清華大學物理系
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Quantum Phases and Phase Transitions in Open Quantum Systems

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Y. Guo, K. Ding, and S. Yang #, Rep. Prog. Phys. 88, 118001 (2025)

Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion

Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Y. Guo and S. Yang #, arXiv: 2602.21979

Outline

- imaginary-time Lindbladian evolution

  Y. Guo, K. Ding, and S. Yang #, Rep. Prog. Phys. 88, 118001 (2025)

  Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion

- locally purified density operator (LPDO) and purification perspective

  Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

  Y. Guo and S. Yang #, arXiv: 2602.21979

Part one

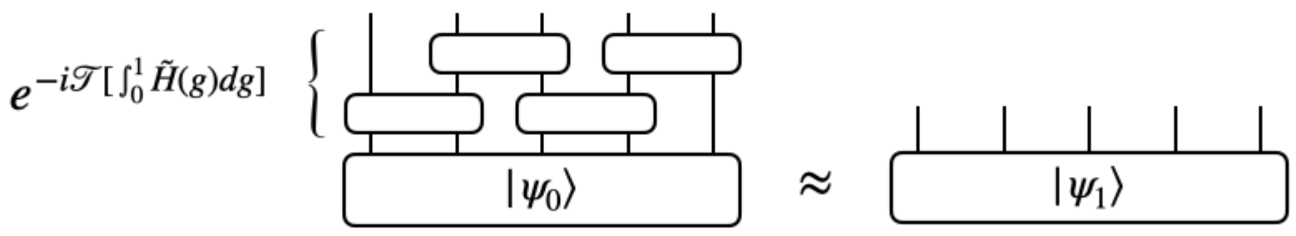
Imaginary-time Lindbladian evolution

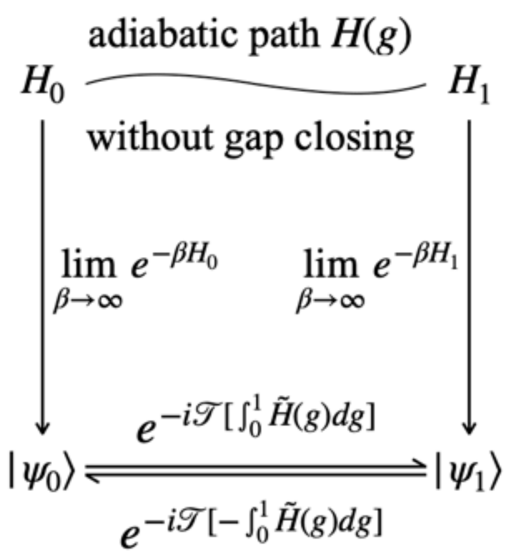
Y. Guo, K. Ding, and S. Yang #, Rep. Prog. Phys. 88, 118001 (2025)
Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion

Classification in closed systems

- equivalence relation for Hamiltonians

  two local, gapped Hamiltonians H_0 and H_1 belong to the same quantum phase if and only if there is a smooth path $H(g)$ connecting H_0 and H_1, the energy gap $\Delta(g)$ is always finite along the path





- equivalence relation for ground states

  two local, gapped Hamiltonians H_0 and H_1 belong to the same quantum phase if and only if their corresponding ground states $|\psi_0\rangle$ and $|\psi_1\rangle$ can be connected by finite-time local evolution $|\psi_1\rangle = e^{-i\mathcal{T}[\int_0^1 \tilde{H}(g)dg]}\,|\psi_0\rangle$ and $|\psi_0\rangle = e^{-i\mathcal{T}[-\int_0^1 \tilde{H}(g)dg]}\,|\psi_1\rangle$

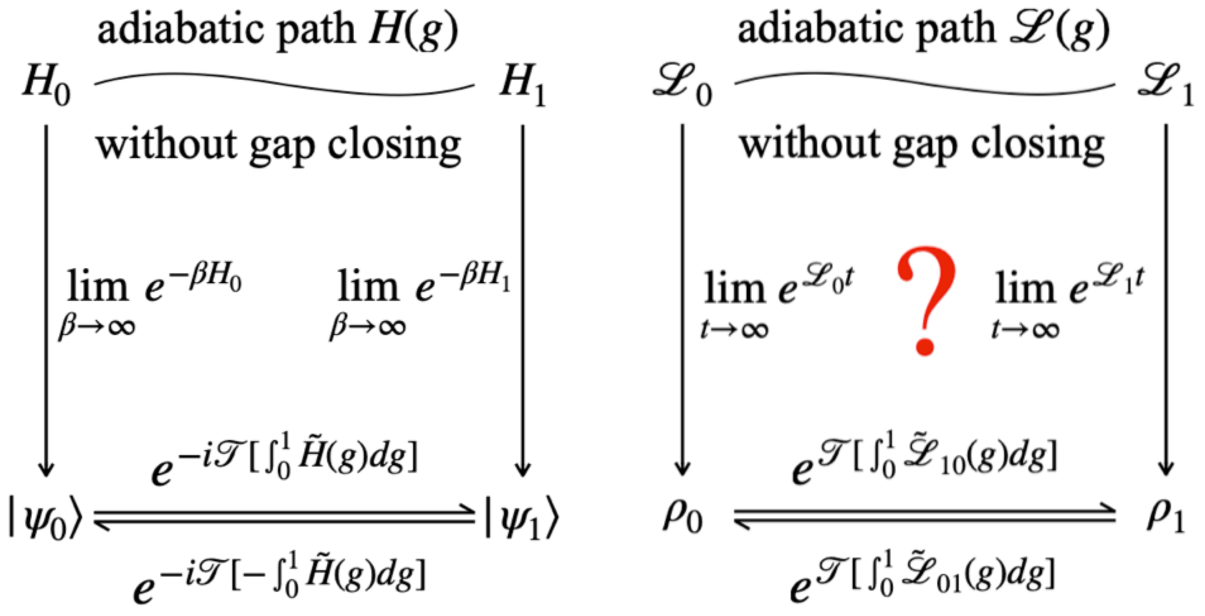
X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010)

Real-time Lindbladian evolution in open systems

- time evolution of a mixed state in an open system is described by the Lindbladian master equation
  $\frac{d\rho}{dt} = -i[H,\rho] + \sum_k \left[L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right]$
- density matrix $\rho = \sum_k \lambda_k |\psi_k\rangle\langle\psi_k| \Rightarrow$ supervector form $|\rho\rangle\rangle = \sum_k \lambda_k |\psi_k\rangle \otimes |\psi_k^*\rangle$
- Liouville supervector $\mathcal{L}|\rho\rangle\rangle = (-iH_{\text{eff}} \otimes I + iI \otimes H_{\text{eff}}^* + \sum_k L_k \otimes L_k^*)|\rho\rangle\rangle$
  $H_{\text{eff}} = H - \sum_k \frac{i}{2} L_k^\dagger L_k$
- the eigen-decomposition of Liouville superoperator $\mathcal{L}|\rho_i\rangle\rangle = \lambda_i|\rho_i\rangle\rangle$
  $\text{Re}(\lambda_i) \leq 0$, and $\text{Im}(\lambda_i) = 0$ if ρ_i is Hermitian
- the steady state is obtained by real-time Lindbladian evolution $\rho_{\text{ss}} = \lim_{t\to+\infty} e^{\mathcal{L}t}\rho(0)$, satisfying
  $\mathcal{L}|\rho_{\text{ss}}\rangle\rangle = 0$
- Liouville gap is $\Delta = -\text{Re}(\lambda_1)$, where λ_i is arranged as $0 = \lambda_0 \geq \text{Re}(\lambda_1) \geq \text{Re}(\lambda_2) \geq \ldots$

H.-P. Breuer and F. Petruccione, The Theory of Open Quantum Systems (Oxford University Press, 2007)

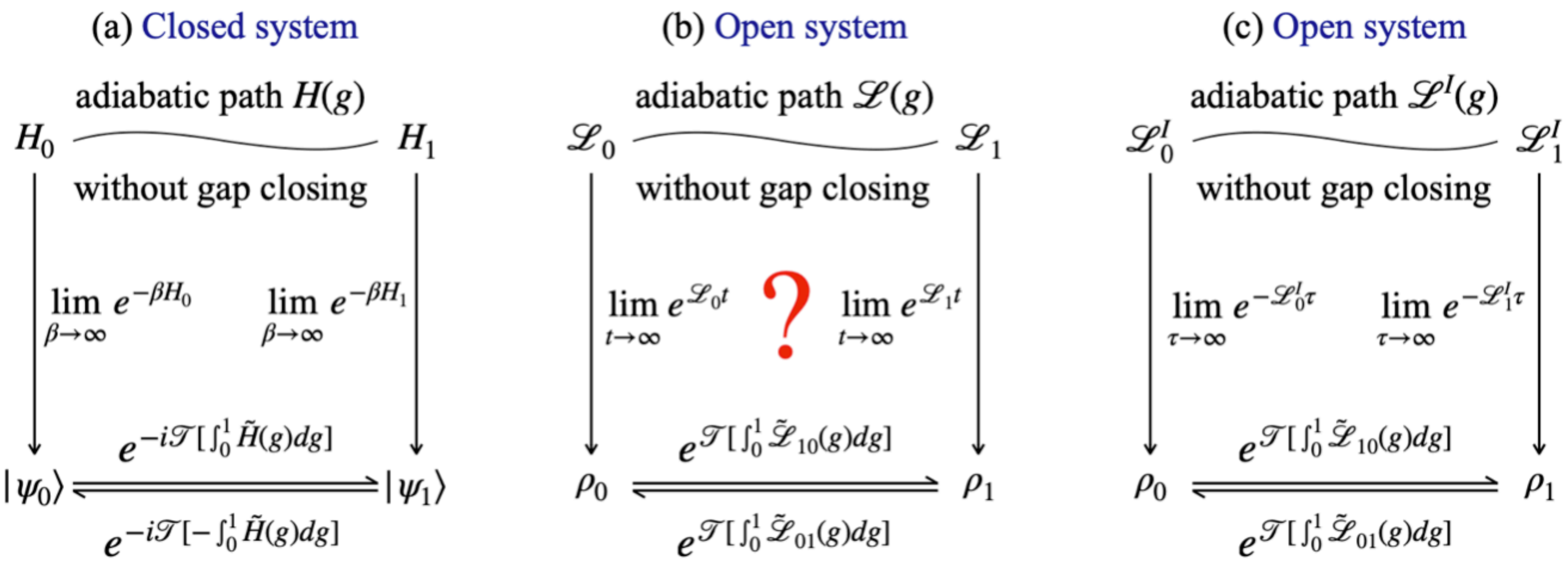
Paradox in open systems



- inconsistency when degrading to the pure state

1. take the jump operators $L_k = 0$, Lindbladian equation $\rightarrow$ Schrodinger equation
   all eigenstates of H become the steady states of $e^{\mathcal{L}dt}$ with zero eigenvalues $\rightarrow$ not degrade to ground state
2. although the Liouville gap may close along a path $\mathcal{L}(g)$ connecting $\mathcal{L}(0)$ and $\mathcal{L}(1)$
   it is still possible to connect $\rho(0)$ and $\rho(1)$ via finite-time Lindbladian evolutions $\tilde{\mathcal{L}}_{01} = \mathcal{L}(0), \tilde{\mathcal{L}}_{10} = \mathcal{L}(1)$
   because $\rho(i)$ can be achieved through $\mathcal{L}(i)$ from any initial states not orthogonal to it
   all $\rho(i)$ in the same phase $\rightarrow$ cannot distinguish between distinct phases

New framework: Imaginary-time Lindbladian evolution



- consider the time evolution for the entire system (system and environment) then trace out environment

real-time	$\frac{d\rho}{dt} = -i[H,\rho] + \sum_k \left[L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right] \equiv \mathscr{L}(\rho)$	$\mathscr{L}\|\rho\rangle\rangle = \left(-iH_{\text{eff}} \otimes I + iI \otimes H_{\text{eff}}^* + \sum_k L_k \otimes L_k^*\right)\|\rho\rangle\rangle$	$H_{\text{eff}} = H - \frac{i}{2}\sum_k L_k^\dagger L_k$
imaginary-time	$\frac{d\rho}{d\tau} = -\{H,\rho\} + \sum_k \left[L_k\rho L_k^\dagger + \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right] \equiv -\mathscr{L}^I(\rho)$	$\mathscr{L}^I\|\rho\rangle\rangle = \left(H_{\text{eff}}^I \otimes I + I \otimes H_{\text{eff}}^{I*} - \sum_k L_k \otimes L_k^*\right)\|\rho\rangle\rangle$	$H_{\text{eff}}^I = H - \frac{1}{2}\sum_k L_k^\dagger L_k$

- imaginary-Liouville superoperator $\mathcal{L}^I$

Classification in open systems

- equivalence relation for imaginary-Liouville superoperator

  two local, gapped open systems $\mathcal{L}^I(0)$ and $\mathcal{L}^I(1)$ belong to the same quantum phase if and only if there is a smooth path $\mathcal{L}^I(g)$, the imaginary-Liouville gap $\Delta^I(g)$ is always finite along the path

$$
\begin{tikzcd}
\mathscr{L}_0^I \arrow[r, dash, "\text{adiabatic path } \mathscr{L}^I(g)", "\text{without gap closing}"'] \arrow[d, "\lim\limits_{\tau\to\infty} e^{-\mathscr{L}_0^I\tau}"] & \mathscr{L}_1^I \arrow[d, "\lim\limits_{\tau\to\infty} e^{-\mathscr{L}_1^I\tau}"'] \\
\rho_0 \arrow[r, shift left, "e^{\mathcal{T}[\int_0^1 \tilde{\mathscr{L}}_{10}(g)dg]}"] & \rho_1 \arrow[l, shift left, "e^{\mathcal{T}[\int_0^1 \tilde{\mathscr{L}}_{01}(g)dg]}"]
\end{tikzcd}
$$

- equivalence relation for mixed states

  two local, gapped open systems $\mathcal{L}^I(0)$ and $\mathcal{L}^I(1)$ belong to the same quantum phase if and only if their corresponding steady states $\rho(0)$ and $\rho(1)$ can be connected by finite-time local Lindbladian evolutions $|\rho(0)\rangle\rangle = e^{\mathcal{T}[\int_0^1 \tilde{L}_{01}(g)dg]}\,|\rho(1)\rangle$ and $|\rho(1)\rangle\rangle = e^{\mathcal{T}[\int_0^1 \tilde{L}_{10}(g)dg]}\,|\rho(0)\rangle$

Phase transitions in open systems

- imaginary-Liouville superoperator $\mathcal{L}^I$

  $\frac{d\rho}{d\tau} = -\{H, \rho\} + \sum_k \left[L_k \rho L_k^\dagger + \frac{1}{2}\{L_k^\dagger L_k, \rho\} \right] \equiv -\mathcal{L}^I(\rho)$

- diagonalize $\mathcal{L}^I$, $\mathcal{L}^I(\rho) = E\rho$

  sort the spectrum according to real parts, $\mathrm{Re}(E_0) \leq \mathrm{Re}(E_1) \leq \mathrm{Re}(E_2) \leq \dots$

  the imaginary-time Lindbladian evolution has a steady state if and only if E_0 is real

- imaginary-Liouville gap $\Delta^I = \mathrm{Re}(E_1) - E_0$

  Δ^I gap closing $\Rightarrow$ phase transition in open system

  naturally degrades to closed systems

Phase diagram

- four imaginary-Liouvilles $\mathcal{L}^I$ in an open system with $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\tau$ symmetry

phase (α,β)	Hamiltonian	jump operator	quantum state
trivial pure (0,0)	$H^{00} = -\sum_i \tau^x_{i+1/2} - \sum_i \sigma^x_i$	$L^{00}=0$	$\vert\cdots \rightarrow_\sigma \rightarrow_\tau \rightarrow_\sigma \rightarrow_\tau \cdots\rangle$
SPT (0,1)	$H^{01} = -\sum_i \sigma^z_i \tau^x_{i+1/2}\sigma^z_{i+1} - \sum_i \tau^z_{i-1/2}\sigma^x_i\tau^z_{i+1/2}$	$L^{01}=0$	$\frac{1}{2^{N/2}}\sum_{\{\sigma_i\}} \vert\cdots \uparrow_\sigma \rightarrow_\tau \uparrow_\sigma \leftarrow_\tau \downarrow_\sigma \cdots\rangle \equiv \frac{1}{2^{N/2}}\sum_{\{\sigma_i\}} \vert\psi^{\mathrm{SPT}}_{\{\sigma_i\}}\rangle$
trivial mixed (1,0)	$H^{10} = -\sum_i \tau^x_{i+1/2}$	$L_i^{[1]10} = \sigma^z_i,\ L_i^{[2]10} = \sigma^x_i$	$\cdots\left[\frac{1}{2}(\vert\uparrow\rangle\langle\uparrow\vert + \vert\downarrow\rangle\langle\downarrow\vert)_\sigma\right]\otimes\vert\rightarrow\rangle\langle\rightarrow\vert_\tau]\cdots$
ASPT (1,1)	$H^{11} = -\sum_i \sigma^z_i\tau^x_{i+1/2}\sigma^z_{i+1}$	$L_i^{[1]11} = \sigma^z_i,\ L_i^{[2]11} = \tau^z_{i-1/2}\sigma^x_i\tau^z_{i+1/2}$	$\frac{1}{2^N}\sum_{\{\sigma_i\}} \vert\psi^{\mathrm{SPT}}_{\{\sigma_i\}}\rangle\langle\psi^{\mathrm{SPT}}_{\{\sigma_i\}}\vert$

- $(\rho_{00},\rho_{01},\rho_{10},\rho_{11}) \Rightarrow (H^{00},H^{01},H^{10},H^{11}),(L^{00},L^{01},L^{10},L^{11}) \Rightarrow (\mathcal{L}^I_{00},\mathcal{L}^I_{01},\mathcal{L}^I_{10},\mathcal{L}^I_{11})$
- linearly interpolation between these four systems, calculate the ground state of $\mathcal{L}^I(\alpha,\beta)$

$$\mathcal{L}^I(\alpha,\beta) = (1-\alpha)(1-\beta)\mathcal{L}^I_{00} + (1-\alpha)\beta\mathcal{L}^I_{01} + \alpha(1-\beta)\mathcal{L}^I_{10} + \alpha\beta\mathcal{L}^I_{11}, \quad \mathcal{L}^I(\alpha,\beta) \Rightarrow \rho_{\alpha,\beta}$$

SPT phase: cluster state (Haldane phase)

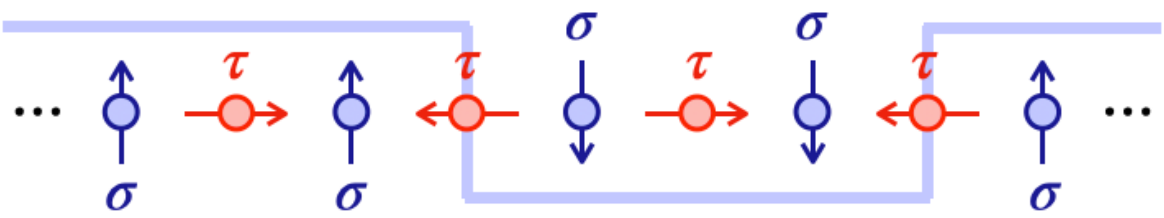
- cluster state in the closed system (Haldane phase)

  $$H = -\sum_i \left(\sigma_i^z \tau_{i+1/2}^x \sigma_{i+1}^z + \tau_{i-1/2}^z \sigma_i^x \tau_{i+1/2}^z\right)$$

- two global $\mathbb{Z}_2$ symmetries $U = \prod_i \sigma_i^x$ and $K = \prod_i \tau_{i+1/2}^x$

- ground state follows the decorated domain wall construction

  $$|\psi^{\mathrm{SPT}}\rangle = \frac{1}{2^{N/2}} \sum_{\{\sigma_i\}} |\cdots \uparrow_\sigma \rightarrow_\tau \uparrow_\sigma \leftarrow_\tau \downarrow_\sigma \rightarrow_\tau \downarrow_\sigma \leftarrow_\tau \uparrow_\sigma \ldots\rangle \equiv \frac{1}{2^{N/2}} \sum_{\{\sigma_i\}} |\psi_{\{\sigma_i\}}^{\mathrm{SPT}}\rangle$$



- excitations of τ spins are placed at the domain wall of σ spins

  all components $|\psi_{\{\sigma_i\}}^{\mathrm{SPT}}\rangle$ span the degenerate ground states manifold of $H_1 \equiv -\sum_i \sigma_i^z \tau_{i+1/2}^x \sigma_{i+1}^z$

  $H_2 \equiv -\sum_i \tau_{i-1/2}^z \sigma_i^x \tau_{i+1/2}^z$ introduces a map between different components

  X. Chen, Y.-M. Lu, & A. Vishwanath, Nat. Commun. 5, 3507 (2014)

Average symmetry-protected topological (ASPT) phase

- decohered Haldane phase

  one of the $\mathbb{Z}_2$ symmetry is broken to weak symmetry due to decoherence

  density matrix is constructed as $\rho^{\mathrm{ASPT}} = \frac{1}{2^N}\sum_{\{\sigma_i\}} |\psi^{\mathrm{SPT}}_{\{\sigma_i\}}\rangle\langle\psi^{\mathrm{SPT}}_{\{\sigma_i\}}|$

  where $|\psi^{\mathrm{SPT}}_{\{\sigma_i\}}\rangle = |\cdots\uparrow_\sigma\rightarrow_\tau\uparrow_\sigma\leftarrow_\tau\downarrow_\sigma\rightarrow_\tau\downarrow_\sigma\leftarrow_\tau\uparrow_\sigma\cdots\rangle$

- $K = \prod_i \tau^x_{i+1/2}$ remains a strong symmetry , $K\rho = e^{i\theta}\rho$

  $U = \prod_i \sigma^x_i$ becomes a weak symmetry , $U\rho \neq e^{i\theta}\rho$ but $U\rho U^\dagger = \rho$

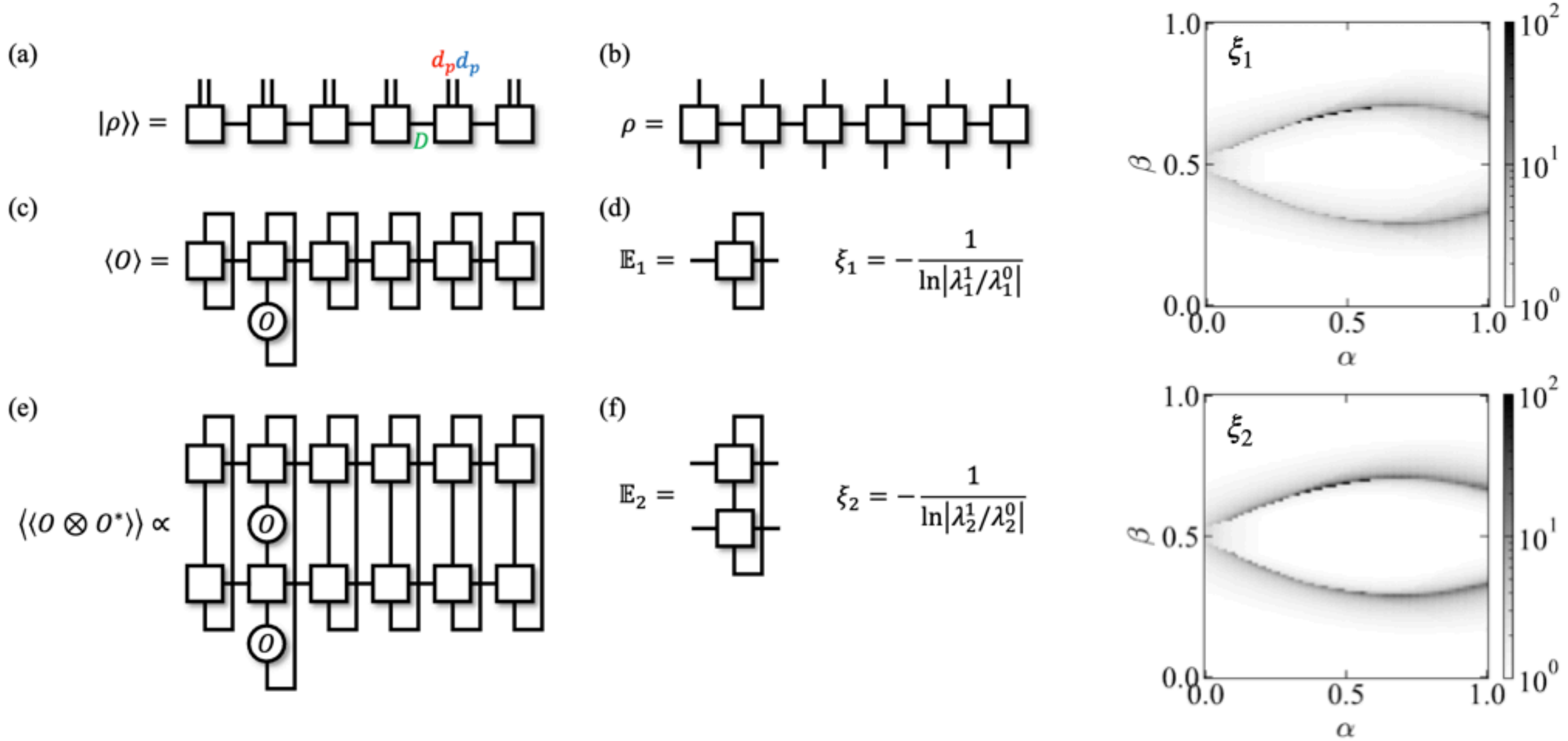
  nontrivial topological properties jointly protected by strong and weak symmetries

R. Ma and C. Wang, Phys. Rev. X 13, 031016 (2023)

Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Correlation lengths

- expectation value over the density matrix $\langle O \rangle \equiv \mathrm{Tr}[\rho O]$, detect strong symmetry
  expectation value over the supervector $\langle\langle O \otimes O^* \rangle\rangle = \frac{\langle\langle \rho | O \otimes O^* | \rho \rangle\rangle}{\langle\langle \rho | \rho \rangle\rangle} = \frac{\mathrm{Tr}[\rho O \rho O^\dagger]}{\mathrm{Tr}[\rho^2]}$, detect weak symmetry
- linear correlation function $C^{(1)}(i,j) \equiv \langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle \sim e^{-|i-j|/\xi_1}$
  Renyi-2 correlation function $C^{(2)}(i,j) \equiv \langle\langle O_i O_j \otimes O_i^* O_j^* \rangle\rangle - \langle\langle O_i \otimes O_i^* \rangle\rangle \langle\langle O_j \otimes O_j^* \rangle\rangle \sim e^{-|i-j|/\xi_2}$



Symmetry properties

- strong symmetry indicator $|\langle K\rangle| = |\,\mathrm{Tr}(\rho\prod_i \tau^x_{i+1/2})| = 1$
  the steady state always preserves the Z_2^τ symmetry

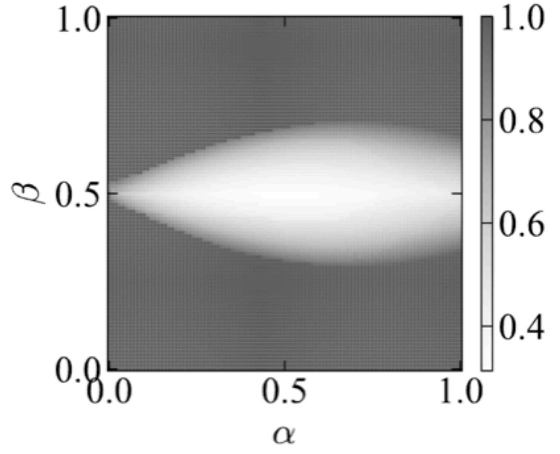
- weak symmetry indicator $\langle\langle U\otimes U^*\rangle\rangle = \frac{\mathrm{Tr}[\rho(\prod_i \sigma_i^x)\rho(\prod_i \sigma_i^x)]}{\mathrm{Tr}[\rho^2]}$

- string order $\langle O_{\mathrm{str}}\rangle \equiv \lim_{|i-j|\to\infty}\langle \sigma_i^z \tau^x_{i+1/2}\tau^x_{i+3/2}\cdots\tau^x_{j-3/2}\tau^x_{j-1/2}\sigma_j^z\rangle$

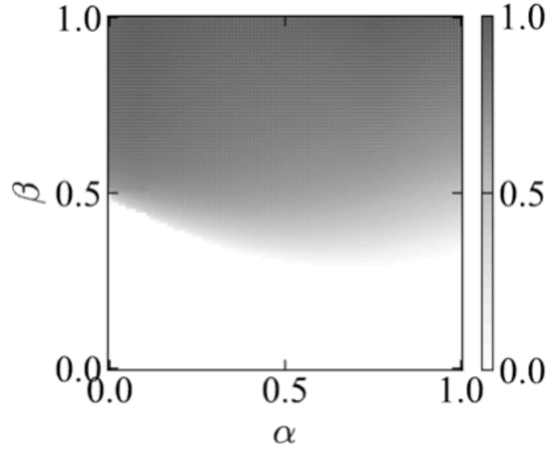
$$\langle O_{\mathrm{str}}\rangle \sim \begin{cases} O(1), & \text{domain wall decoration (ASPT)} \\ \langle \sigma_i^z\sigma_j^z\rangle, & \text{trivial decoration} \sim \begin{cases} O(1), & \mathbb{Z}_2^\sigma \text{ SSB} \\ 0, & \mathbb{Z}_2^\sigma \text{ trivial} \end{cases} \end{cases}$$

- imaginary Liouville gap $\Delta^I(\alpha, 1-\beta) = \Delta^I(\alpha,\beta)$

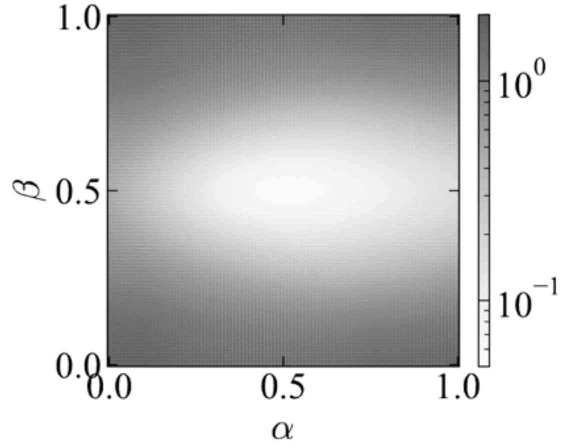
symmetry indicator of Z_2^σ



string order $\langle O_{\mathrm{str}}\rangle$

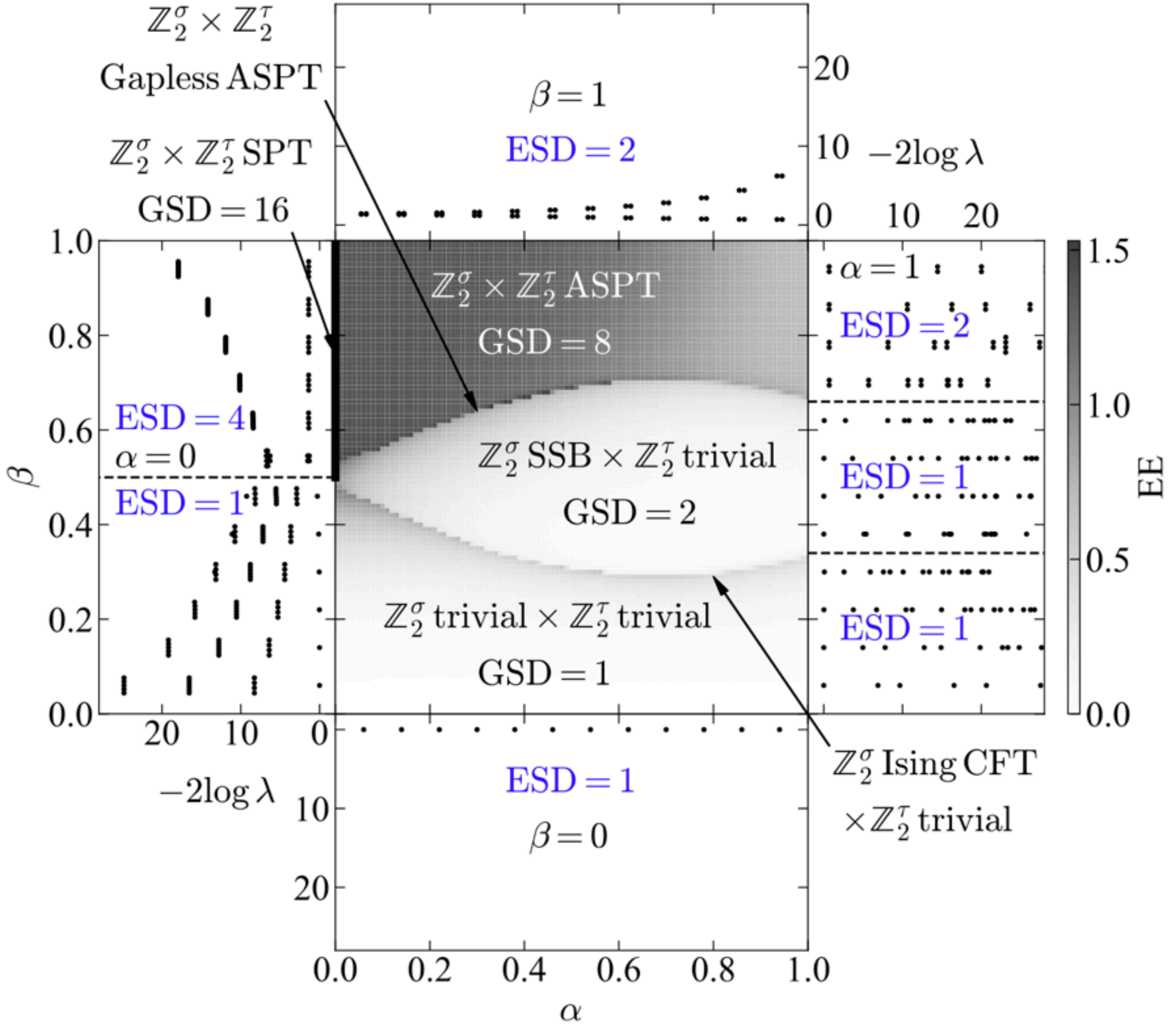


imaginary Liouville gap Δ^I



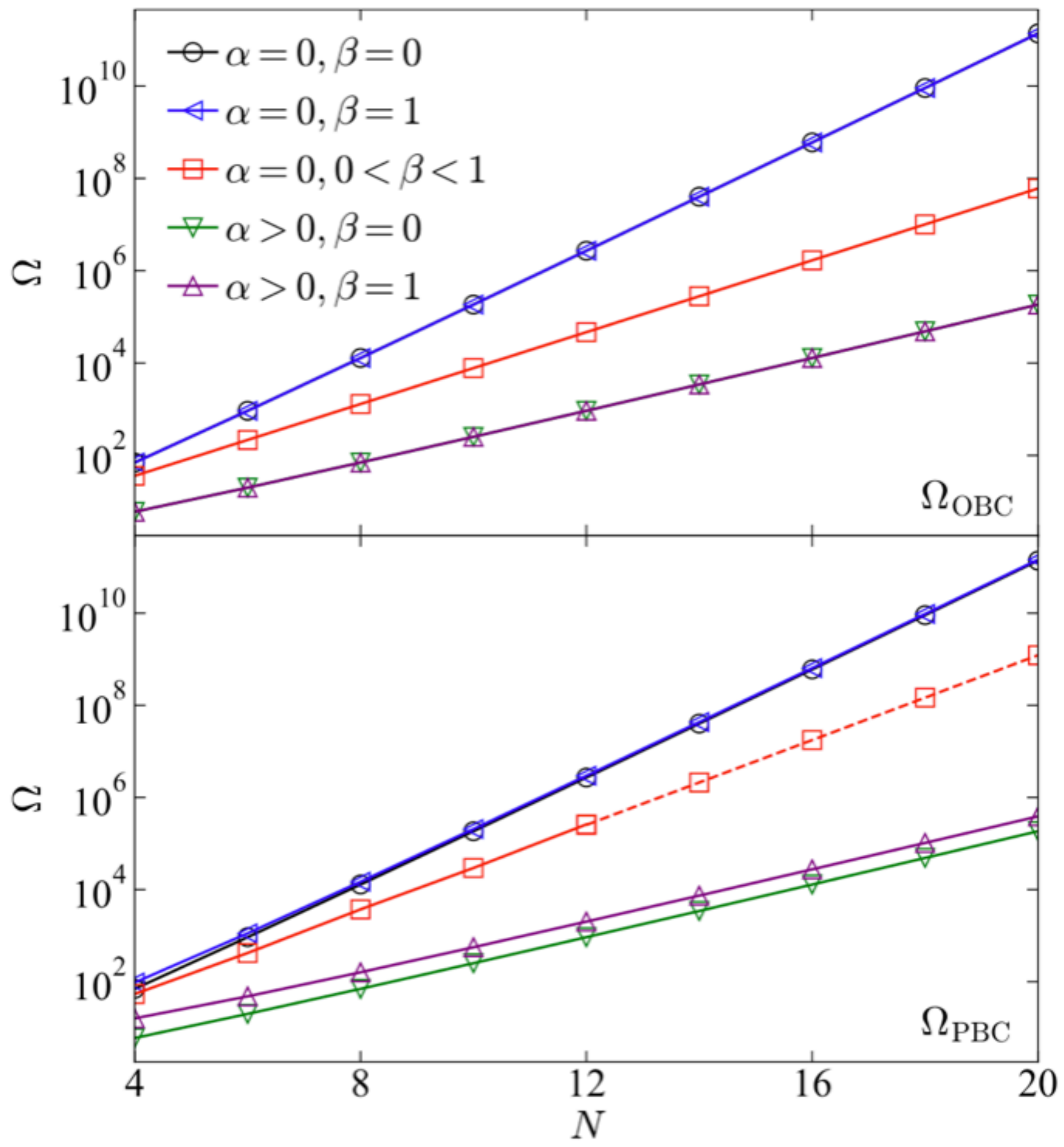
Phase diagram

- entanglement entropy and entanglement spectrum for the supervector $|\rho\rangle\rangle$
- entanglement spectrum degeneracy (ESD)
  4-fold degeneracy for SPT, 2-fold for ASPT
- ground state degeneracy (GSD) under OBC, $N = 6$
  2-fold degeneracy for SSB
  8-fold for ASPT
  totally 3 independent edge modes, $\mathrm{GSD} = 2^3 = 8$
- duality map $U_{\mathrm{DW}} \equiv \prod_i \mathrm{CZ}^{\tau,\sigma}_{i-1/2,i}\mathrm{CZ}^{\sigma,\tau}_{i,i+1/2}$
  gapless SPT $\leftrightarrow$ conventional Ising criticality
  trivial product state $\leftrightarrow$ cluster state
  phase diagram has symmetry $\beta \leftrightarrow 1-\beta$
  $U_{\mathrm{DW}} \otimes U^*_{\mathrm{DW}} \mathcal{L}^I(\alpha,\beta) = \mathcal{L}^I(\alpha, 1-\beta) U_{\mathrm{DW}} \otimes U^*_{\mathrm{DW}}$



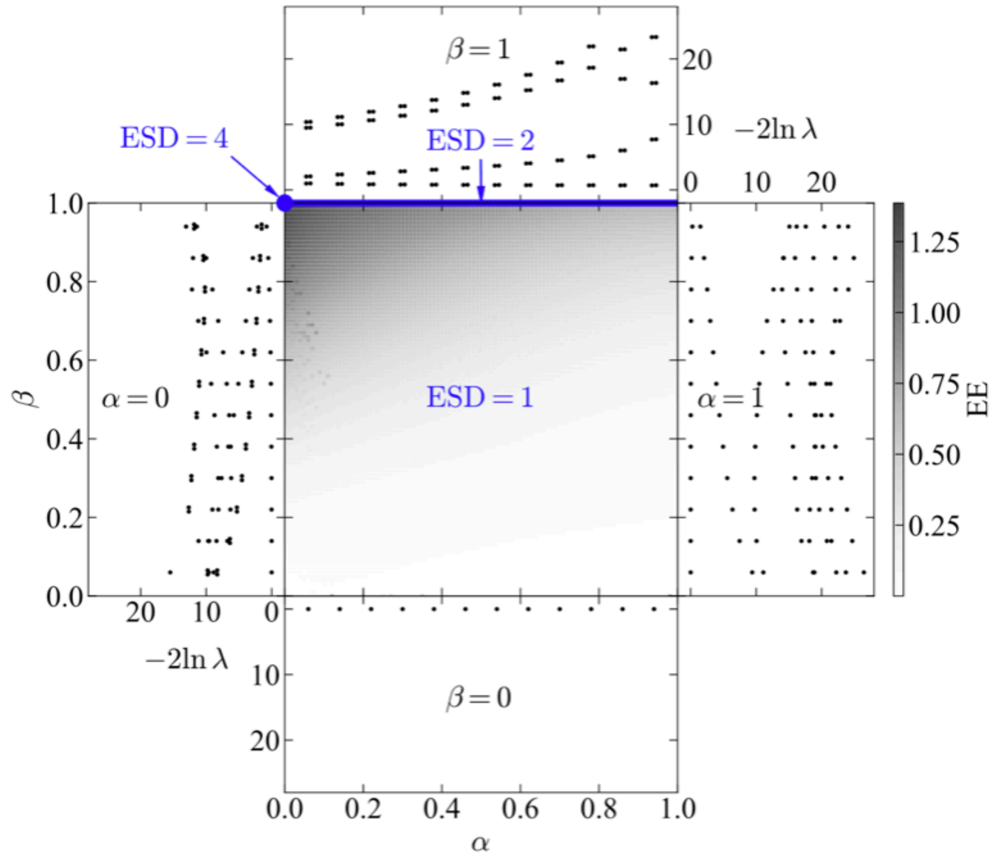
Comparison with real-time Lindbladian evolution (1)

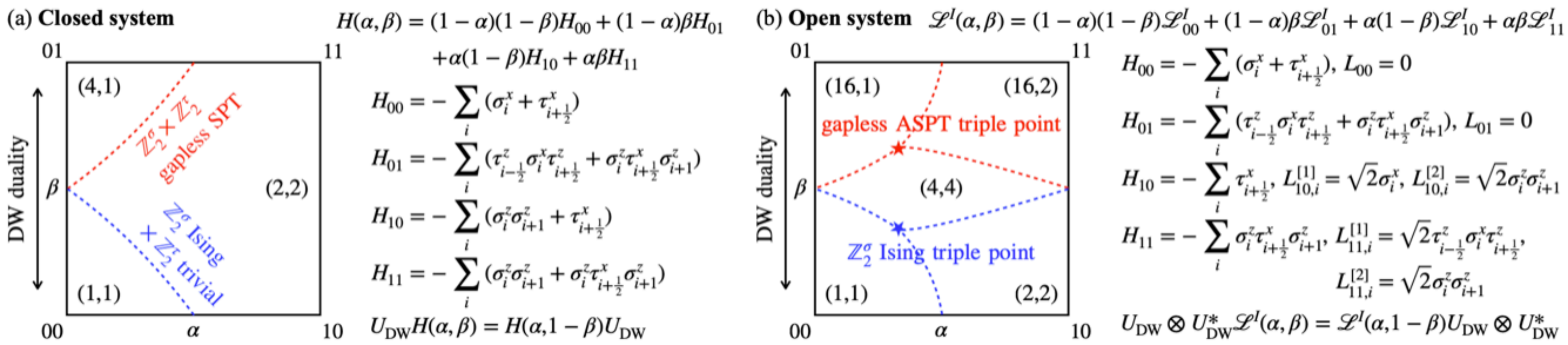
- method 1: use the same set of H and L_k
- in the pure-state limit, degrades to the Schroedinger equation
- all eigenstates of H are steady states of $e^{\mathcal{L}t}$
  $\Rightarrow$ highly degenerate steady-state subspace
- unable to characterize the phase transition between trivial state and SPT state
- this degeneracy survives when stepping into the mixed state regime

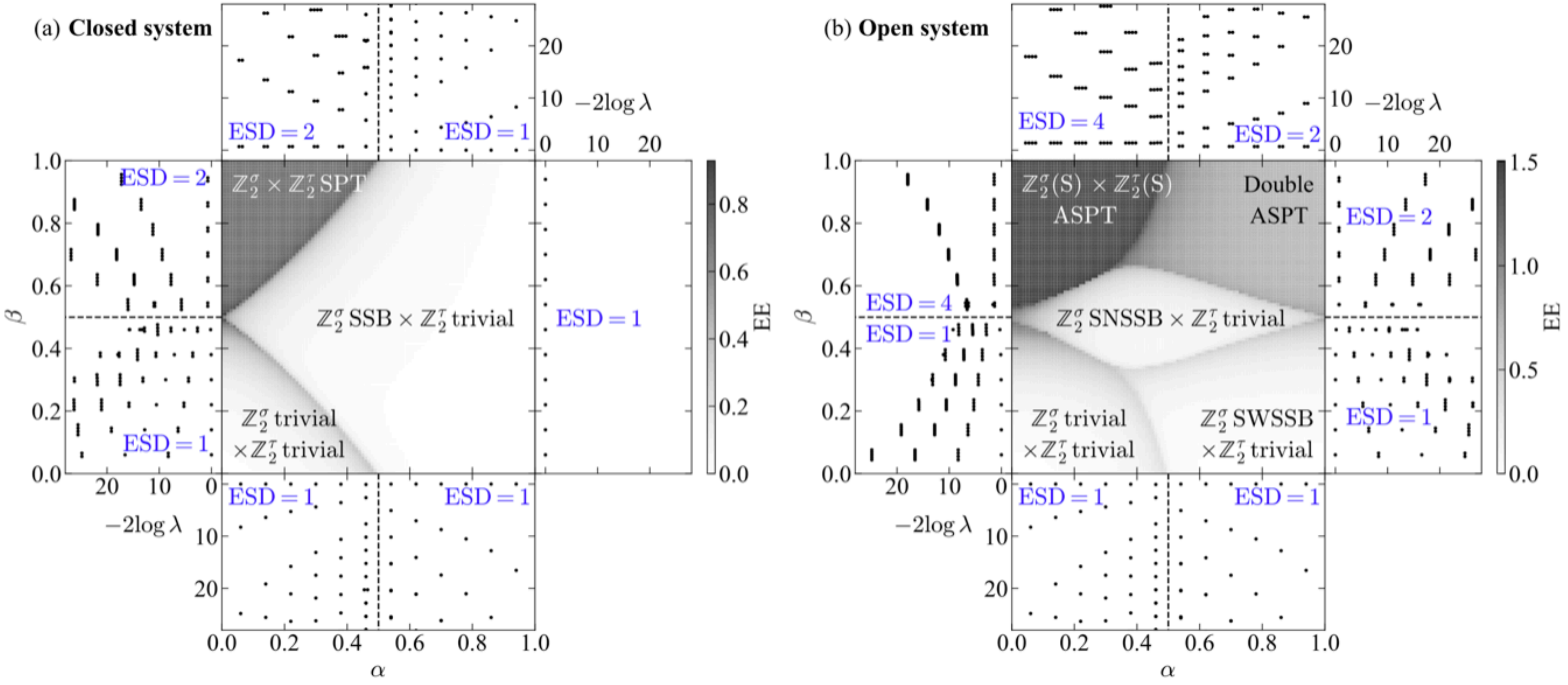


Comparison with real-time Lindbladian evolution (2)

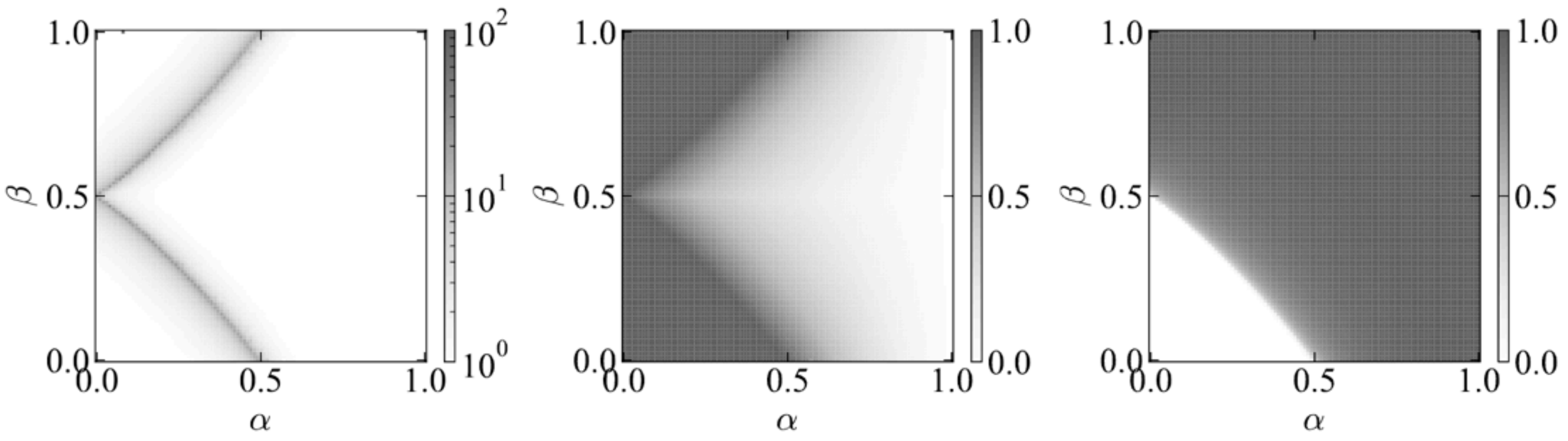
- method 2: use the same steady-state density matrix to construct real-time Lindbladian $\Rightarrow$ different H and L_k
- $\mathcal{L}$ for four corner states
- trivial pure: $L_i^{[1]00} = \sigma_i^z \frac{1-\sigma_i^x}{2} \sigma_{i+1}^z, L_i^{[2]00} = \tau_{i-1/2}^z \frac{1-\tau_{i-1/2}^x}{2} \tau_{i+1/2}^z$
- SPT: $L_i^{[1]01} = \sigma_i^z \frac{1-\tau_{i-1/2}^z \sigma_i^x \tau_{i+1/2}^z}{2} \sigma_{i+1}^z, L_i^{[2]01} = \tau_{i-1/2}^z \frac{1-\sigma_{i-1}^z \tau_{i-1/2}^x \sigma_i^z}{2} \tau_{i+1/2}^z$
- trivial mixed: $L_i^{[1]10} = \sigma_i^z, L_i^{[2]10} = \sigma_i^x, L_i^{[3]10} = \tau_{i-1/2}^z \frac{1-\tau_{i-1/2}^x}{2} \tau_{i+1/2}^z$
- ASPT: $L_i^{[1]11} = \sigma_i^z$, $L_i^{[2]11} = \tau_{i-1/2}^z \sigma_i^x \tau_{i+1/2}^z$,
  $L_i^{[3]11} = \tau_{i-1/2}^z \frac{1-\sigma_{i-1}^z \tau_{i-1/2}^x \sigma_i^z}{2} \tau_{i+1/2}^z$.
- entanglement entropy and entanglement spectrum are smoothly connected between different phases $\Rightarrow$ real-time Lindbladian formalism fails to capture the essential phase transitions





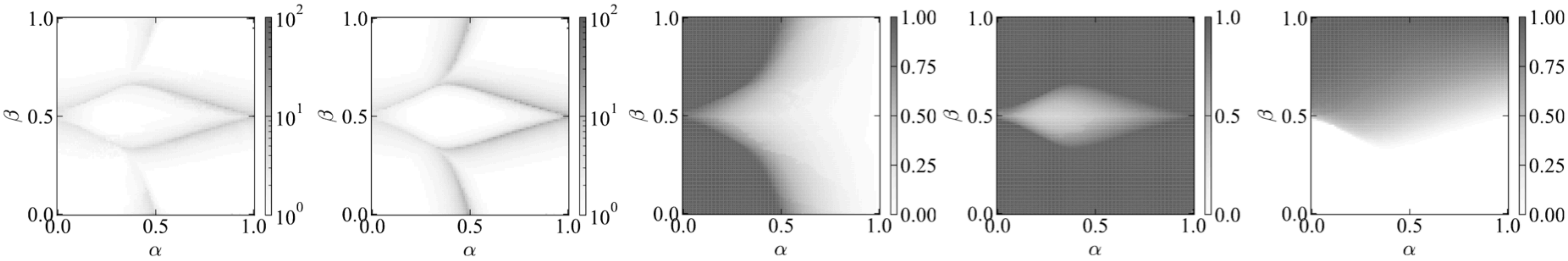


Strong-to-weak SSB meets ASPT order, Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion



(a) Linear corr-length ξ_1 (b) Symmetry indicator of $\mathbb{Z}_2^\sigma$ (c) String order $\langle O_{\mathrm{str}} \rangle$

Phase	Edge modes	SSB	$\mathrm{GSD}_{\mathrm{OBC}}$	$\mathrm{GSD}_{\mathrm{PBC}}$
$\mathbb{Z}_2^\sigma$ trivial $\times$ $\mathbb{Z}_2^\tau$ trivial	1	I	1	1
$\mathbb{Z}_2^\sigma$ SWSSB $\times$ $\mathbb{Z}_2^\tau$ trivial	1	$\mathbb{Z}_2^\sigma$	2	2
$\mathbb{Z}_2^\sigma$ SNSSB $\times$ $\mathbb{Z}_2^\tau$ trivial	1	$\left(\mathbb{Z}_2^\sigma\right)^{\otimes 2}$	4	4
$\mathbb{Z}_2^\sigma$ (S) $\times$ $\mathbb{Z}_2^\tau$ (S) ASPT	$2^4 = 16$	I	16	1
Double ASPT	$2^3 = 8$	$\mathbb{Z}_2^\sigma$	16	2

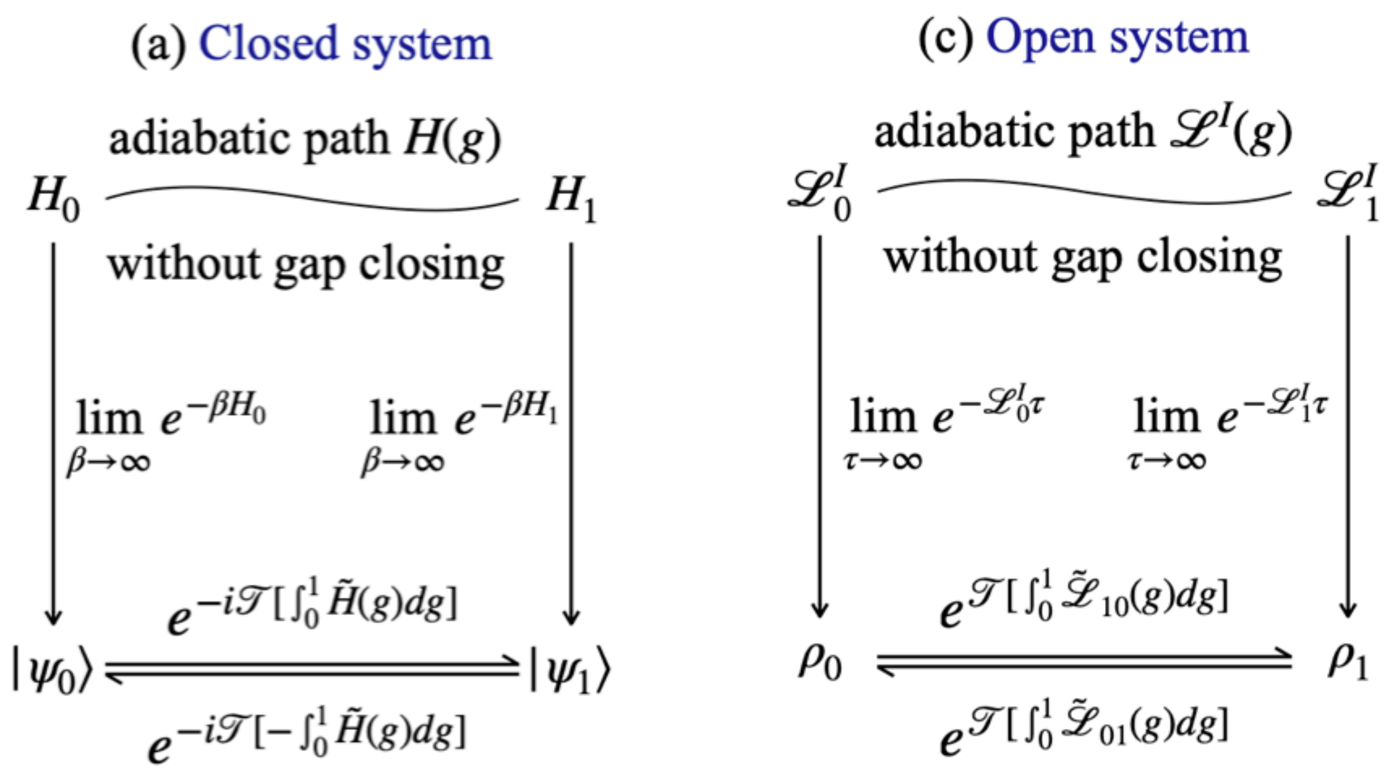


(a) Linear corr-length ξ_1 (b) Rényi-2 corr-length ξ_2 (c) Symmetry indicator of strong $\mathbb{Z}_2^\sigma$ (d) Symmetry indicator of weak $\mathbb{Z}_2^\sigma$ (e) String order $\langle O_{\mathrm{str}} \rangle$

Strong-to-weak SSB meets ASPT order, Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion

Summary of part one

- new framework to understand quantum phases and phase transitions in open systems
- imaginary-Liouville superoperator $\mathcal{L}^I \rightarrow$ imaginary-time Lindbladian evolution $e^{-\mathcal{L}^I\tau} \rightarrow$ steady state ρ
- completing the last piece of the puzzle for open-system quantum phases, open a new avenue



Y. Guo, K. Ding, and S. Yang #, Rep. Prog. Phys. 88, 118001 (2025)

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Part two

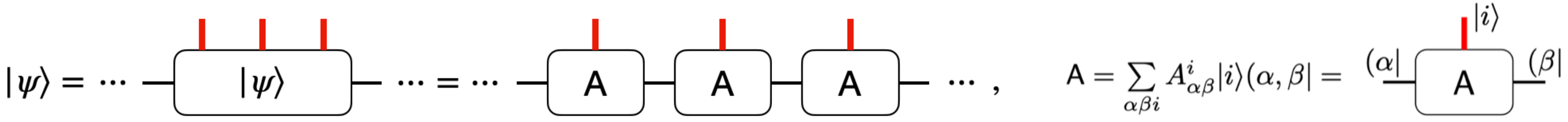
Locally purified density operator and purification perspective

Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Y. Guo and S. Yang #, arXiv: 2602.21979

Matrix product states for SPT phases

- $(1+1)D$ system, $|\psi\rangle = \sum_{\{i_j\}} \mathrm{Tr}\left(A^{i_1}\cdots A^{i_N}\right)|i_1 \ldots i_N\rangle$



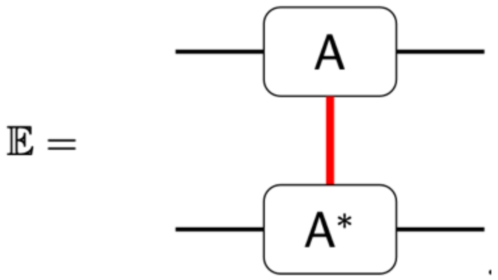
- injective MPS , $A^i_{\alpha\beta} : (\mathbb{C}^D)^{\otimes 2} \to (\mathbb{C}^{d_p})^{\otimes L}$ forms an injective map

  this implies that the physical space of the local tensor can span the entire virtual space

- an injective MPS characterizes a short-range correlated state

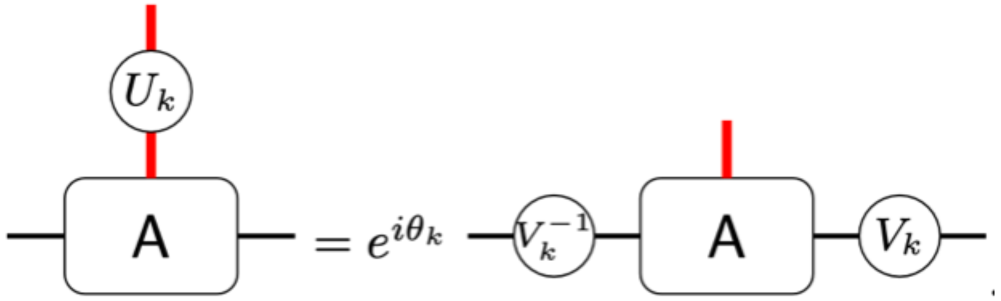
  $C(i,j) \equiv \langle\psi|O_iO_j|\psi\rangle - \langle\psi|O_i|\psi\rangle\langle\psi|O_j|\psi\rangle \sim e^{-|i-j|/\xi}$

- equivalent definition, transfer matrix $\mathbb{E}$ has a unique dominant eigenvalue



Projective representation on virtual indices

- impose symmetry constraints on short-range correlated states $\rightarrow$ SPT states
- symmetry group K , the onsite symmetry action on the physical index forms a linear representation
  $U(k_1)U(k_2) = U(k_1 k_2), \quad k_1, k_2 \in K$
- the symmetry action on the physical index induces a gauge transformation $V(k)$ on the virtual indices



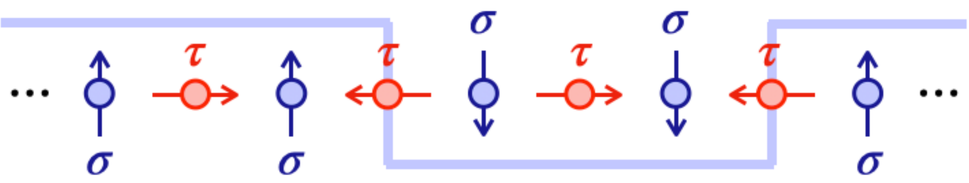
- sequentially applying $U(k_2)$ and $U(k_1)$, the resulting tensor should be identical to that for $U(k_1 k_2)$
  the transformations on the virtual index should be equivalent up to a $U(1)$ phase
  $V(k_1)V(k_2) = \mu_2(k_1, k_2)V(k_1 k_2), \quad \mu_2(k_1, k_2) \in U(1)$
  projective representation of the symmetry K

X. Chen, Z.-C. Gu, Z.-X. Liu, and X.-G. Wen, Phys. Rev. B 87, 155114 (2013)

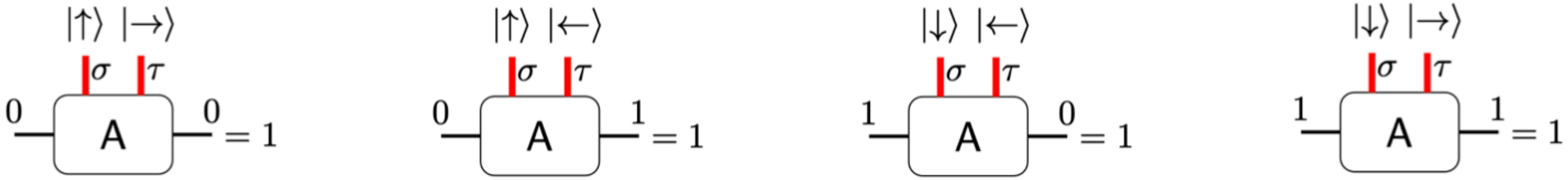
SPT phase: cluster state (Haldane phase)

- decorated domain wall construction

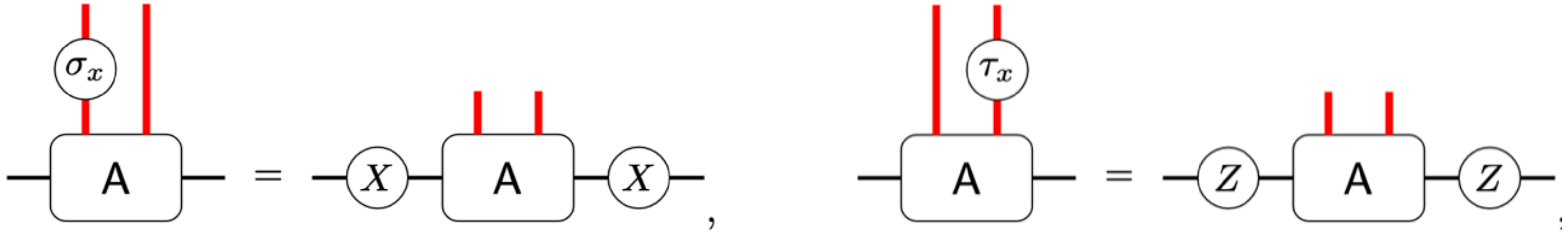
$$|\psi^{\mathrm{SPT}}\rangle = \frac{1}{2^{N/2}} \sum_{\{\sigma_i\}} |\cdots \uparrow_\sigma \rightarrow_\tau \uparrow_\sigma \leftarrow_\tau \downarrow_\sigma \rightarrow_\tau \downarrow_\sigma \leftarrow_\tau \uparrow_\sigma \cdots\rangle \equiv \frac{1}{2^{N/2}} \sum_{\{\sigma_i\}} |\psi^{\mathrm{SPT}}_{\{\sigma_i\}}\rangle$$



- cluster state can be represented by an injective MPS with $D = 2$



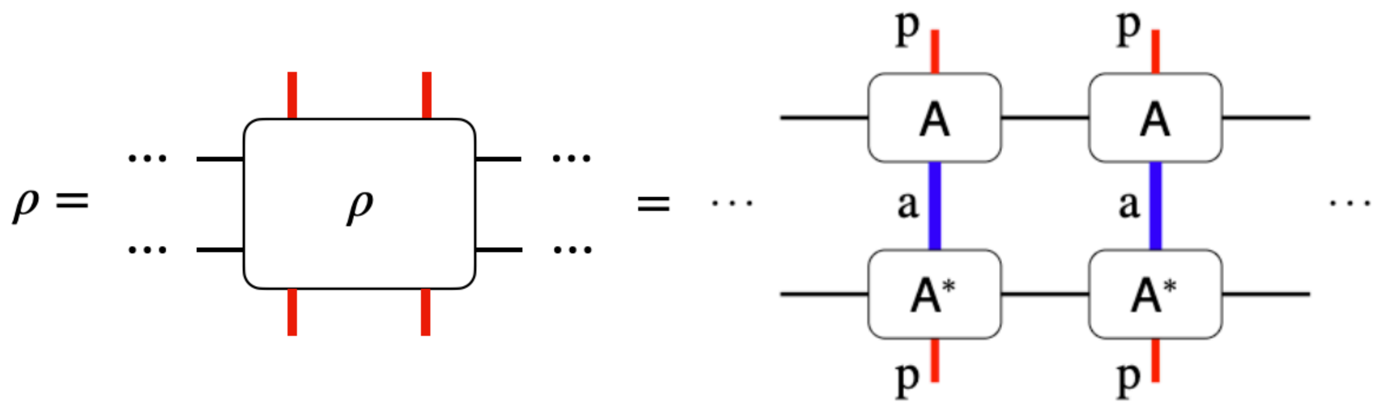
- symmetry actions



- nontrivial SPT phase characterized by the projective representation on virtual indices

Locally purified density operator (LPDO)

- tensor network representation for mixed state, with physical (p) and ancillary (a) degrees of freedom



- LPDO is Hermitian and positive semidefinite by design

  natural and efficient representation for mixed states

- quantum state tomography with locally purified density operators

  Y. Guo and S. Yang #, Commun. Phys. 7, 322 (2024)

- locally purified density operators for noisy quantum circuits

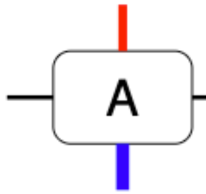
  Y. Guo and S. Yang #, Chin. Phys. Lett. 41, 120302 (2024), Editors' Suggestion

- locally purified density operators for symmetry-protected topological phases in mixed states

  Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Weak Injectivity condition

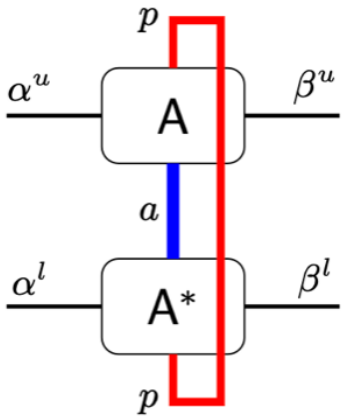
- requires this map be injective



$: \mathbb{C}^D \times \mathbb{C}^D \to \mathbb{C}^{d_p} \times \mathbb{C}^{d_\kappa}$

- linear correlation function $\mathcal{C}^{(1)}(O, i, j) \equiv \mathrm{Tr}[\rho O_i O_j] - \mathrm{Tr}[\rho O_i]\mathrm{Tr}[\rho O_j] \sim e^{-|i-j|/\xi_1}$

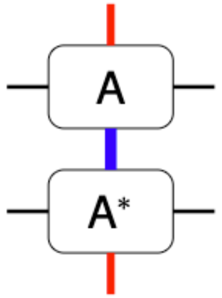
- transfer matrix has a unique dominant eigenvalue



Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Strong injectivity condition

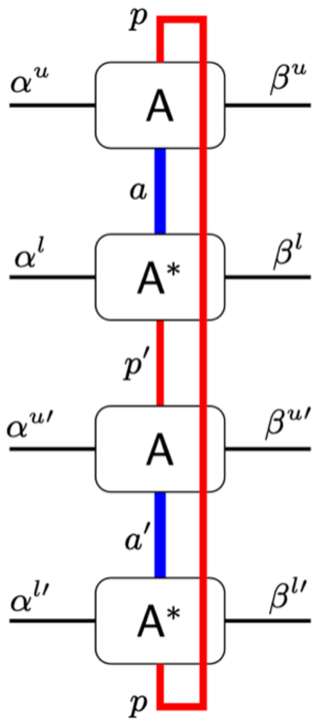
- requires this map be injective



$$: \mathbb{C}^D \times \mathbb{C}^D \times \mathbb{C}^D \times \mathbb{C}^D \to \mathbb{C}^{d_p} \times \mathbb{C}^{d_p}$$

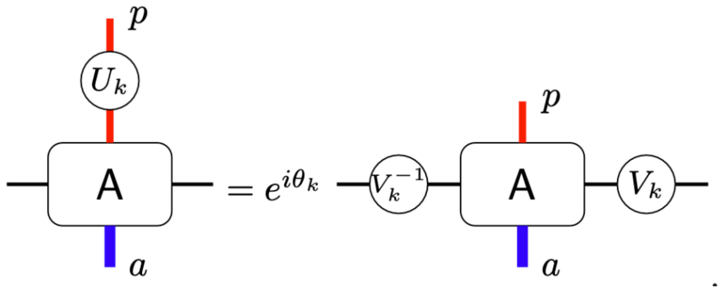
- Renyi-2 correlation function $\mathcal{C}^{(2)}(O,i,j) \equiv \frac{\mathrm{Tr}[\rho O_i O_j \rho O_i O_j]}{\mathrm{Tr}[\rho^2]} - \frac{\mathrm{Tr}[\rho O_i \rho O_i]}{\mathrm{Tr}[\rho^2]} \frac{\mathrm{Tr}[\rho O_j \rho O_j]}{\mathrm{Tr}[\rho^2]} \sim e^{-|i-j|/\xi_2}$

- transfer matrix has a unique dominant eigenvalue

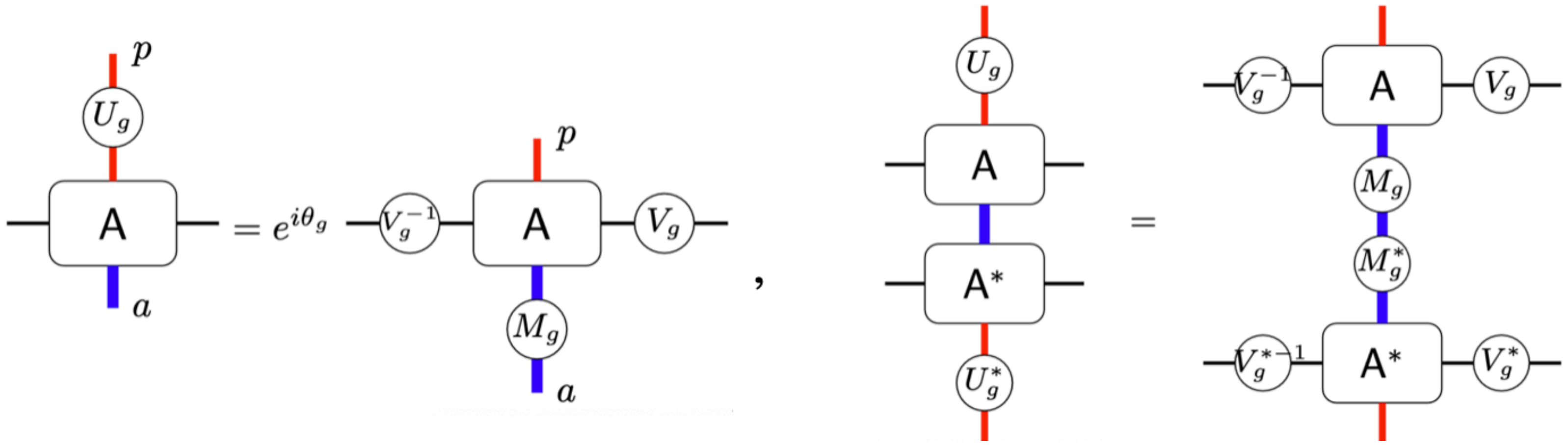


Symmetries in the LPDO representation

- strong symmetry $U(k)\rho = e^{i\theta}\rho$



- strong symmetry indicator $\langle O \rangle \equiv \mathrm{Tr}[\rho O]$
- weak symmetry $U(g)\rho U(g)^\dagger = \rho$



- weak symmetry indicator $\mathrm{Tr}[\rho O \rho O]/\mathrm{Tr}[\rho^2]$

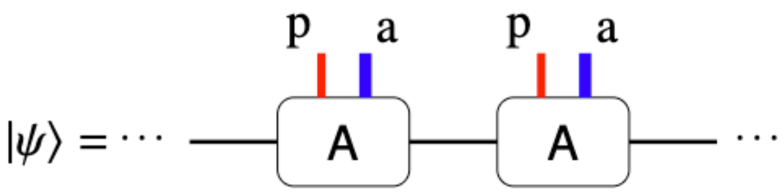
Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Purification framework

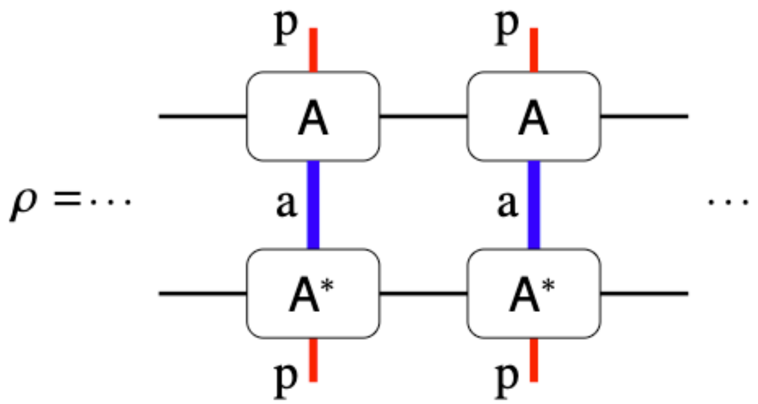
- mixed-state $\rho \to$ enlarged pure state $|\psi\rangle$
- classify all 1D mixed-state phases with $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

  by mapping them to a pure-state model with $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry

  Y. Guo and S. Yang #, arXiv: 2602.21979
- LPDO naturally encodes a purification as a matrix product state (MPS)



- each physical index is locally purified by an ancillary index, satisfying $\rho = \mathrm{Tr}_{\mathrm{a}}[|\psi\rangle\langle\psi|]$



Trivial weakly symmetric state

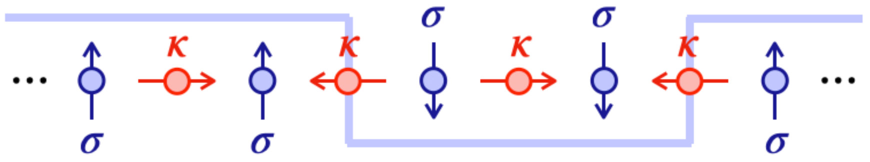
- trivial product state $\rho_\sigma^{\mathrm{W}} = \frac{1}{2^N} \prod_i \sigma_i^0$
  does not satisfy the strong symmetry, $K\rho_\sigma^{\mathrm{W}} \neq \rho_\sigma^{\mathrm{W}}$
  satisfies the weak symmetry, $U\rho_\sigma^{\mathrm{W}} U^\dagger = \rho_\sigma^{\mathrm{W}}$

- purification given by the product of Bell pairs
  $|\psi_{\kappa\sigma}^{\mathrm{Bell}}\rangle = \frac{1}{2^{N/2}} \prod_i \left(|{\uparrow_{\sigma_i}}{\uparrow_{\kappa_i}}\rangle + |{\downarrow_{\sigma_i}}{\downarrow_{\kappa_i}}\rangle\right)$
  tracing out environment κ yields ρ_σ^{W}

- environment $\rightarrow$ local charge reservior, keeps track of symmetry transformation

Pure state $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\kappa$ SPT $\leftrightarrow$ Mixed state $\mathbb{Z}_2^\sigma$ SWSSB

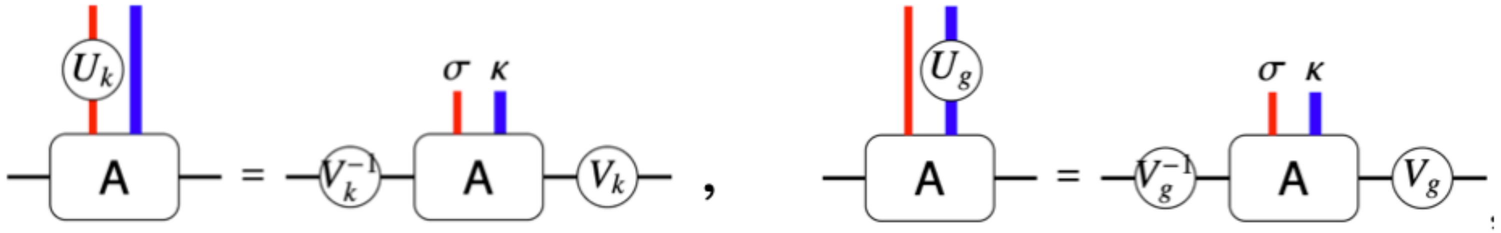
- the fixed-point density matrix characterizing $\mathbb{Z}_2$ strong-to-weak spontaneous symmetry breaking (SWSSB)
  $\rho_\sigma^{\mathrm{SWSSB}} = \frac{1}{2^N}\left(\prod_i \sigma_i^0 + \prod_i \sigma_i^x\right) = \rho_+ + \rho_-$
- $K = \prod_i \sigma_i^x$, the entire state has strong symmetry $K\rho_\sigma^{\mathrm{SWSSB}} = \rho_\sigma^{\mathrm{SWSSB}}$
  each term only has weak symmetry $K\rho_+ = \rho_-, \quad K\rho_- = \rho_+$
- $\rho_\sigma^{\mathrm{SWSSB}}$ can be purified to a 1D cluster state with SPT order protected by $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\kappa$ symmetry
  decorated domain-wall (DW) construction
  $|\psi_{\kappa\sigma}^{\mathrm{SPT}}\rangle = \frac{1}{2^{N/2}}\sum_{\{\sigma_i\}} |\cdots \uparrow_\sigma \rightarrow_\kappa \uparrow_\sigma \leftarrow_\kappa \downarrow_\sigma \leftarrow_\kappa \uparrow_\sigma \cdots\rangle = \frac{1}{2^{N/2}}\sum_{\{\sigma_i\}} |\psi_{\kappa\sigma}^{\mathrm{DW}}\rangle_{\{\sigma_i\}}$
  excitations of κ spins are placed on the domain wall of σ spins



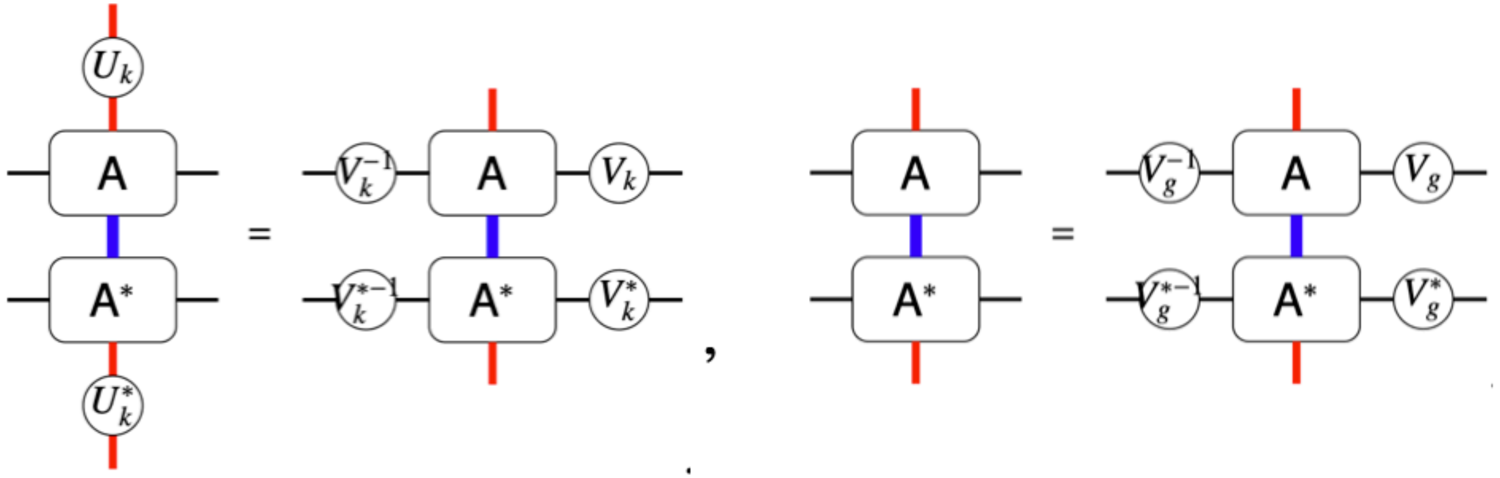
- environment $\rightarrow$ stores nonlocal domain-wall information
  long-range order appears in fidelity correlators and Renyi-2 correlator
  microscopic mechanism of SWSSB from the purification view

Pure state $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\kappa$ SPT $\leftrightarrow$ Mixed state $\mathbb{Z}_2^\sigma$ SWSSB

- $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\kappa$ SPT state, the symmetry transformation of the local tensor



- after tracing out κ, the local tensor satisfies



- $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\kappa$ SPT $\leftrightarrow$ nontrivial mixed anomaly between K and G $\leftrightarrow$ the resulting LPDO violates the strong injectivity condition $\leftrightarrow$ long-range Renyi-2 correlator $\leftrightarrow$ the mixed state has SWSSB order

$$\mathcal{C}^{(2)}(O,i,j) \equiv \frac{\mathrm{Tr}[\rho O_i O_j \rho O_i O_j]}{\mathrm{Tr}[\rho^2]} - \frac{\mathrm{Tr}[\rho O_i \rho O_i]}{\mathrm{Tr}[\rho^2]} \frac{\mathrm{Tr}[\rho O_j \rho O_j]}{\mathrm{Tr}[\rho^2]}$$

Domain wall (DW) duality map with controlled-Z (CZ) gates

- $U_{\sigma\tau}^{\mathrm{DW}} \equiv \prod_i \mathrm{CZ}_{\tau_{i-1}\sigma_i}\mathrm{CZ}_{\sigma_i\tau_i}$ exchanges the trivial symmetric state and the cluster state

  $U_{\sigma\tau}^{\mathrm{DW}}|\psi_{\sigma\tau}^{\mathrm{Trivial}}\rangle = |\psi_{\sigma\tau}^{\mathrm{SPT}}\rangle, \quad U_{\sigma\tau}^{\mathrm{DW}}|\psi_{\sigma\tau}^{\mathrm{SPT}}\rangle = |\psi_{\sigma\tau}^{\mathrm{Trivial}}\rangle$

  $|\psi_{\sigma\tau}^{\mathrm{Trivial}}\rangle = |\psi_{\sigma}^{\mathrm{Trivial}}\rangle \otimes |\psi_{\tau}^{\mathrm{Trivial}}\rangle = \prod_i |{\rightarrow_{\sigma_i}}\rangle \otimes \prod_i |{\rightarrow_{\tau_i}}\rangle$

ASPT

- pure state $|\psi_{\sigma\tau\kappa}^{\sigma-\tau\ \mathrm{SPT},\ \sigma-\kappa\ \mathrm{Parallel}}\rangle = U_{\sigma\tau}^{\mathrm{DW}}\left(|\psi_{\kappa\sigma}^{\mathrm{Bell}}\rangle \otimes |\psi_{\tau}^{\mathrm{Trivial}}\rangle\right)$
- mixed state $\rho_{\sigma\tau}^{\mathrm{ASPT}} = U_{\sigma\tau}^{\mathrm{DW}}\left(\rho_{\sigma}^{\mathrm{W}} \otimes |\psi_{\tau}^{\mathrm{Trivial}}\rangle\langle\psi_{\tau}^{\mathrm{Trivial}}|\right)U_{\sigma\tau}^{\mathrm{DW}\dagger}$

R. Ma and C. Wang, Phys. Rev. X 13, 031016 (2023)

Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Double ASPT

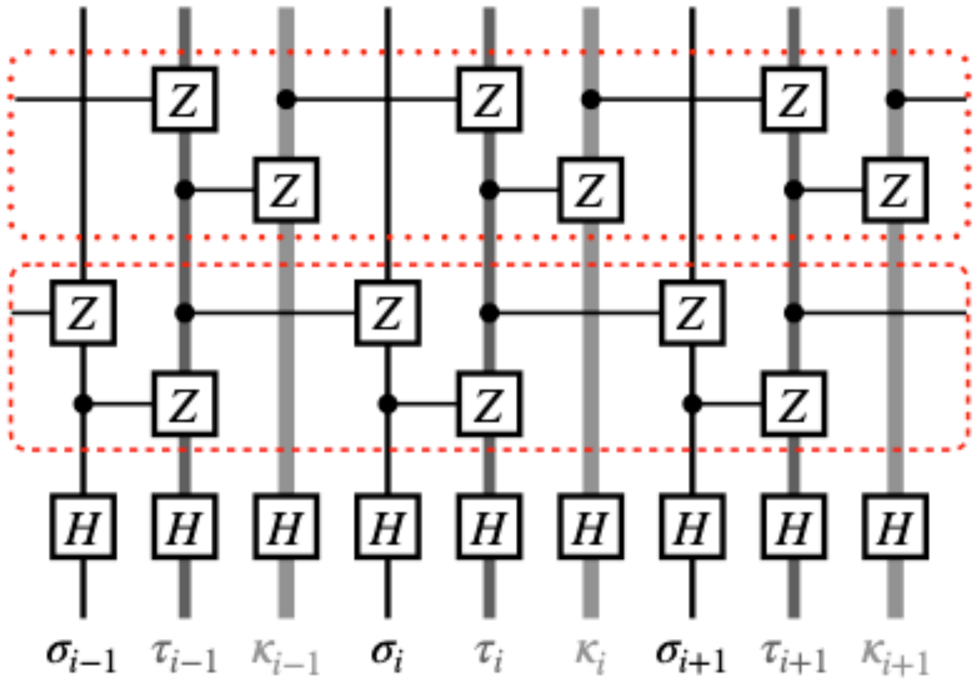
- pure state $|\psi_{\sigma\tau\kappa}^{\sigma-\tau\ \mathrm{SPT},\ \sigma-\kappa\ \mathrm{SPT}}\rangle = U_{\kappa\sigma}^{\mathrm{DW}}\left(|\psi_{\sigma\tau}^{\mathrm{SPT}}\rangle \otimes |\psi_{\kappa}^{\mathrm{Trivial}}\rangle\right) = U_{\kappa\sigma}^{\mathrm{DW}}U_{\sigma\tau}^{\mathrm{DW}}|\psi_{\sigma\tau\kappa}^{\mathrm{Trivial}}\rangle$
- mixed state $\rho_{\sigma\tau}^{\sigma\text{-double ASPT}} = U_{\sigma\tau}^{\mathrm{DW}}\left(\rho_{\sigma}^{\mathrm{SWSSB}} \otimes |\psi_{\tau}^{\mathrm{Trivial}}\rangle\langle\psi_{\tau}^{\mathrm{Trivial}}|\right)U_{\sigma\tau}^{\mathrm{DW}\dagger}$

Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion

Minimal $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\tau \times \mathbb{Z}_2^\kappa$ purified model

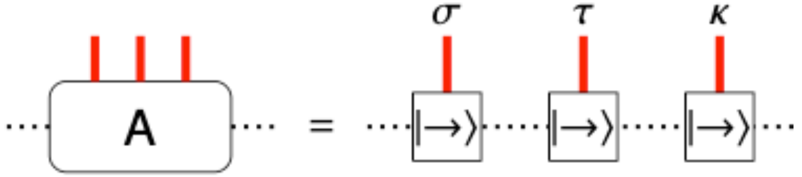
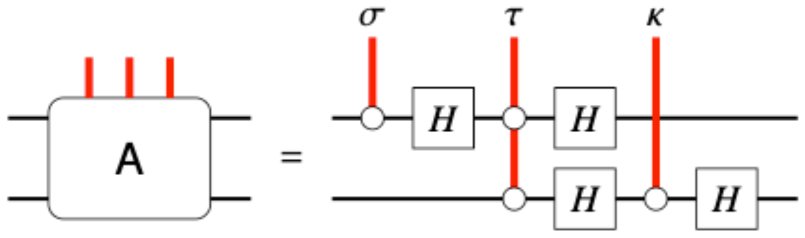
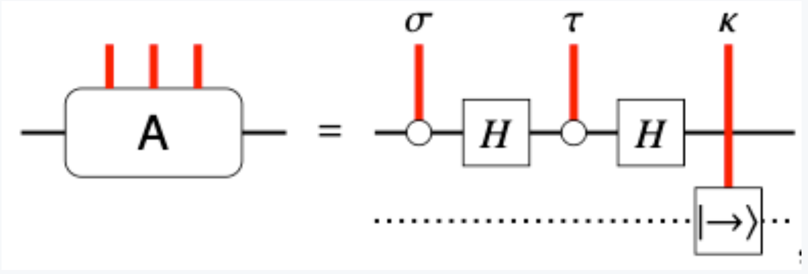
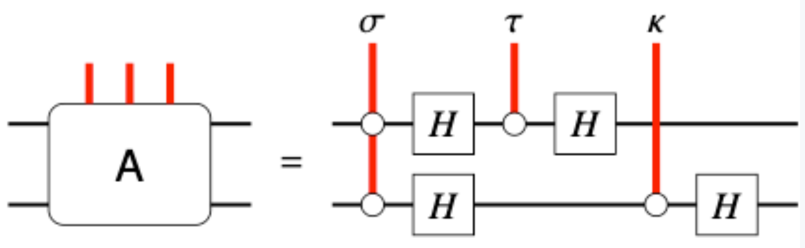
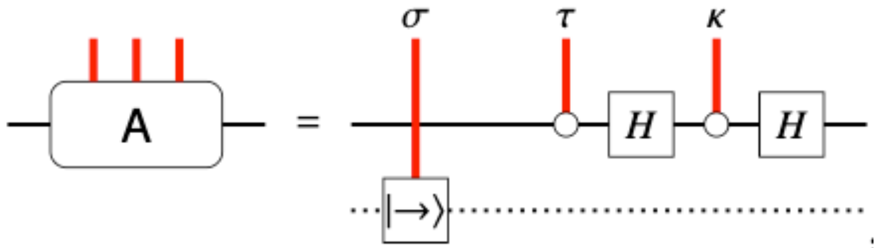
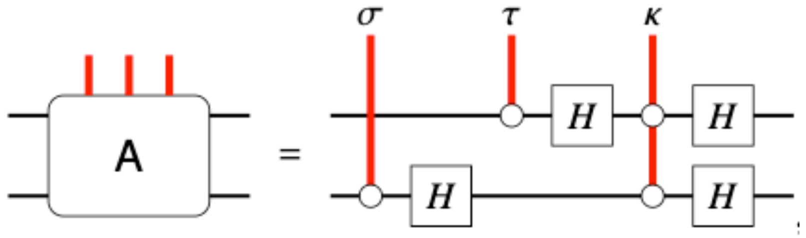
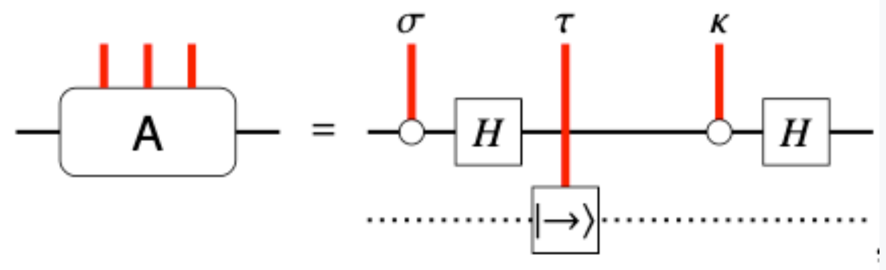
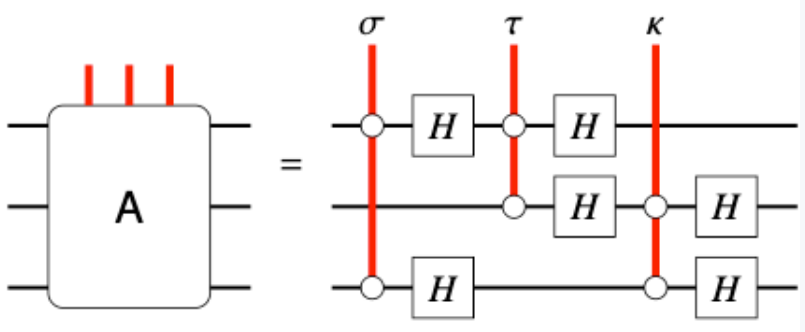
- pure-state SPT phases protected by $\mathbb{Z}_2^\sigma \times \mathbb{Z}_2^\tau \times \mathbb{Z}_2^\kappa$ symmetry are labeled by $\mu = (\mu_{\sigma\tau}, \mu_{\tau\kappa}, \mu_{\kappa\sigma}) \in \{\pm 1\}^3$
  $\mu_{\alpha\beta} = -1$ indicates the mixed anomaly between α and β, with $\alpha, \beta \in \{\sigma, \tau, \kappa\}$

- eight fixed point wavefunctions can be constructed by decorating domain walls with controlled-Z gates
  $|\psi^\mu_{\sigma\tau\kappa}\rangle = \left(U^{\mathrm{DW}}_{\sigma\tau}\right)^{\frac{1-\mu_{\sigma\tau}}{2}} \left(U^{\mathrm{DW}}_{\tau\kappa}\right)^{\frac{1-\mu_{\tau\kappa}}{2}} \left(U^{\mathrm{DW}}_{\kappa\sigma}\right)^{\frac{1-\mu_{\kappa\sigma}}{2}} |\psi^{\mathrm{Trivial}}_{\sigma\tau\kappa}\rangle$



- quantum circuit to generate $|\psi^{(-1,-1,+1)}_{\sigma\tau\kappa}\rangle$
  starting from the product state $\prod_i |\uparrow\rangle_i$, one layer of Hadamard gates (H) generates $|\psi^{(+1,+1,+1)}_{\sigma\tau\kappa}\rangle = \prod_i |\rightarrow\rangle_i$

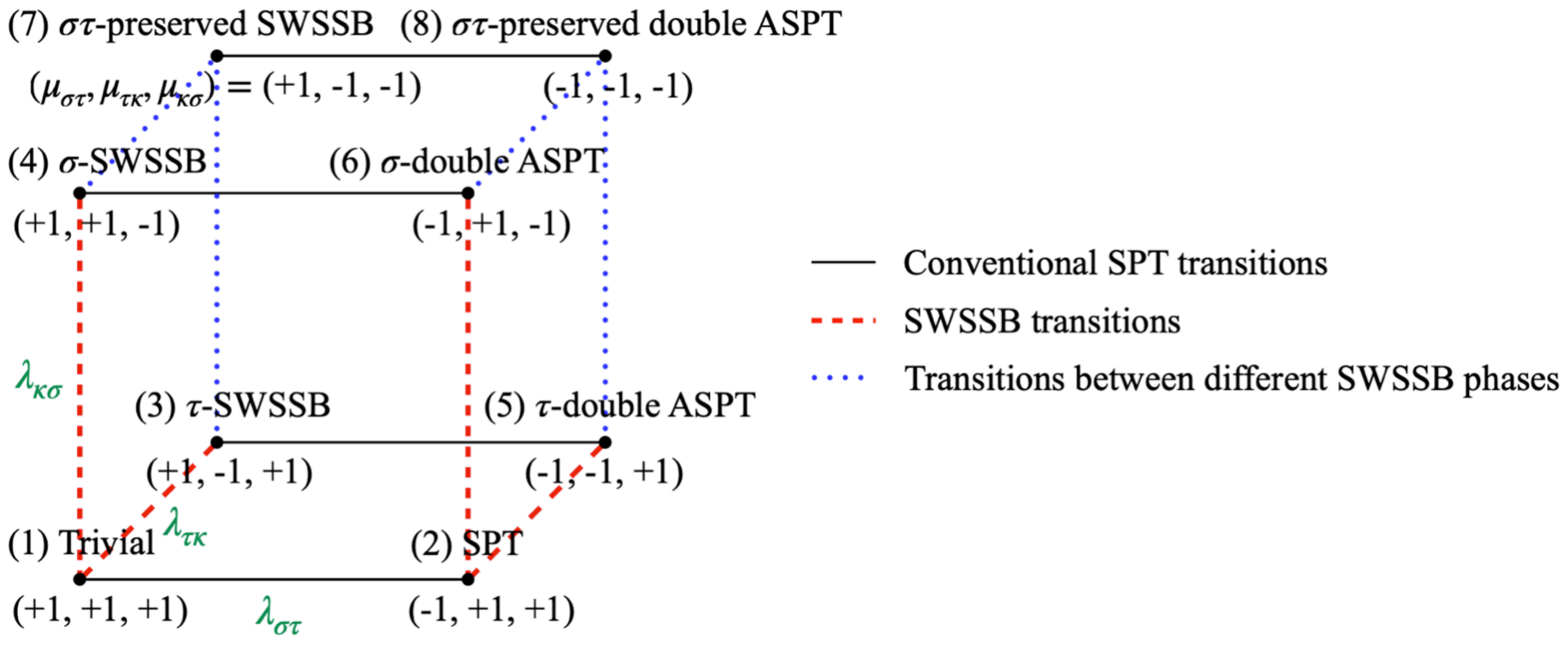
Eight fixed points labeled by $(\mu_{\sigma\tau}, \mu_{\tau\kappa}, \mu_{\kappa\sigma}) \in \{\pm 1\}^3$

mixed state	pure state	mixed state	pure state
(1) Trivial $\mu = (+1, +1, +1)$		(5) τ-double ASPT $\mu = (-1, -1, +1)$	
(2) SPT $\mu = (-1, +1, +1)$		(6) σ-double ASPT $\mu = (-1, +1, -1)$	
(3) τ-SWSSB $\mu = (+1, -1, +1)$		(7) $\sigma\tau$-preserved SWSSB $\mu = (+1, -1, -1)$	
(4) σ-SWSSB $\mu = (+1, +1, -1)$		(8) $\sigma\tau$-preserved double ASPT $\mu = (-1, -1, -1)$	

Phase cube

- we implement a trilinear interpolation between eight fixed-point Hamiltonians

$$
\begin{aligned}
H(\lambda_{\sigma\tau},\lambda_{\tau\kappa},\lambda_{\kappa\sigma}) &= (1-\lambda_{\sigma\tau})(1-\lambda_{\tau\kappa})(1-\lambda_{\kappa\sigma})H_{\sigma\tau\kappa}^{(+1,+1,+1)} + \lambda_{\sigma\tau}(1-\lambda_{\tau\kappa})(1-\lambda_{\kappa\sigma})H_{\sigma\tau\kappa}^{(-1,+1,+1)} \\
&+(1-\lambda_{\sigma\tau})\lambda_{\tau\kappa}(1-\lambda_{\kappa\sigma})H_{\sigma\tau\kappa}^{(+1,-1,+1)} + (1-\lambda_{\sigma\tau})(1-\lambda_{\tau\kappa})\lambda_{\kappa\sigma}H_{\sigma\tau\kappa}^{(+1,+1,-1)} \\
&+\lambda_{\sigma\tau}\lambda_{\tau\kappa}(1-\lambda_{\kappa\sigma})H_{\sigma\tau\kappa}^{(-1,-1,+1)} + \lambda_{\sigma\tau}(1-\lambda_{\tau\kappa})\lambda_{\kappa\sigma}H_{\sigma\tau\kappa}^{(-1,+1,-1)} \\
&+(1-\lambda_{\sigma\tau})\lambda_{\tau\kappa}\lambda_{\kappa\sigma}H_{\sigma\tau\kappa}^{(+1,-1,-1)} + \lambda_{\sigma\tau}\lambda_{\tau\kappa}\lambda_{\kappa\sigma}H_{\sigma\tau\kappa}^{(-1,-1,-1)}
\end{aligned}
$$

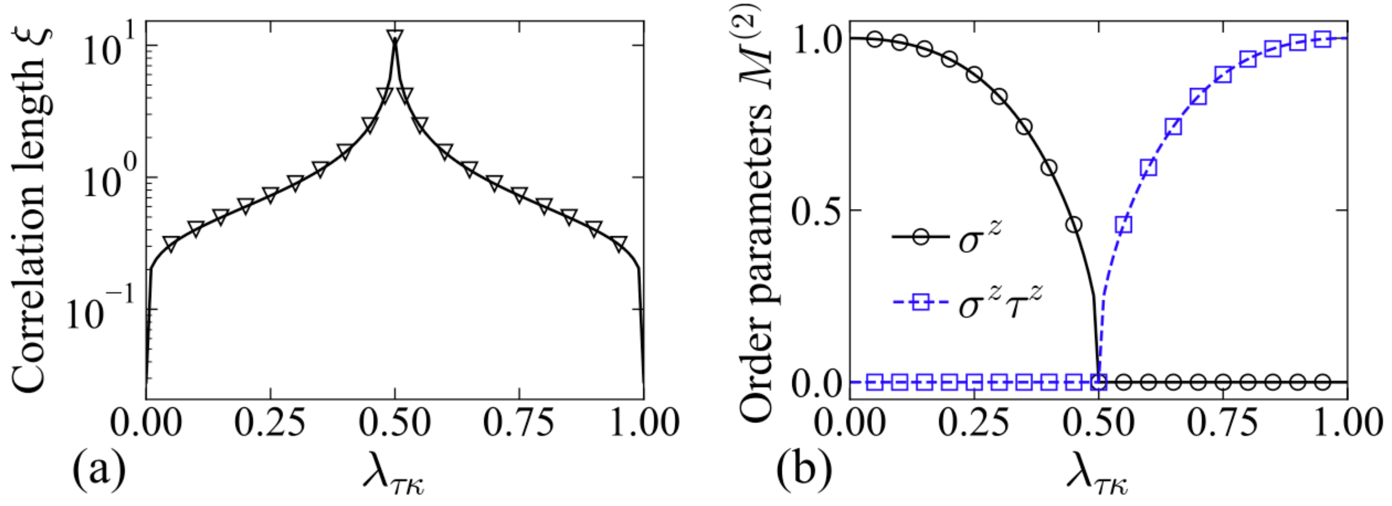


σ-SWSSB $\leftrightarrow$ $\sigma\tau$-preserved SWSSB

- linear interpolation $H(\lambda_{\tau\kappa}) = (1-\lambda_{\tau\kappa})H_{\sigma\tau\kappa}^{(+1,+1,-1)} + \lambda_{\tau\kappa}H_{\sigma\tau\kappa}^{(+1,-1,-1)}$
- duality relation $U_{\tau\kappa}^{\mathrm{DW}}H(\lambda_{\tau\kappa})U_{\tau\kappa}^{\mathrm{DW}\dagger} = H(1-\lambda_{\tau\kappa})$, self-dual critical point at $\lambda_c = 0.5$
- compute the ground state using the VUMPS method

  trace out κ spins to obtain mixed states, calculate Renyi-2 correlators to detect SWSSB

  $M_\sigma^{(2)} = \lim_{|i-j|\to\infty}\mathcal{C}^{(2)}(\sigma^z,i,j),\quad M_{\sigma\tau}^{(2)} = \lim_{|i-j|\to\infty}\mathcal{C}^{(2)}(\sigma^z\tau^z,i,j)$



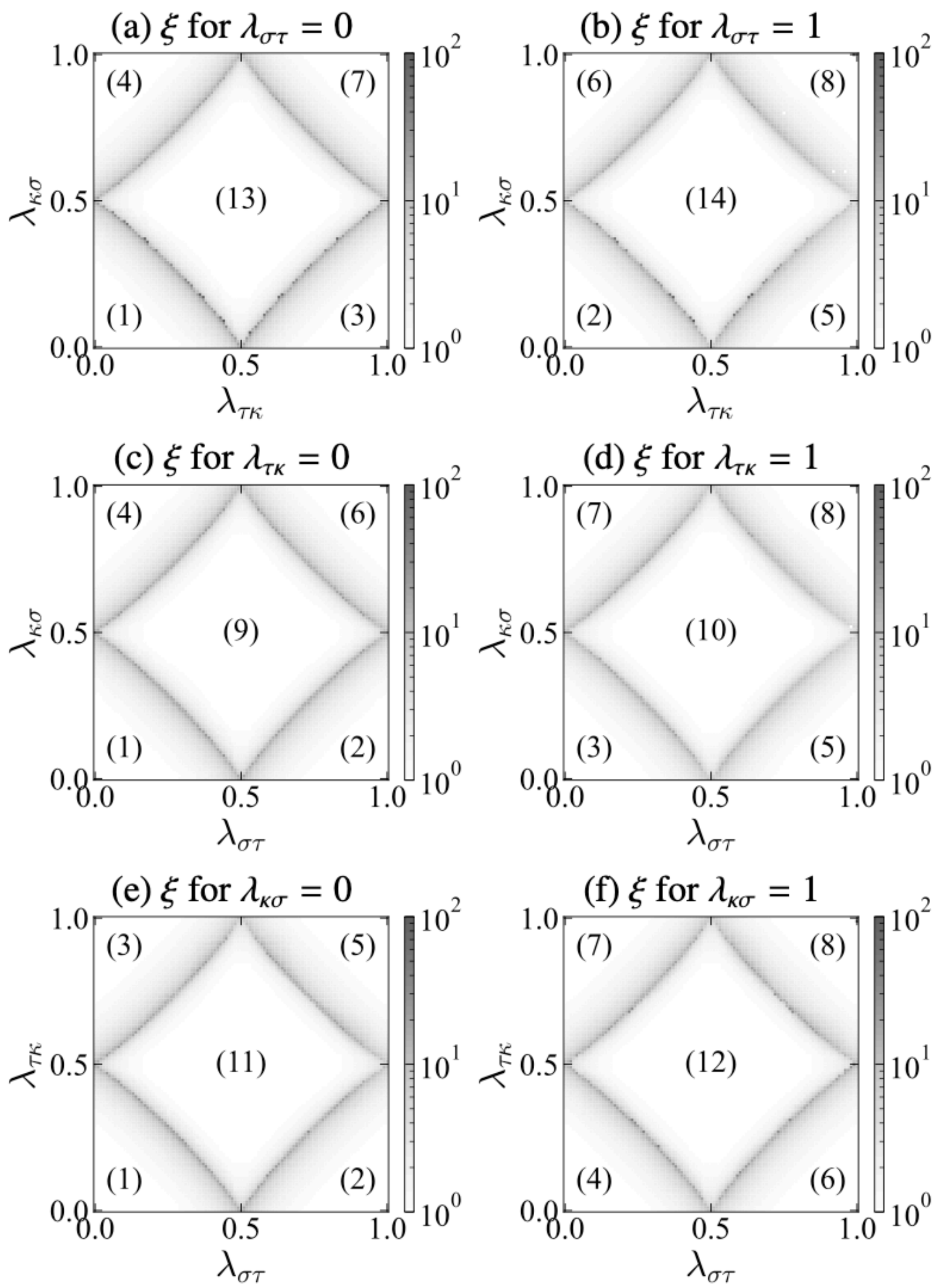
- σ-SWSSB phase ($\lambda_{\tau\kappa} < \lambda_c$): $M_\sigma^{(2)} \neq 0, \mathbb{Z}_2^\sigma$ weak, $\mathbb{Z}_2^\tau$ strong, $\mathbb{Z}_2^{\sigma\tau}$ weak

  $\sigma\tau$-preserved SWSSB phase ($\lambda_{\tau\kappa} > \lambda_c$): $M_{\sigma\tau}^{(2)} \neq 0, \mathbb{Z}_2^\sigma$ weak, $\mathbb{Z}_2^\tau$ weak, $\mathbb{Z}_2^{\sigma\tau}$ strong

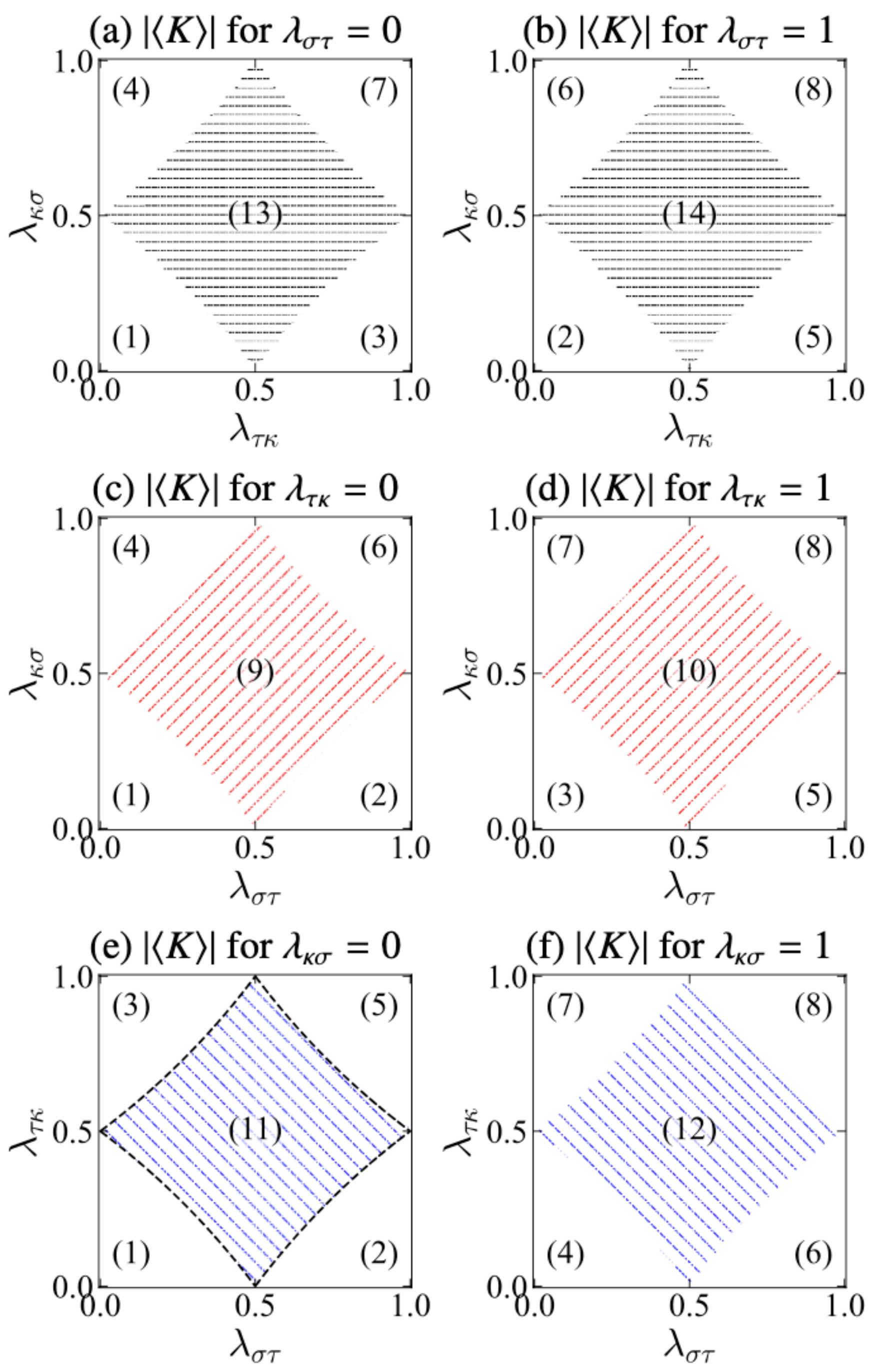
  only in open systems, weak symmetry is hidden under SWSSB rather than eliminated

Face phase diagrams

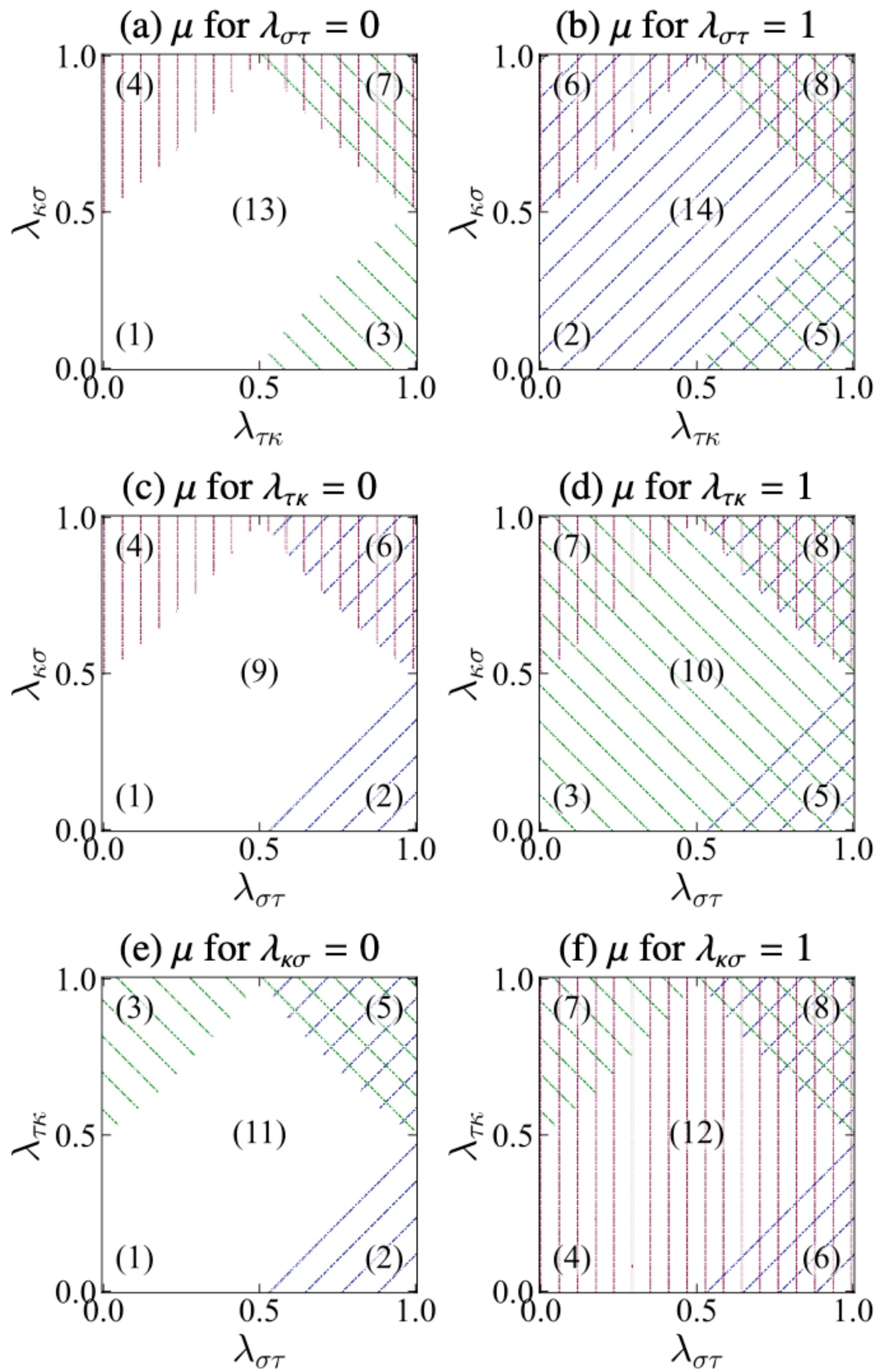
correlation length



symmetry indictor

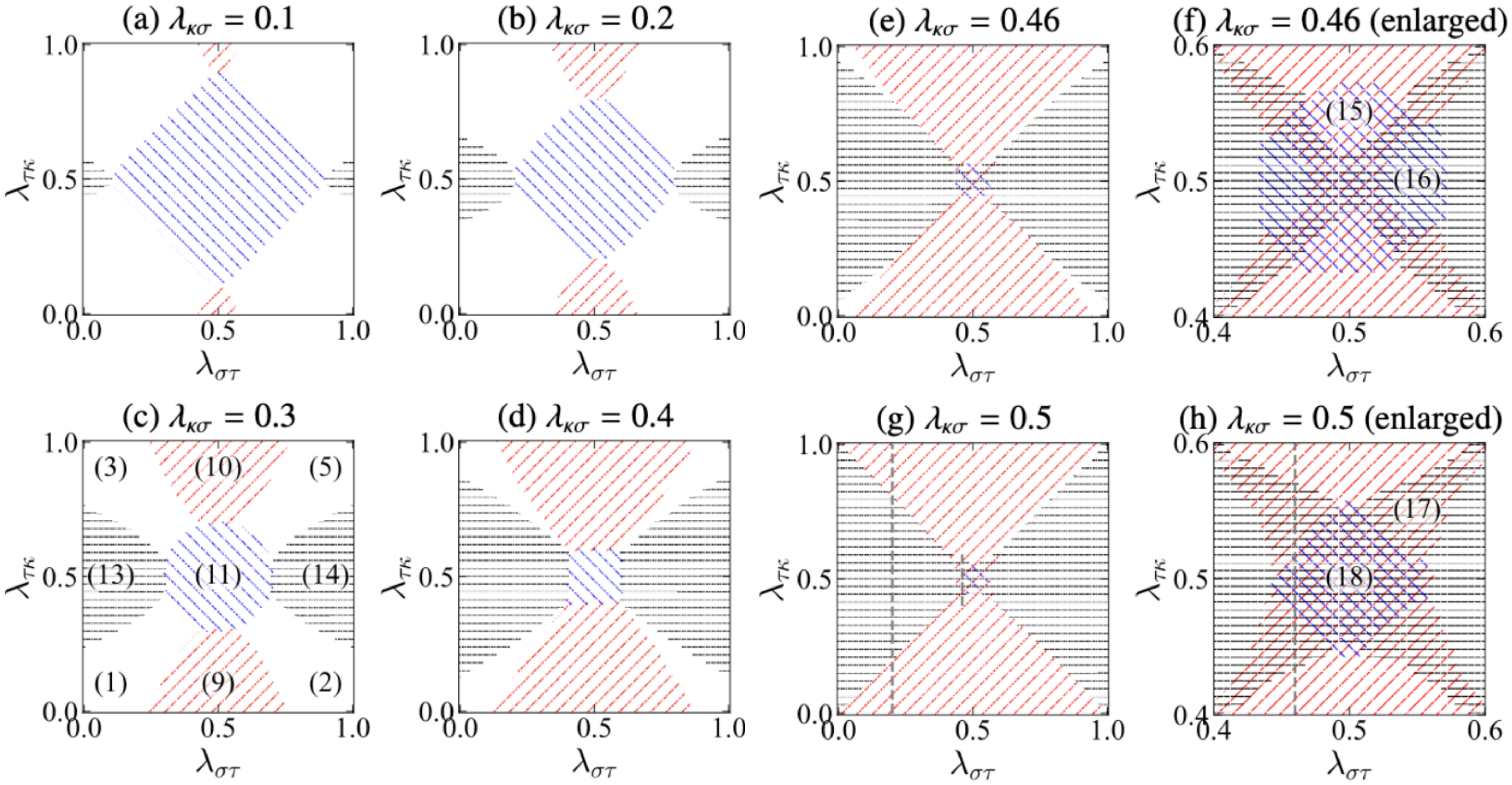


topological invariant



Inside the phase cube

- each SSB region is rooted on one face and extends into the bulk, occupying a pyramid-like volume
- complete symmetry breaking at the cube center

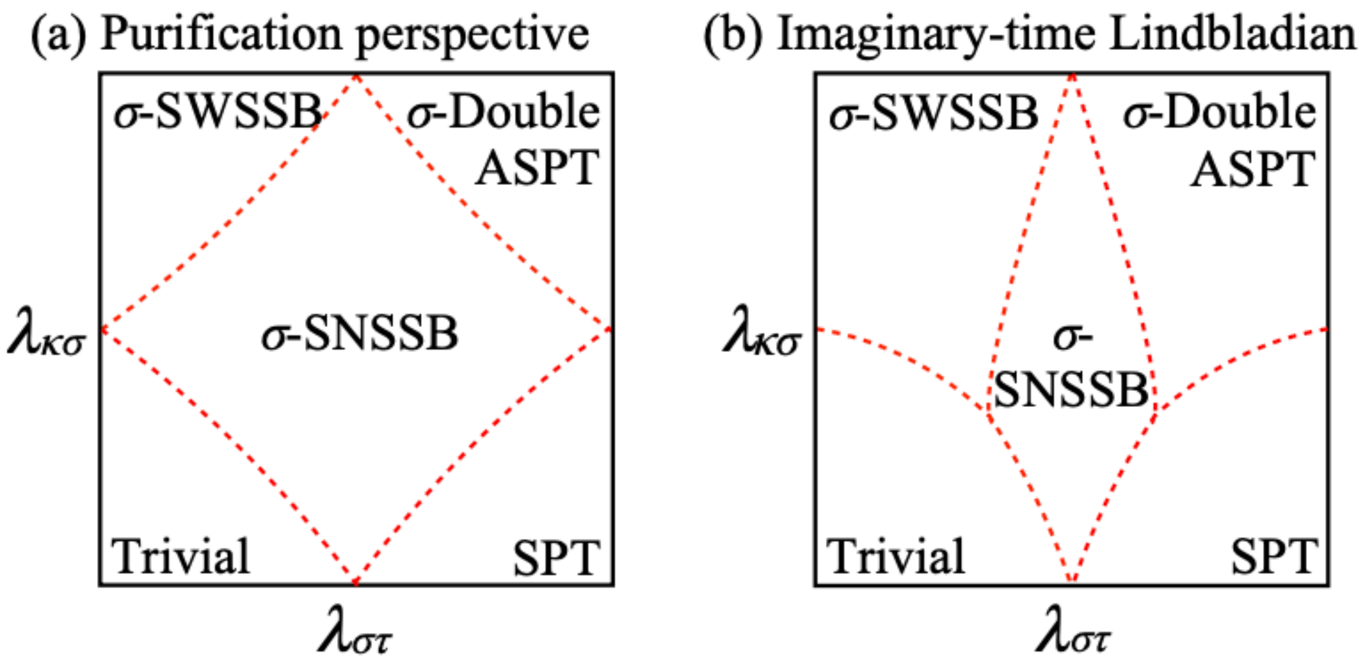


All phases in the phase cube

No.	Pure-state phase								Mixed-state phase				
	$\mathbb{Z}_2^\sigma$	$\mathbb{Z}_2^\tau$	$\mathbb{Z}_2^\kappa$	Hatch$_{\vert\langle K\rangle\vert}$	$\mu_{\sigma\tau}$	$\mu_{\tau\kappa}$	$\mu_{\kappa\sigma}$	Hatch$_\mu$	$\mathbb{Z}_2^\sigma$	$\mathbb{Z}_2^\tau$	$\mathbb{Z}_2^{\sigma\tau}$	$\mu_{\sigma\tau}$	Phase name
1	Symmetric	Symmetric	Symmetric	None	+1	+1	+1	None	Symmetric	Symmetric	Symmetric	+1	Trivial
2	Symmetric	Symmetric	Symmetric	None	−1	+1	+1	/	Symmetric	Symmetric	Symmetric	−1	SPT
3	Symmetric	Symmetric	Symmetric	None	+1	−1	+1	\	Symmetric	SWSSB	SWSSB	+1	τ-SWSSB
4	Symmetric	Symmetric	Symmetric	None	+1	+1	−1	|	SWSSB	Symmetric	SWSSB	+1	σ-SWSSB
5	Symmetric	Symmetric	Symmetric	None	−1	−1	+1	/ \	Symmetric	SWSSB	SWSSB	−1	τ-double ASPT
6	Symmetric	Symmetric	Symmetric	None	−1	+1	−1	/ |	SWSSB	Symmetric	SWSSB	−1	σ-double ASPT
7	Symmetric	Symmetric	Symmetric	None	+1	−1	−1	\ |	SWSSB	SWSSB	Symmetric	+1	$\sigma\tau$-preserved SWSSB
8	Symmetric	Symmetric	Symmetric	None	−1	−1	−1	/ \ |	SWSSB	SWSSB	Symmetric	−1	$\sigma\tau$-preserved double ASPT
9	SSB	Symmetric	Symmetric	//	+1	+1	+1	None	SNSSB	Symmetric	SNSSB	+1	σ-SNSSB
10	SSB	Symmetric	Symmetric	//	+1	−1	+1	\	SNSSB	SWSSB	SNSSB	+1	σ-SNSSB τ-SWSSB
11	Symmetric	SSB	Symmetric	\\	+1	+1	+1	None	Symmetric	SNSSB	SNSSB	+1	τ-SNSSB
12	Symmetric	SSB	Symmetric	\\	+1	+1	−1	|	SWSSB	SNSSB	SNSSB	+1	σ-SWSSB τ-SNSSB
13	Symmetric	Symmetric	SSB	=	+1	+1	+1	None	Symmetric	Symmetric	Symmetric	+1	Trivial
14	Symmetric	Symmetric	SSB	=	−1	+1	+1	/	Symmetric	Symmetric	Symmetric	−1	SPT
15	SSB	SSB	Symmetric	// \\	+1	+1	+1	None	SNSSB	SNSSB	SNSSB	+1	σ-SNSSB τ-SNSSB
16	Symmetric	SSB	SSB	\\ =	+1	+1	+1	None	Symmetric	SNSSB	SNSSB	+1	τ-SNSSB
17	SSB	Symmetric	SSB	// =	+1	+1	+1	None	SNSSB	Symmetric	SNSSB	+1	σ-SNSSB
18	SSB	SSB	SSB	// \\ =	+1	+1	+1	None	SNSSB	SNSSB	SNSSB	+1	σ-SNSSB τ-SNSSB

Comparison of the phase diagrams

- imaginary-time Lindbladian → an infinite-size reservoir with a fast relaxation process
  repeated reinitialization of ancillae at each time step
- this purified model → only involves a single ancilla on each site



- while there exists a one-to-one correspondence between the phases in the two approaches
  the precise locations of the phase boundaries do not coincide

Summary of part two

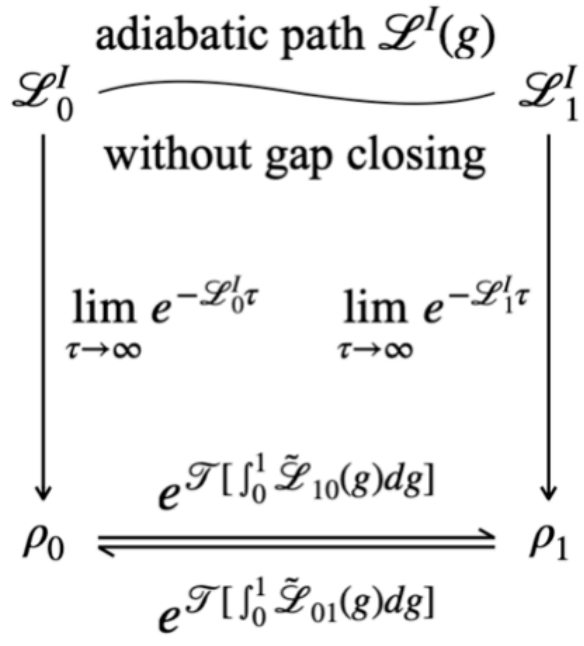
- propose LPDO and purification-based framework

  $\rightarrow$ construct, classify, and diagnose quantum phases and criticality in open quantum systems
- eight fixed points form a phase cube $\rightarrow$ recover the full structure of mixed-state phases and transitions
- a unique class of critical behavior that connects distinct SWSSB phases, only exists in open systems

  Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

  Y. Guo and S. Yang #, arXiv: 2602.21979

Quantum phase transitions in open quantum systems



- imaginary-time Lindbladian evolution

$$\frac{d\rho}{d\tau} = -\{H,\rho\} + \sum_k \left[L_k\rho L_k^\dagger + \frac{1}{2}\{L_k^\dagger L_k,\rho\}\right] \equiv -\mathcal{L}^I(\rho)$$

Y. Guo, K. Ding, and S. Yang #, Rep. Prog. Phys. 88, 118001 (2025)

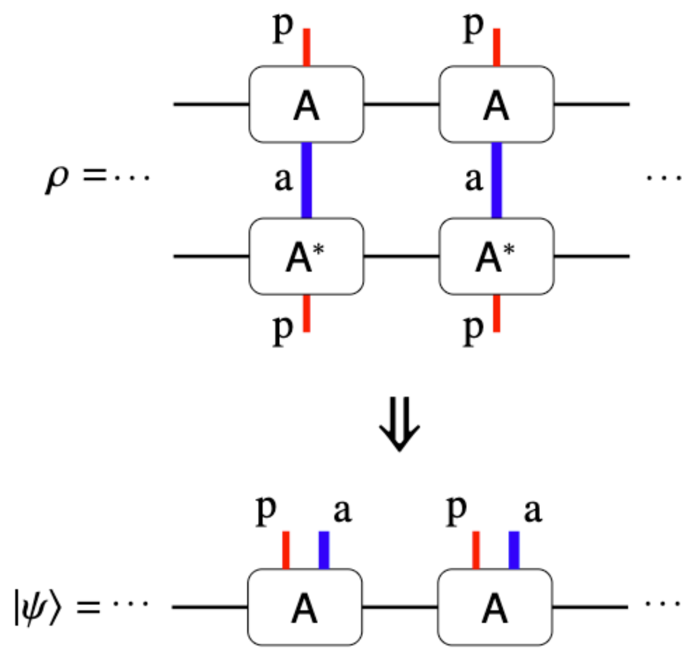
Y. Guo and S. Yang #, Phys. Rev. B 111, L201108 (2025), Editors' Suggestion

- locally purified density operator and purification perspective

LPDO $\Rightarrow$ MPS

Y. Guo *, J.-H. Zhang *, H.-R. Zhang, S. Yang #, and Z. Bi #, Phys. Rev. X 15, 021060 (2025)

Y. Guo and S. Yang #, arXiv: 2602.21979



Thank you!

Real-time and imaginary-time Lindblandian evolution

real-time evolution	imaginary-time evolution
$\rho(t+\Delta t)_s = \mathrm{Tr}_a\Big[e^{-iH_s\otimes I_a\Delta t - iL_{ks}\otimes\sigma_a^+\sqrt{\Delta t} - iL_{ks}^\dagger\otimes\sigma_a^-\sqrt{\Delta t}}$ $\rho(t)_s\otimes\|0\rangle\langle 0\|_a\, e^{iH_s\otimes I_a\Delta t + iL_{ks}^\dagger\otimes\sigma_a^-\sqrt{\Delta t} + iL_{ks}\otimes\sigma_a^+\sqrt{\Delta t}}\Big]$ $= \mathrm{Tr}_a\big[(\rho(t)_s - iH_s\rho(t)_s\Delta t + i\rho(t)_s H_s\Delta t)\otimes\|0\rangle\langle 0\|_a$ $- iL_{ks}\rho(t)_s\otimes\|1\rangle\langle 0\|_a\sqrt{\Delta t} + i\rho(t)_s L_{ks}^\dagger\otimes\|0\rangle\langle 1\|_a\sqrt{\Delta t}$ $+ L_{ks}\rho(t)_s L_{ks}^\dagger\otimes\|1\rangle\langle 1\|_a\,\Delta t$ $- \frac{1}{2}\left(L_{ks}^\dagger L_{ks}\rho(t)_s + \rho(t)_s L_{ks}^\dagger L_{ks}\right)\otimes\|0\rangle\langle 0\|_a\,\Delta t\big]$ $= \rho(t)_s - i\left[H_s, \rho(t)_s\right]\Delta t$ $+ \sum_k\left[L_{ks}\rho(t)_s L_{ks}^\dagger - \frac{1}{2}\left\{L_{ks}^\dagger L_{ks}, \rho(t)_s\right\}\right]\Delta t,$	$\rho(\tau+\Delta\tau)_s = \mathrm{Tr}_a\Big[e^{-H_s\otimes I_a\Delta\tau - L_{ks}\otimes\sigma_a^+\sqrt{\Delta\tau} - L_{ks}^\dagger\otimes\sigma_a^-\sqrt{\Delta\tau}}$ $\rho(\tau)_s\otimes\|0\rangle\langle 0\|_a\, e^{-H_s\otimes I_a\Delta\tau - L_{ks}^\dagger\otimes\sigma_a^-\sqrt{\Delta\tau} - L_{ks}\otimes\sigma_a^+\sqrt{\Delta\tau}}\Big]$ $= \mathrm{Tr}_a\big[(\rho(\tau)_s - H_s\rho(\tau)_s\Delta\tau - \rho(\tau)_s H_s\Delta\tau)\otimes\|0\rangle\langle 0\|_a$ $- L_{ks}\rho(\tau)_s\otimes\|1\rangle\langle 0\|_a\sqrt{\Delta\tau} - \rho(\tau)_s L_{ks}^\dagger\otimes\|0\rangle\langle 1\|_a\sqrt{\Delta\tau}$ $+ L_{ks}\rho(\tau)_s L_{ks}^\dagger\otimes\|1\rangle\langle 1\|_a\,\Delta\tau$ $+ \frac{1}{2}\left(L_{ks}^\dagger L_{ks}\rho(\tau)_s + \rho(\tau)_s L_{ks}^\dagger L_{ks}\right)\otimes\|0\rangle\langle 0\|_a\,\Delta\tau\big]$ $= \rho(\tau)_s - \left\{H_s, \rho(\tau)_s\right\}\Delta\tau$ $+ \sum_k\left[L_{ks}\rho(\tau)_s L_{ks}^\dagger + \frac{1}{2}\left\{L_{ks}^\dagger L_{ks}, \rho(\tau)_s\right\}\right]\Delta\tau,$
$\frac{d\rho}{dt} = -i[H,\rho] + \sum_k\left[L_k\rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right]$ $\equiv \mathcal{L}(\rho)$	$\frac{d\rho}{d\tau} = -\{H,\rho\} + \sum_k\left[L_k\rho L_k^\dagger + \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right]$ $\equiv -\mathcal{L}^I(\rho)$

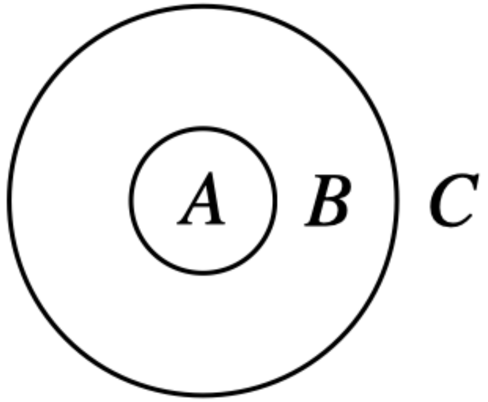
Different correlation functions for a mixed state

- linear correlator $C(O,i,j) \equiv \langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle \sim e^{-|i-j|/\xi}$
- Renyi-n correlator $C^{(n)}(O,i,j) \equiv \langle O_i O_j \rangle^{(n)} - \langle O_i \rangle^{(n)} \langle O_j \rangle^{(n)} \sim e^{-|i-j|/\xi_n}, \langle O \rangle^{(n)} \equiv \frac{\mathrm{Tr}\left[\rho^{\frac{n}{2}} O \rho^{\frac{n}{2}} O\right]}{\mathrm{Tr}[\rho^n]}$
- conditional mutual information (CMI)

  $I(A:C|B) = S(AB) + S(BC) - S(B) - S(ABC) \sim e^{-L_B/\xi_M}$

  $S(X) = -\mathrm{Tr}[\rho_X \ln \rho_X], \xi_M$ is Markov length

S. Sang and T. H. Hsieh, Phys. Rev. Lett. 134, 070403 (2025)

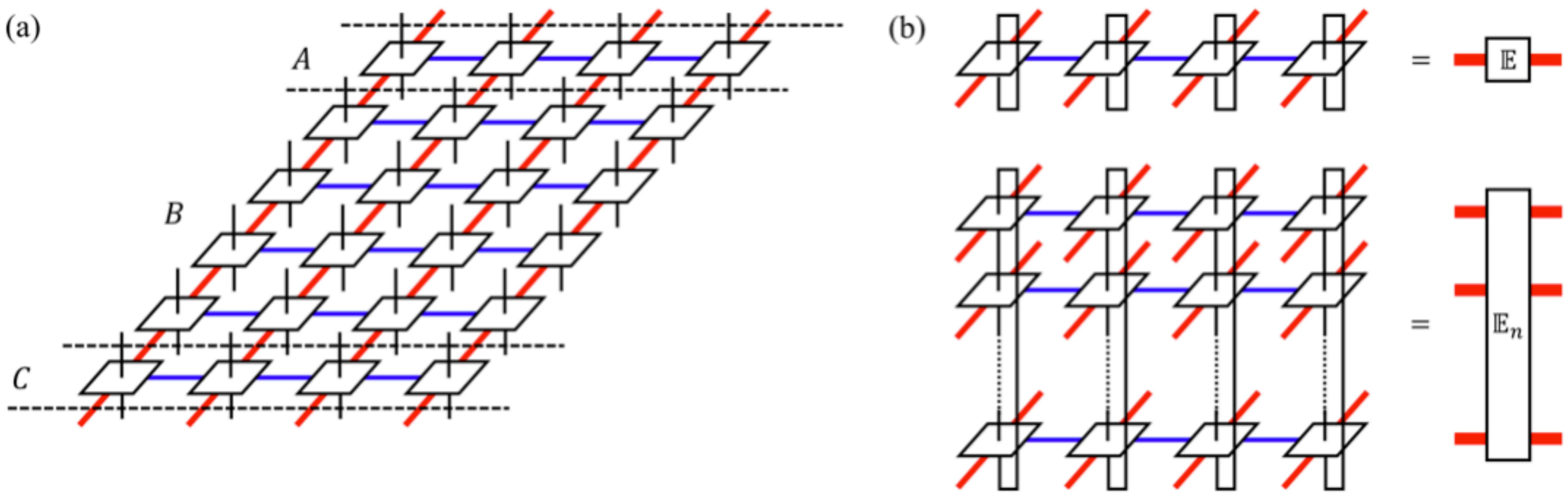


- divergence of Markov length is an indirector for phase transition

  Markov length remains finite in the same phase

Calculation of Markov length

- calculate conditional mutual information with Renyi-n entropy and extrapolate to $n \to 1$
  $S^{(n)}(X) = \frac{1}{1-n} \ln \mathrm{Tr}\left[\rho_X^n\right]$
- linear or Renyi-n correlation lengths are determined by the spectrum of the transfer operators $\mathbb{E}$ and $\mathbb{E}_n$ for even n
- Spectral decomposition $\mathbb{E}_n = \sum_i \lambda_n^i |r_n^i)(l_n^i|$ with decreasing order $|\lambda_n^0| \geq |\lambda_n^1| \geq \dots$
  Renyi-n correlation length $\xi_n = -1/\ln|\lambda_n^1/\lambda_n^0|$



Calculation of Markov length

- Renyi- n conditional mutual information

  $\mathrm{Tr}[\rho_X^n] = \sum_i (\lambda_n^i)^{L_X} (l^{0\otimes n}|r_n^i)(l_n^i|r^{0\otimes n}) \equiv \sum_i (\lambda_n^i)^{L_X} f_n^i$

  $S^{(n)}(X) = \frac{1}{1-n} \ln \mathrm{Tr}[\rho_X^n] \approx \frac{1}{1-n} \ln \left\{ (\lambda_n^0)^{L_X} f_n^0 \left[1 + \left(\frac{\lambda_n^1}{\lambda_n^0} \right)^{L_X} \frac{f_n^1}{f_n^0} \right] \right\} \approx \frac{1}{1-n} \left[L_X \ln \lambda_n^0 + \ln f_n^0 + \frac{f_n^1}{f_n^0} e^{-L_X/\xi_n} \right]$

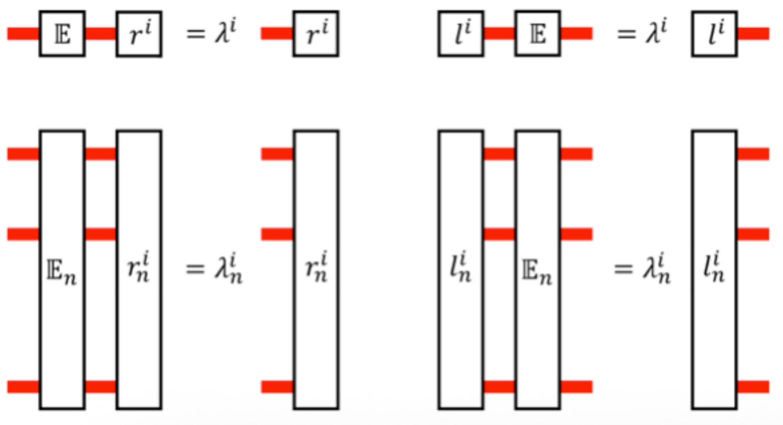
  $I^{(n)}(A:C|B) = \frac{1}{1-n} \frac{f_n^1}{f_n^0} \left[e^{-L_{AB}/\xi_n} + e^{-L_{BC}/\xi_n} - e^{-L_B/\xi_n} - e^{-L_{ABC}/\xi_n} \right]$

  $= \frac{1}{n-1} \frac{f_n^1}{f_n^0} \left(1 - e^{-L_A/\xi_n}\right) \left(1 - e^{-L_C/\xi_n}\right) e^{-L_B/\xi_n} \propto e^{-L_B/\xi_n}$

- Renyi-n Markov length $\xi_M^{(n)}$ is just Renyi-n correlation length ξ_n for even n

- in 1D systems, finite $\xi_2 \Rightarrow$ finite ξ

- $\mathcal{L}^I$ is gapless $\Rightarrow$ divergent $\xi_2 \Rightarrow$ divergent $\xi_{2^k} \overset{n \to 1}{\Rightarrow}$ divergent ξ_M



Parent Haimiltonian and mixed state

- the stabilizer Hamiltonian for the trivial state $|\psi^{(+1,+1,+1)}_{\sigma\tau\kappa}\rangle = \prod_i |\rightarrow\rangle_i$ is $H^{\text{Trivial}}_{\sigma\tau\kappa} = -\sum_i(\sigma^x_i + \tau^x_i + \kappa^x_i)$
- the parent Hamiltonian for $|\psi^{\mu}_{\sigma\tau\kappa}\rangle$ can be generated by conjugating with the corresponding DW maps

$$H^{\mu}_{\sigma\tau\kappa} = \left[\left(U^{\text{DW}}_{\sigma\tau}\right)^{\frac{1-\mu_{\sigma\tau}}{2}}\left(U^{\text{DW}}_{\tau\kappa}\right)^{\frac{1-\mu_{\tau\kappa}}{2}}\left(U^{\text{DW}}_{\kappa\sigma}\right)^{\frac{1-\mu_{\kappa\sigma}}{2}}\right] H^{\text{Trivial}}_{\sigma\tau\kappa}\left[\left(U^{\text{DW}}_{\sigma\tau}\right)^{\frac{1-\mu_{\sigma\tau}}{2}}\left(U^{\text{DW}}_{\tau\kappa}\right)^{\frac{1-\mu_{\tau\kappa}}{2}}\left(U^{\text{DW}}_{\kappa\sigma}\right)^{\frac{1-\mu_{\kappa\sigma}}{2}}\right]^{\dagger}$$

- using the relations $U^{\text{DW}}_{\sigma\tau}\sigma^x_i U^{\text{DW}\dagger}_{\sigma\tau} = \tau^z_{i-1}\sigma^x_i\tau^z_i$ and $U^{\text{DW}}_{\sigma\tau}\tau^x_i U^{\text{DW}\dagger}_{\sigma\tau} = \sigma^z_i\tau^x_i\sigma^z_{i+1}$
- the parent Hamiltonian has a compact form $H^{\mu}_{\sigma\tau\kappa} = -\sum_i \left(S^{\mu}_{\sigma_i} + S^{\mu}_{\tau_i} + S^{\mu}_{\kappa_i}\right)$

$$S^{\mu}_{\sigma_i} = \sigma^x_i\left(\tau^z_{i-1}\tau^z_i\right)^{\frac{1-\mu_{\sigma\tau}}{2}}\left(\kappa^z_{i-1}\kappa^z_i\right)^{\frac{1-\mu_{\kappa\sigma}}{2}}$$
$$S^{\mu}_{\tau_i} = \tau^x_i\left(\sigma^z_i\sigma^z_{i+1}\right)^{\frac{1-\mu_{\sigma\tau}}{2}}\left(\kappa^z_{i-1}\kappa^z_i\right)^{\frac{1-\mu_{\tau\kappa}}{2}}$$
$$S^{\mu}_{\kappa_i} = \kappa^x_i\left(\tau^z_i\tau^z_{i+1}\right)^{\frac{1-\mu_{\tau\kappa}}{2}}\left(\sigma^z_i\sigma^z_{i+1}\right)^{\frac{1-\mu_{\kappa\sigma}}{2}}$$

- tracing out the κ degrees of freedom yields mixed states $\rho^{\mu}_{\sigma\tau} = \text{Tr}_{\kappa}\left[|\psi^{\mu}_{\sigma\tau\kappa}\rangle\langle\psi^{\mu}_{\sigma\tau\kappa}|\right]$

String order and fidelity correlator

- parent Hamiltonian of the cluster state $|\psi_{\kappa\sigma}^{\mathrm{SPT}}\rangle$ is $H_{\kappa\sigma}^{\mathrm{SPT}} = -\sum_i \left[\kappa_{i-1}^z \sigma_i^x \kappa_i^z + \sigma_i^z \kappa_i^x \sigma_{i+1}^z\right]$
- $|\psi_{\kappa\sigma}^{\mathrm{SPT}}\rangle$ exhibits nonlocal string order , $\lim_{|i-j|\to\infty} |\langle S(\sigma^z, \kappa^x, i, j)\rangle| = 1$
  where $\langle S(\sigma^z, \kappa^x, i, j)\rangle = \langle\psi_{\kappa\sigma}^{\mathrm{SPT}}|\sigma_i^z \left(\prod_{l=i}^{j-1} \kappa_l^x\right) \sigma_j^z |\psi_{\kappa\sigma}^{\mathrm{SPT}}\rangle$
- fidelity correlator of the mixed state $\mathcal{C}^{\mathcal{F}}(\sigma^z, i, j) \equiv \mathcal{F}\left(\sigma_i^z \sigma_j^z \rho \sigma_i^z \sigma_j^z, \rho\right)$
  $\mathcal{F}$ denotes the fidelity between two mixed states $\mathcal{F}(\rho_1, \rho_2) = \left(\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right)^2$
- we introduce two pure states $|\psi_2\rangle \equiv \sigma_i^z \sigma_j^z |\psi_{\kappa\sigma}^{\mathrm{SPT}}\rangle$ and $|\psi_2\rangle \equiv \prod_{l=i}^{j-1} \kappa_l^x |\psi_{\kappa\sigma}^{\mathrm{SPT}}\rangle$
- string order parameter can be expressed using the fidelity between these two states
  $|\langle S(\sigma^z, \kappa^x, i, j)\rangle| = |\langle\psi_1|\psi_2\rangle| = \sqrt{\mathcal{F}(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|)} \leq \sqrt{\mathcal{F}(\mathrm{Tr}_\kappa[|\psi_1\rangle\langle\psi_1|], \mathrm{Tr}_\kappa[|\psi_2\rangle\langle\psi_2|])}$
  $= \sqrt{\mathcal{F}\left(\sigma_i^z \sigma_j^z \rho_\sigma^{\mathrm{SWSSB}} \sigma_i^z \sigma_j^z, \rho_\sigma^{\mathrm{SWSSB}}\right)} = \sqrt{\mathcal{C}^{\mathcal{F}}(\sigma^z, i, j)}$
- if the original pure state has a long-range string order $\lim_{|i-j|\to\infty} |\langle S(\sigma^z, \kappa^x, i, j)\rangle| \to O(1)$
  the mixed state is long-range correlated in terms of the fidelity correlator $\lim_{|i-j|\to\infty} \mathcal{C}^{\mathcal{F}}(\sigma^z, i, j) \to O(1)$