Private correlations and mixed state complexity

Nordita, July 2026

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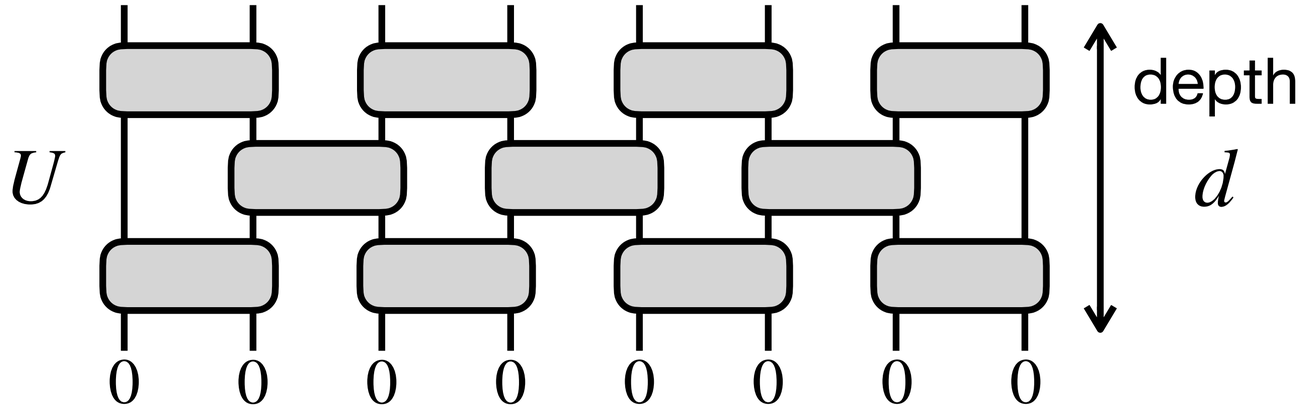
arXiv:2502.13218/PRL 2026, 2510.02275

work with Max McGinley, Cambridge → UBC

Characterization of pure states based on quantum complexity

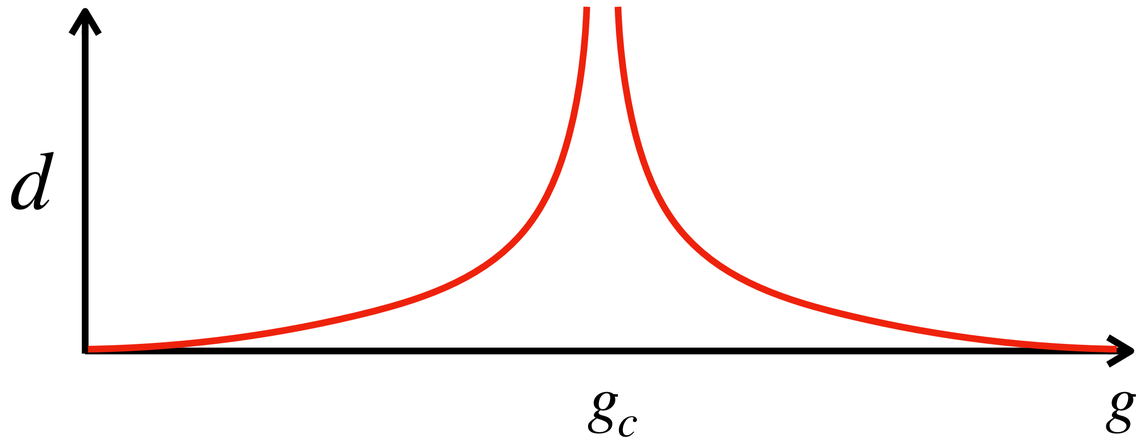
$$|\psi\rangle \approx U|0\ldots0\rangle$$



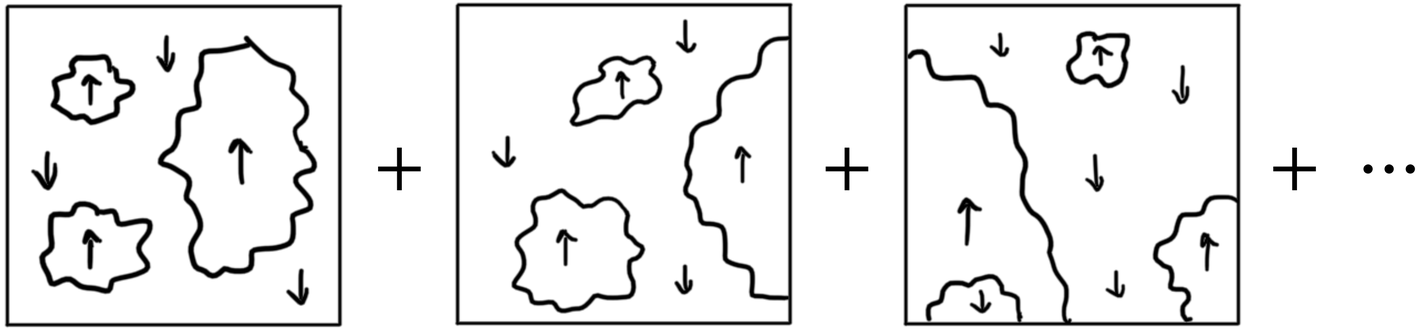
d finite in topologically trivial ground states

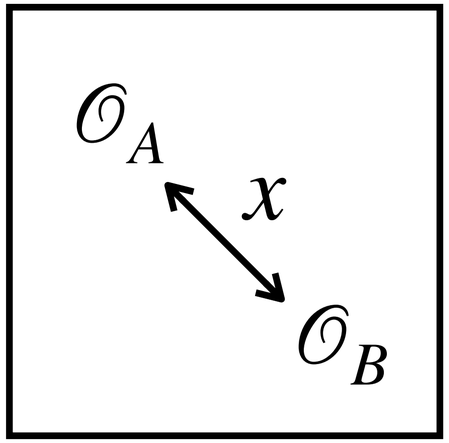
Chen, Gu, Wen PRB 2011

d diverges at continuous quantum phase transitions



Unitary circuit depth is powerful as it is related to physical properties

Pure state $|\psi\rangle =$  $+ \cdots$



$$\langle \mathcal{O}_A \mathcal{O}_B \rangle_c \neq 0 \implies I(A:B)$$

$$\implies \text{depth}\, d \geq x/2$$

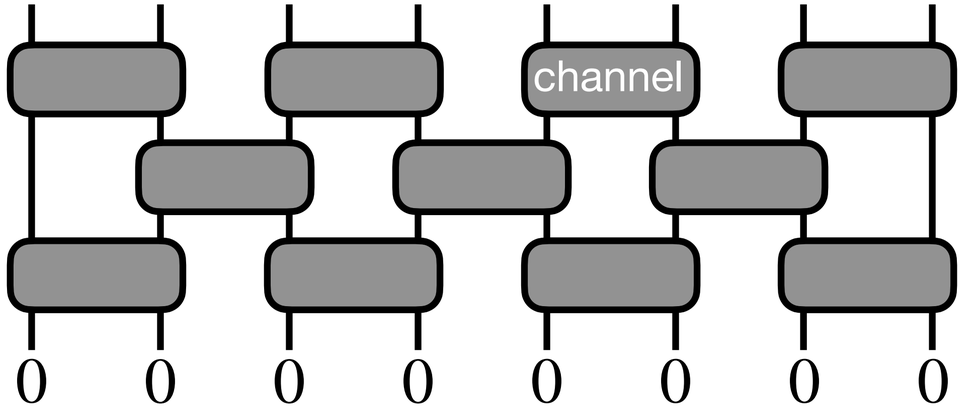
Characterization of mixed-state phases

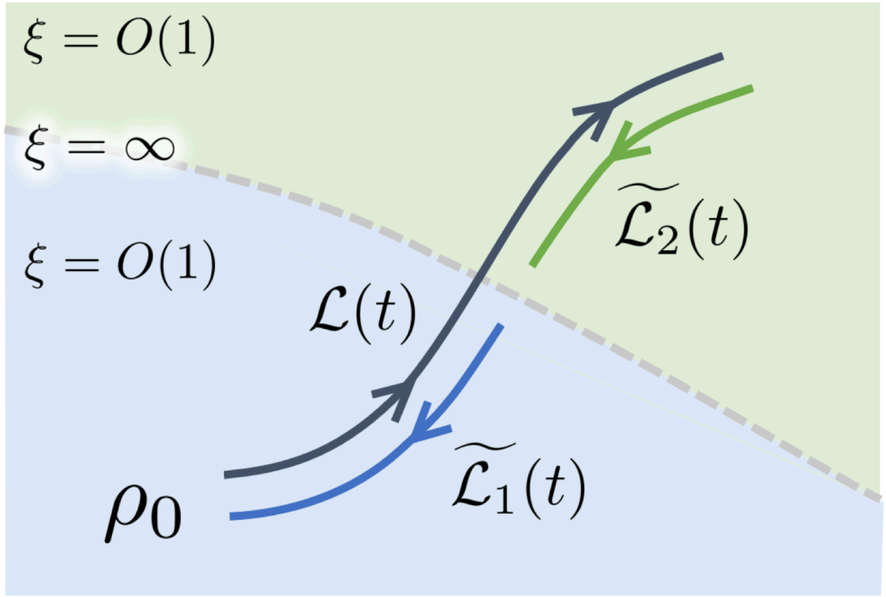
In pure-state phase have connectivity under finite-depth circuits

c.f. mixed-state phase: connectivity under finite-depth circuits of channels

Hastings PRL 2011, Coser, Perez-Garcia Quantum 2019, Sang, Lessa, Mong, Grover, Wang, Hsieh 2507.02292

$$\rho \approx \mathscr{E} \circ \cdots \circ \mathscr{E}'\big(|0\ldots0\rangle\langle 0\ldots0| \big)$$



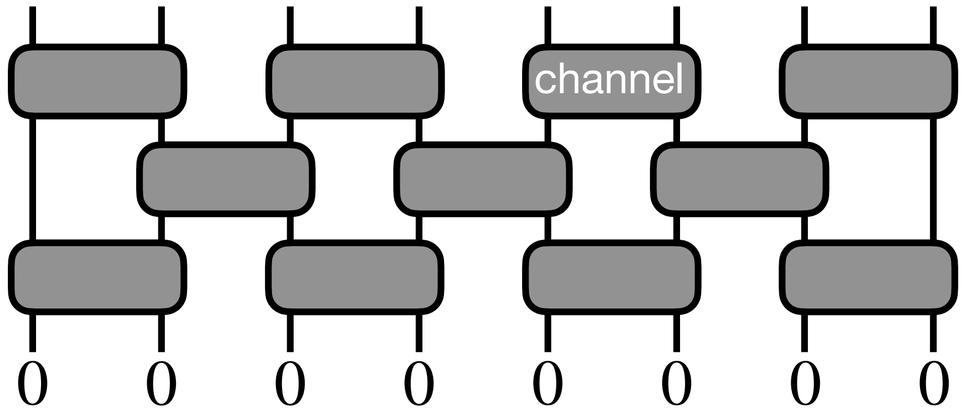


Sang, Hsieh PRL 2025

For mixed states there are various notions of complexity

Circuit of local channels

$$\rho \approx \mathscr{E} \circ \cdots \circ \mathscr{E}'\big(\,|0\ldots0\rangle\langle 0\ldots0|\,\big)$$

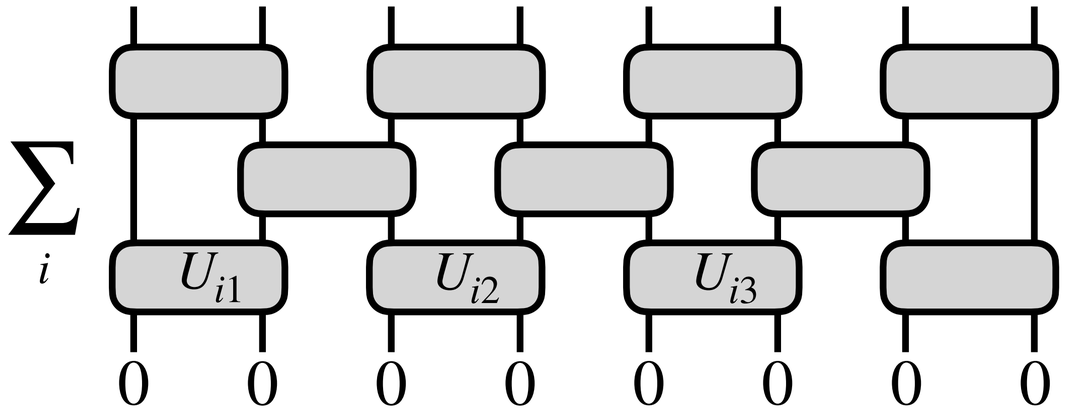


Classical distributions with long-range correlations are ‘complex’ in this definition

Ensemble of unitary circuits

$$\rho \approx \sum_i p_i U_i |0\ldots\rangle\langle 0\ldots| U_i^\dagger$$

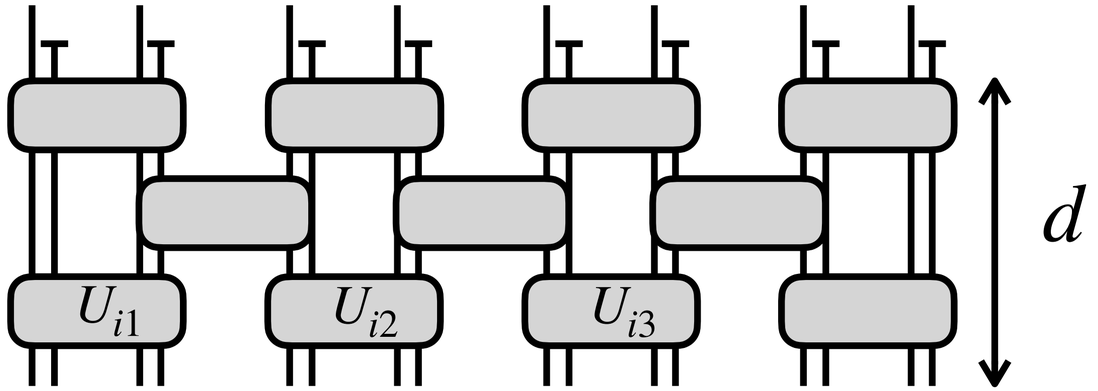
‘Mod out’ arbitrary classical correlations



Local operations and shared randomness (LOSR)

Ensemble of unitary circuits with ancillas

$$\rho \approx \sum_i p_i \mathrm{Tr}_{\text{ancillas}} \left(U_i |0\ldots\rangle\langle 0\ldots| U_i^\dagger \right)$$



Including ancilla + tracing out allows for implementation of non-unital channels

Here LOSR are free; c.f. LOCC, where measurement + feedforward are free

Buscemi PRL 2012

Quantumly interesting (= complex relative to LOSR) mixed states?

Expect many quantum states and processes are ‘close’ to classical

e.g. thermal state $\rho \sim e^{-\beta H}$ of $H = -\sum_{\langle jk \rangle} (Z_j Z_k + g X_j)$

Mixture of spin configurations ‘dressed’ by quantum fluctuations

This talk: simple information-theoretic inequalities to quantify complexity

+ connections to quantum key distillation and ‘private correlations’

+ relations between linear response and thermal state complexity

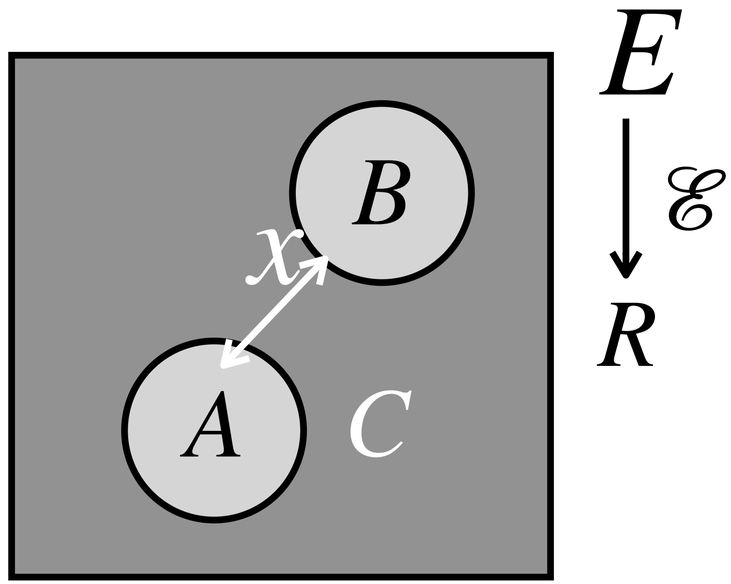
+ smorgasbord of complex mixed states with high entropy

Lower bound on depth

Suppose $\rho_{ABC} = \sum_i p_i U_i |0\rangle\langle 0| U_i^\dagger$ depth $d < x/2$

Non-unique

$$\rho_{ABCR} = \sum_i p_i |i\rangle_R\langle i| \otimes |\phi_i\rangle_{ABC}\langle\phi_i|$$



Since $d < x/2$, $I(A:B|R) = \sum_i p_i I_{\phi_i}(A:B) = 0$

For Ψ_{ABCE} a purification of ρ_{ABC}, $\exists$ channel $\mathscr{E}_{E\to R}(\Psi) = \rho_{ABCR}$

$I(A:B|R) = I(A:BR) - I(A:R) \geq I(A:B) - I(A:E)$ Data processing x 2

If $I(A:B) > I(A:E) \implies d \geq x/2$

Ruling out a low-depth approximation

If $I(A:B) > I(A:E) + \epsilon$ at distance x

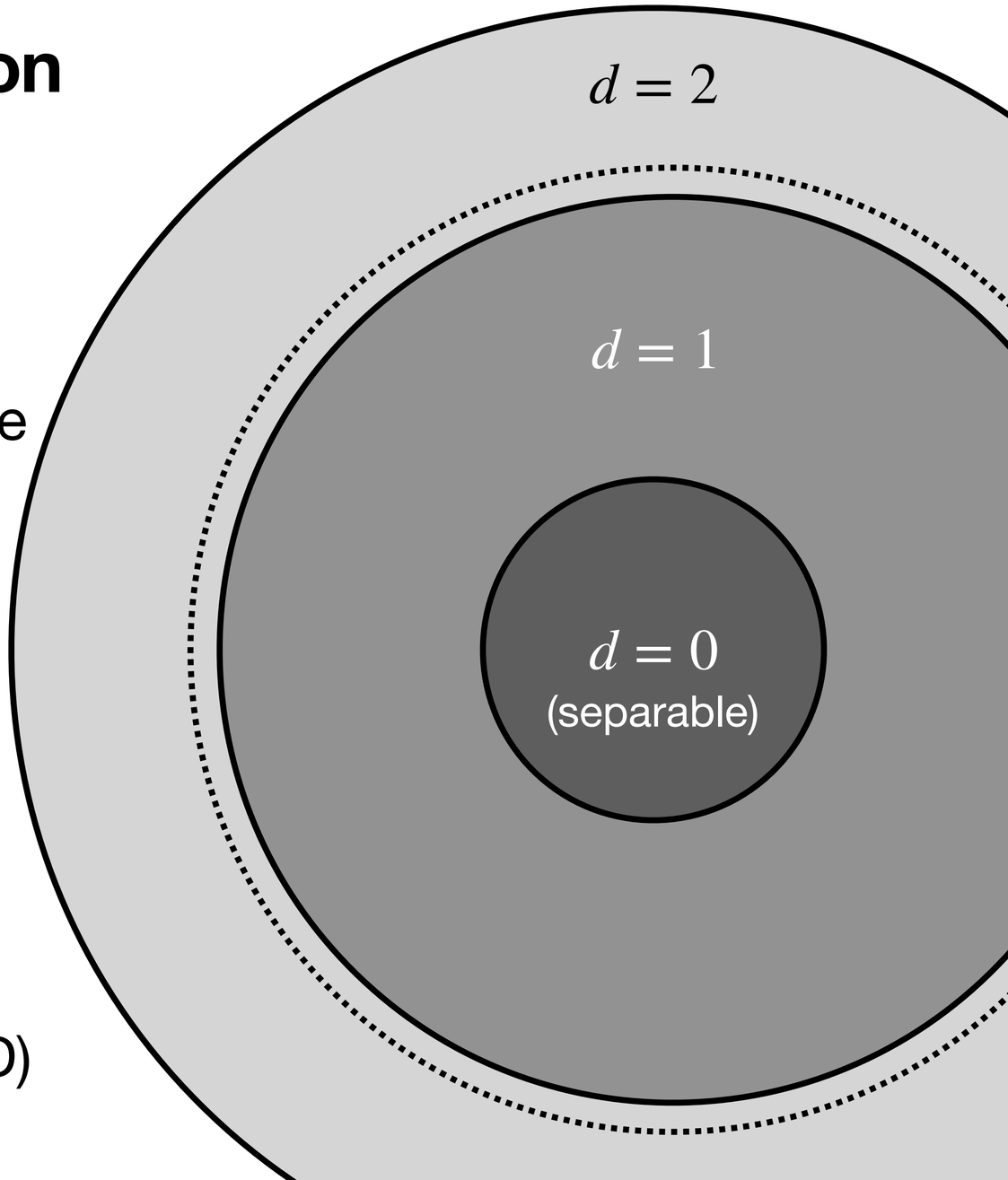
then ‘ $> \epsilon$ away’ from depth $d = x/2$ ensemble

Also holds after channel $A \to A'$

$$I(A':B) > I(A':E) \implies d \geq x/2$$

Next: (another) operational meaning

detour into quantum key distillation (QKD)



Entanglement distillation

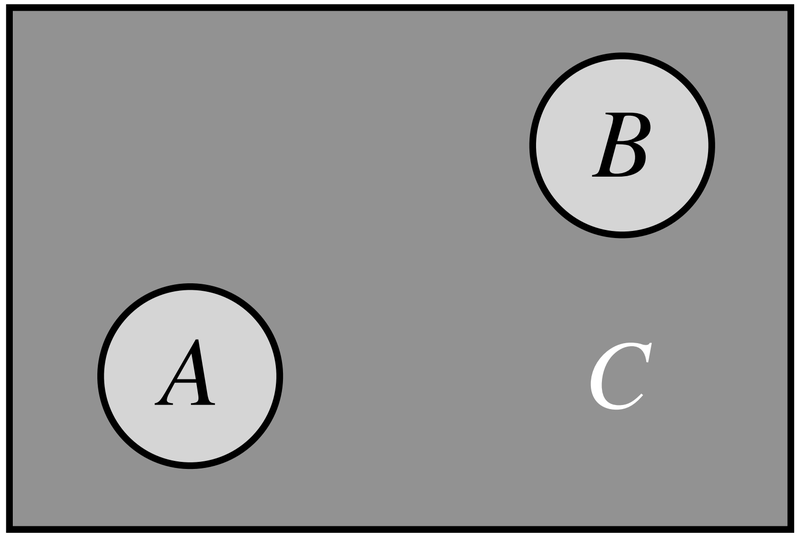
Many-body mixed state $\rho = \rho_{ABC}$

Can Alice and Bob extract EPR pairs?

from many copies of ρ

No! Reduced state ρ_{AB} generically separable

e.g. even in GS of $H = \sum_j \vec{\sigma}_j \cdot \vec{\sigma}_{j+1}$



EPR pair shared by A and B is a correlation that is 'private' from rest of universe

But in a many-body system A and B are more correlated with C than each other

Entanglement distillation fails

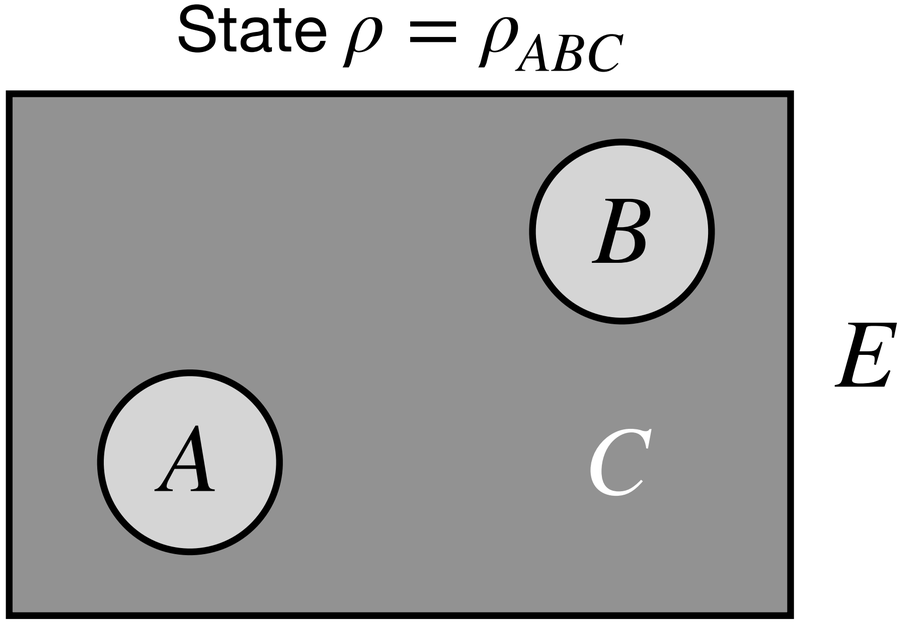
Distilling private classical correlations

Environment E purifies system Ψ_{ABCE}

A performs local measurements $m_1, m_2, m_3, \ldots$

B receives $\rho_{B,m_1} \otimes \rho_{B,m_2} \otimes \cdots$

E receives $\rho_{E,m_1} \otimes \rho_{E,m_2} \otimes \cdots$



B can distill $\chi_B = S(\rho_B) - \sum_m p_m S(\rho_{B,m})$ bits of mutual information with m

$\chi_B > \chi_E \implies$ Classical correlations shared by A and B that are private from E, but not necessarily from C

Cai, Leung, Winter 2005, Devetak, Winter 2005

What are private classical correlations? Part 1

$\chi_B > \chi_E \implies$ Classical correlations shared by A and B that are private from E, but not necessarily from C

$$\Psi_{ABCE} \xrightarrow{\text{QKD}} \sum_i |i\rangle_{AB}\langle i| \otimes \rho_{CE,i}$$

$$\rho_E = \text{Tr}_C\left[\rho_{CE,i}\right]$$

$$\rho_{C,i} = \text{Tr}_E\left[\rho_{CE,i}\right]$$

Classical probability distributions do not have private classical correlations

If $\rho_{ABC} = \sum_s p(s)\,|s\rangle\langle s|$ then $\Psi_{ABCE} = \sum_s \sqrt{p(s)}\,|s\rangle_{ABC} \otimes |s\rangle_E$

and the environment/eavesdropper knows the classical state of the system

What are private classical correlations? Part 2

GHZ state of ABC $\quad \frac{1}{\sqrt{2}}\left(|00\ldots0\rangle + |11\ldots1\rangle \right) \otimes E$

If A measures Z_A, then B can learn A's outcome by measuring Z_B

In this example E is in a tensor product with ABC, so E learns nothing

	A	B	C	E
GHZ	001011101100	001011101100	001011101100	??????????
EPR	001011101100	001011101100	??????????	??????????

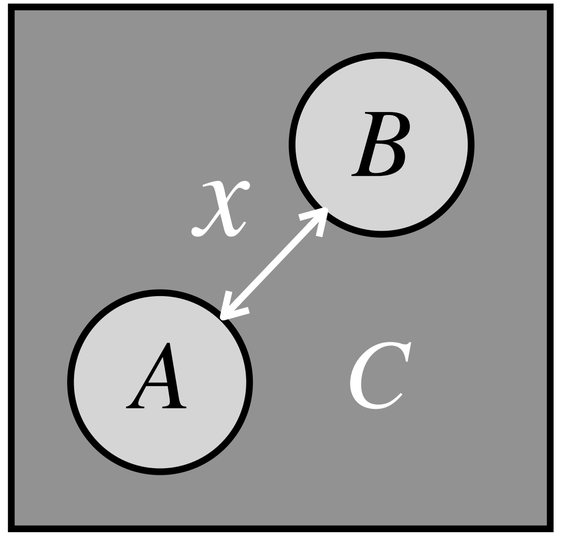
Private classical correlations require entanglement, but are 'weaker' than EPR pairs

State complexity and private classical correlations

Channel converts A into measurement outcome

$$\rho_{ABCE} \to \sum_m p_m |m\rangle_{A'}\langle m| \otimes \rho_{BCE,m}$$

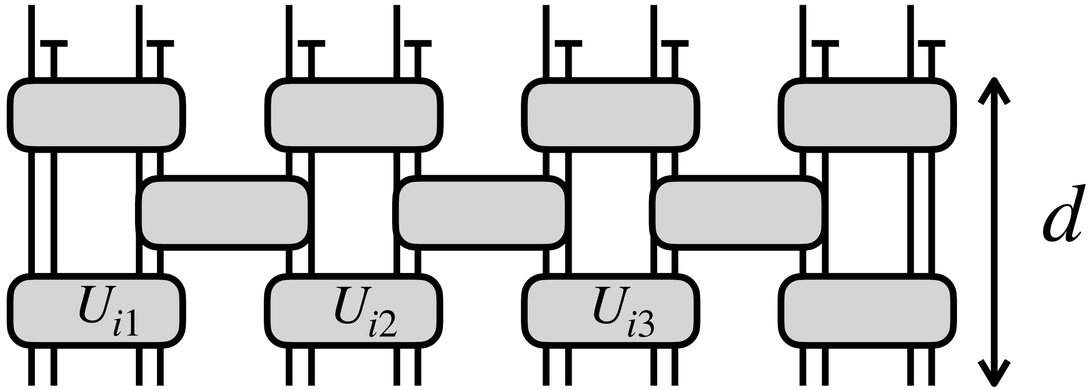
$$I(A':B) > I(A':E) \implies d \geq x/2$$



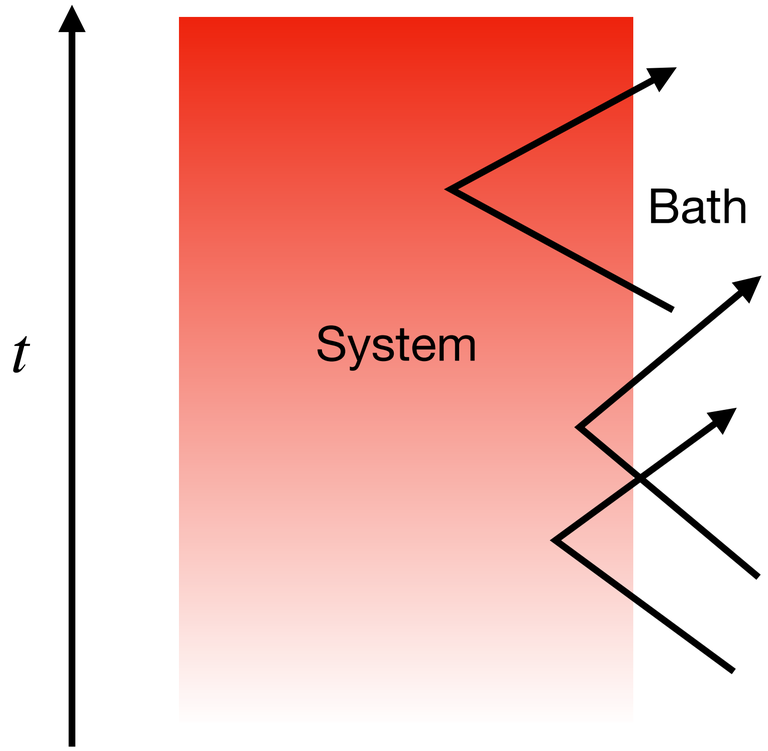
Classical-quantum state: $I(A':B) = \chi_B$

$$\chi_B > \chi_E \implies d \geq x/2$$

+ can distill private bits



When is entanglement important for understanding equilibrium?



$$\rho \sim e^{-H/T}$$

For $T > T_s = O(1)$, ρ is **separable** Bakshi+ 2403.16850

Result applies to spin systems with local interactions

Are (familiar) equilibrium correlations related to d ?

Signatures of entanglement in physical properties

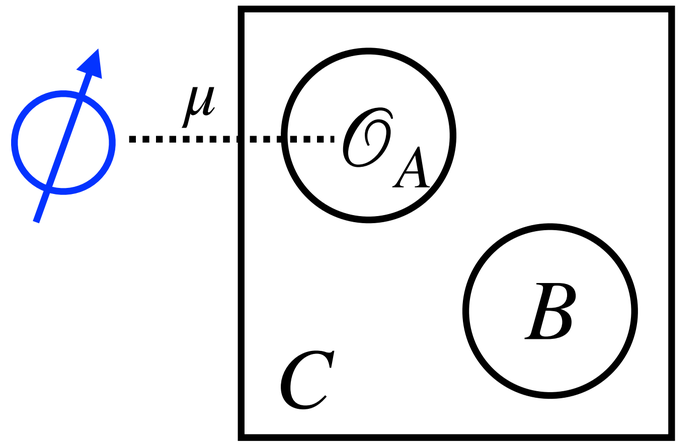
$T = 0$: entanglement and state complexity are related to spatial correlations

$T \neq 0$: entanglement and state complexity are related to *differences* between spatial and temporal correlations

Observed relaxation reveals minimum amount of dressing d

To relate observables to entanglement, consider measuring an observable

Couple a two-level system to $\mathcal{O}_A$, and measure the two-level system



Two outcomes $\quad m = \pm 1$

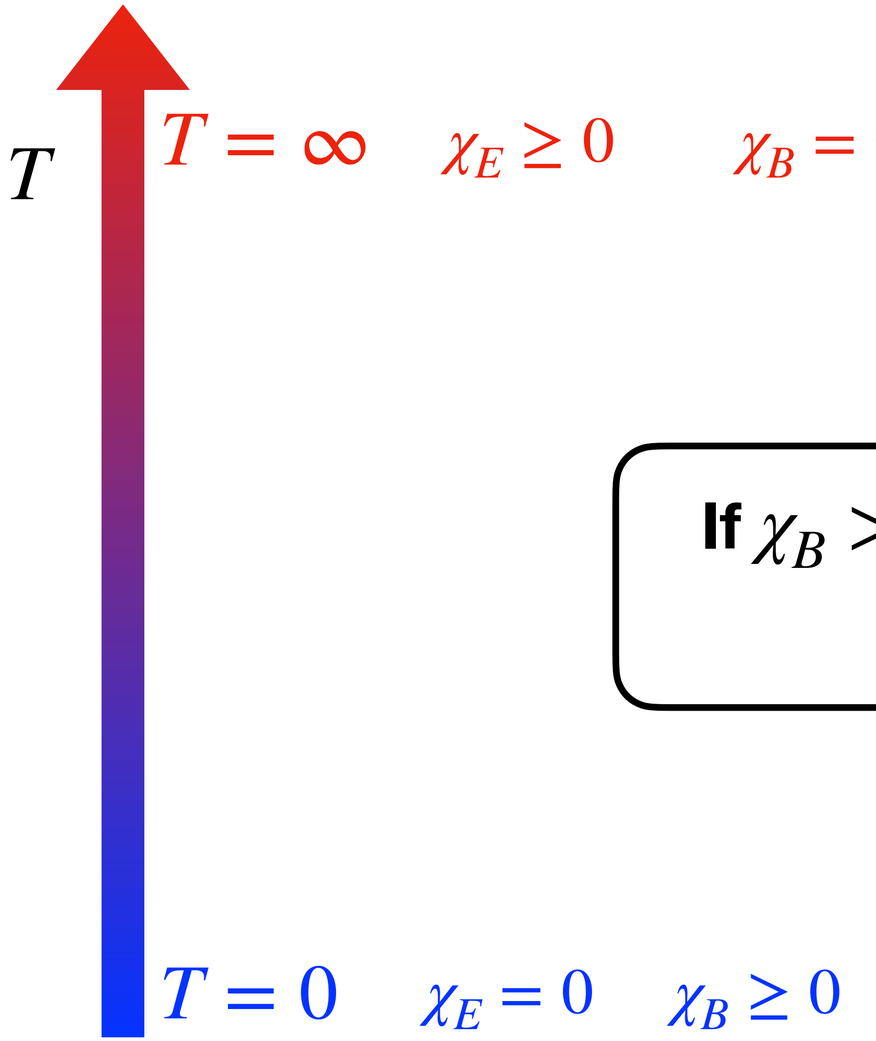
Probabilities $\quad p_m$

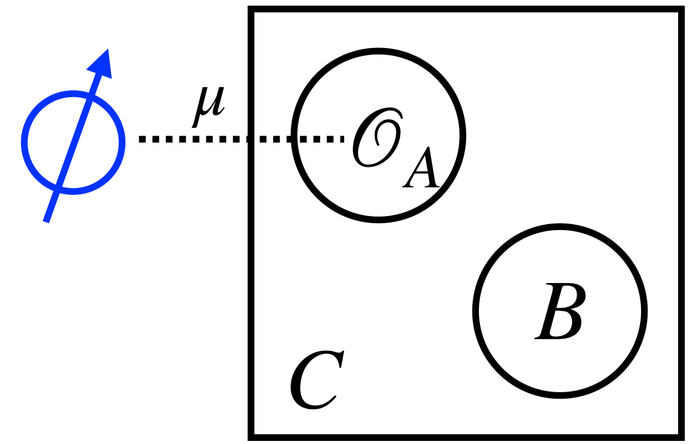
Post-measurement states of ABC $\quad \rho_m$

Post-measurement entropies of e.g. B $\quad S_m(B) = -\,\mathrm{Tr}\left[\rho_{m,B} \ln \rho_{m,B}\right]$

Information revealed to B $\quad \chi_B = S(B) - \sum_m p_m S_m(B)$

Effects of measurement on entropies changes with temperature





If $\chi_B > \chi_E$ then ABC entangled

depth $d \geq x/2$

Considering a 'weak' measurement, we find relations to standard correlations

$$\chi_B > \mu^2 \langle \mathcal{O}_A \mathcal{O}_B \rangle_{T,c}^2 + O(\mu^3)$$

Normalize $||\mathcal{O}_{A,B}||_\infty = 1$

… and, in thermal states, to *linear response!*

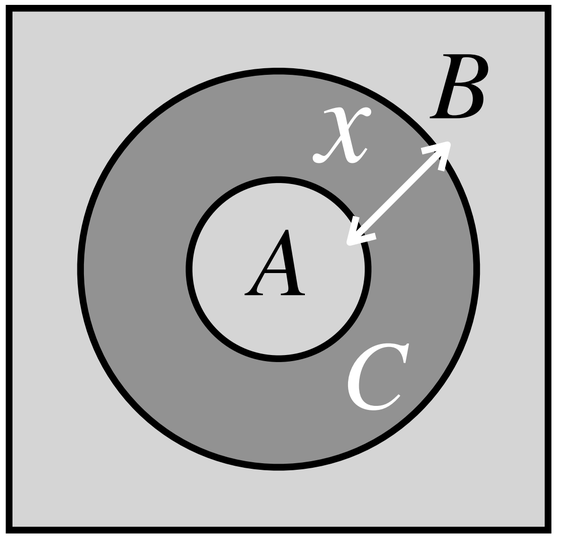
$$\chi_E = \mu^2 T \int_0^\infty dt\, f(Tt) \langle \mathcal{O}(t)\mathcal{O} \rangle_{T,c} = \mu^2 \int \frac{d\omega}{2\pi} \frac{\beta\omega}{e^{\beta\omega} - 1} s_{\mathcal{O}_A}(\omega, \beta)$$

Slow relaxation of $\langle \mathcal{O}(t)\mathcal{O} \rangle_{T,c}$ in time $\Longrightarrow$ large χ_E

Lower bounding the circuit depth does not require knowledge of H

See also Hauke et al. Nat. Phys. 2016: connection between QFI and linear response

Thermal state of quantum critical Hamiltonian



Two length scales: x and β with $x, \beta \gg 1$

$$\frac{I(A:B)}{I(A:E)} = f(x/\beta) \qquad f(x/\beta) > 1 \implies d > x/2$$

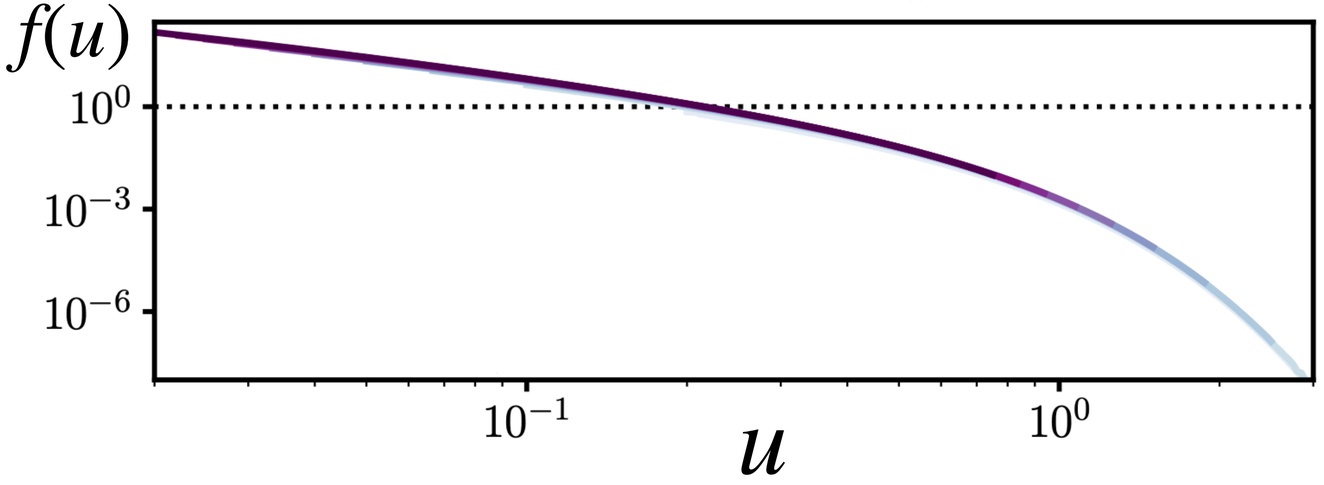
$$x \to \infty \implies I(A:B) \to 0$$

$$\beta \to \infty \implies I(A:E) \to 0$$

& due to data processing ineq.

$$\frac{df(u)}{du} \leq 0 \implies d = \Theta(\beta)$$

Transverse-field Ising chain



Strong symmetry implies entanglement at high temperatures

$T = \infty$ + strong non-Abelian symmetry: long-range entangled

Moharramipour, Lessa, Wang, Hsieh, Sahu 2406.08542

Li, Pollmann, Read, Sala 2406.08567

$T < T_{\mathrm{LRO}}$ + strong Abelian symmetry: long-range entangled

Lu, Zhang, Vijay, Hsieh 2303.15507

$T > T_s = O(1)$ and no symmetry: **exactly separable**

Bakshi, Liu, Moitra, Tang 2403.16850

$T > 0$ + strong Abelian symmetry: **non-separable** (for non-diagonal Hamiltonian)

SG, McGinley 2502.13218

+ Negari, Lessa, Sahu 2508.20166

General idea: strong symmetry obscures E's correlations with charged operators

SG, McGinley 2502.13218

If measure charged $\mathcal{O}$ then $\chi_E = 0 \implies$ any correlation with $\mathcal{O} \implies \chi_B > \chi_E$

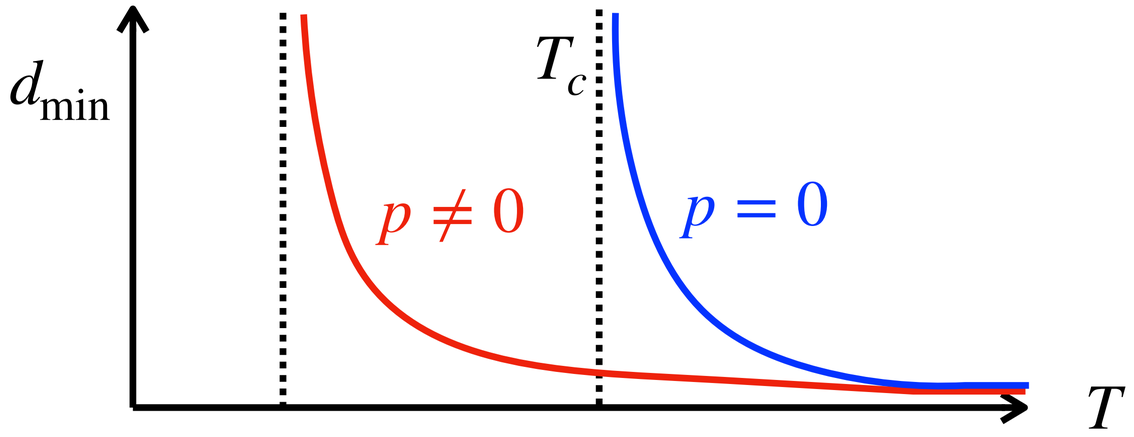
Global constraints and long-range entanglement

Is this long-range entanglement, which requires strong symmetry, **robust?** **Yes*!**

Standard thermal state of classical Ising model $\rho = \frac{1}{2}(\rho_+ + \rho_-)$ $\qquad d = 0$

Suppose we know $\langle \text{Parity} \rangle \neq 0 \implies \rho = p\rho_+ + (1-p)\rho_-$

If measure Z_A then $\chi_B > \chi_E$ at infinite range when $T < T_c - g(p)$



Inequivalence of quantum complexity in thermal ensembles

So far: canonical ensembles are more highly entangled than grand canonical

Now: microcanonical ensemble $\rho \propto \sum_{|E_n - \bar{E}| < W/2} |E_n\rangle\langle E_n|$

For width $W <$ Thouless energy, considering weak measurement, $\chi_E \sim \mu^2 W$

At thermal phase transition $\chi_B \sim \mu^2 x^{-\alpha} \implies d_{\min} \sim W^{-1/\alpha}$

In disordered phase $\chi_B \sim \mu^2 e^{-x/\xi} \implies d_{\min} \sim \log(1/W)$

These are just lower bounds: higher complexity near a thermal phase transition?

Summary

- Bounds relate mutual information to quantum state complexity.
- Definition of quantum complexity ‘mods out’ classical correlations.

$$I(A:B) > I(A:E) \implies d \geq x_{AB}/2 \quad \text{and CMI } I(A:B\,|\,C) > 0$$

- Connection to notion of private classical correlations in quantum key distillation

$$\chi_B > \chi_E \implies d \geq x_{AB}/2 \quad \text{and QKD at rate } K = \chi_B - \chi_E$$

- Complexity of thermal states related to real-time linear response.

$$\langle \mathcal{O}_A \mathcal{O}_B \rangle_c^2 > \int \frac{d\omega}{2\pi} \frac{\beta\omega}{e^{\beta\omega} - 1} s_{\mathcal{O}_A}(\omega, \beta) \implies d \geq x_{AB}/2$$

- Imposing global constraints generates high complexity states.

Outlook

- Extension to probes of mixed-state topological order (let’s discuss!)
- Renyi generalizations and probes of complexity via shadows
- Computable lower bounds on long-range magic?
- Nonequilibrium steady states with high complexity?

  Mechanisms to suppress fluctuations of symmetry charges?