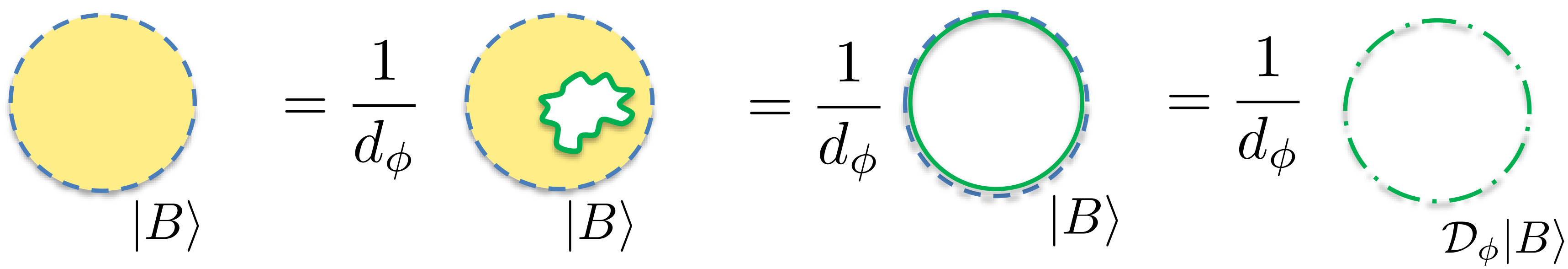
e e e e e e e e e e e e
e + e + e + e + e + e +
+ e + e + e + e + e + e

No conventional symmetry relates them, but self-duality does.

Three ground states for all $\Delta > 1$!

Use #4: Exact g-factor ratios from lattice

Aasen-Fendley-Mong 2020



Consider vector space of all configurations of spins/heights near edge. Each vector $|B\rangle$ corresponds to a boundary condition, e.g. $|\text{fixed up}\rangle = |++++\,+\ldots\rangle$

Acting with $\mathcal{D}_\phi$ absorbs the defect into edge, changing $|B\rangle \to \mathcal{D}_\phi|B\rangle$

$$\frac{Z_{\mathcal{D}_\phi|B\rangle}}{Z_{|B\rangle}} = d_\phi$$

Exact!!

e.g. duality defect in Ising changes b.c. as $\mathcal{D}_\sigma|\text{fixed up}\rangle = |\text{free}\rangle$

$$Z_{\text{free}}(K) = \sqrt{2}\, Z_{\text{fixed}}(\widehat{K})$$

where dual coupling is defined by $\sinh(2K)\sinh(2\widehat{K}) = 1$

For conformal boundary conditions, this ratio of partition functions is

$$\frac{Z_{\mathcal{D}_\phi|B\rangle}}{Z_{|B\rangle}} = d_\phi = \frac{g_{\mathcal{D}_\phi|B\rangle}}{g_{|B\rangle}}$$

where $-T\ln g_{|B\rangle}$ is the boundary free energy. This Affleck-Ludwig g-factor is universal, and computable in RCFT.

Calculation is direct and easy here! Moreover, gives precise lattice expressions for boundary states. Provides precise way of comparing lattice and CFT.

Another way of comparing is to compute the Dehn twist (i.e. modular transformation) in the presence of twisted boundary conditions (vertical topological defect). Yields conformal dimensions up to half-integer shift!

Use #6: 2d quantum Hamiltonians generalizing Kitaev honeycomb model

- The ``categorical" quantum Hamiltonians described above can be generalized to two dimensions by using fusion 2-categories.

Inamura and Ohomori 2023

- Luckily in the cases of most interest, all one needs is a braided fusion category, i.e. what I described above plus braiding, i.e. allowing diagrams with over- and undercrossings.

- I-O show that the Ising category yields the Kitaev honeycomb model. The setup automatically gives local symmetries, the ``plaquette" conserved fluxes.

$$H^{\text{Kitaev}} = -J_x \sum_{a,b \in \text{x-link}} X_a X_b - J_y \sum_{a,c \in \text{y-link}} Y_a Y_c - J_z \sum_{b,d \in \text{z-link}} Z_b Z_d,$$

- Category gives a natural method general way of defining 2d models with such symmetries for any braided fusion category. Time-reversal-symmetry breaking is built in, so get promising candidates for chiral topological order!

Eck and Fendley 2024

Conclusions

- One feature of these lattice calculations is their simplicity. Extract exact and universal results almost by definition
- Provides useful diagnostic tool for identifying lattice with continuum
- Naturally generalizes to higher dimensions -- story just beginning
- Connections to integrability profound, but not well understood
- Moral: if an interesting concept originates on the lattice, probably a good idea to keep thinking about the lattice