Derivation of formulas

Erik Svanberg, Hilma Karlsson, Daniel Voigt, Vaishali Adya

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List of abbreviations

FH - Fundamental Harmonic, 1550 nm
SH - Second Harmonic, 775 nm
SHG - Second Harmonic Generation, 775 nm
ppLN - periodically poled Lithium Niobate
QPM - quasi-phase matching
BHD - Balanced Homodyne Detection
LO - Local oscillator, 1550 nm
BS - Beamsplitter

1 Introduction

The mathematical foundations behind the Second Harmomic Generation (SHG) process is discussed in section 2, the generated squeezing is considered in section 3, and the detection in section 4. There are several text books, a popular one available in the library and also online through KTH accounts is [1].

2 Second Harmonic Generation

The focus throughout this lab work is on the processes based on the χ^2 susceptibility. Both SHG and OPA rely on three-wave mixing processes where the three waves that interact are the pump, signal and idler photons. Due to conservation of momentum

$$\frac{n_i\omega_i}{c_0} + \frac{n_s\omega_s}{c_0} = \frac{n_p\omega_p}{c_0} \tag{1}$$

must be satisfied, where, n_x is the refractive index of the nonlinear medium experienced for the three different modes of light, which are generally represented as idler, signal, and pump and c_0 is the speed of light in vacuum. The energy must also be conserved throughout the nonlinear conversion process. Recalling the energy of a photon being $\hbar\omega$, in other words,

$$\omega_p = \omega_s + \omega_i \tag{2}$$

has to be satisfied.

In case of the second harmonic generation (SHG) two photons at frequency ω are absorbed by the nonlinear material to re-radiate a single photon at frequency 2ω. This χ^2 interaction is called up-conversion. In the case of SHG the energy conservation, equation 2, is satisfied as $\omega + \omega = (2\omega)$.

In nonlinear optics the conservation of momentum is often referred to as phase matching, and in the case of SHG perfect phase matching must satisfy

$$k_\omega + k_\omega = k_{2\omega} \iff \\ \frac{n_\omega \omega}{c_0} + \frac{n_\omega \omega}{c_0} = \frac{n_{2\omega}(2\omega)}{c_0} \tag{3}$$

When utilizing quasi-phase matching (QPM), in which the structure of the crystal is periodically poled, the difference between the wavevectors are:

$$\Delta k = 2k_\omega - k_{2\omega} + k_m \tag{4}$$

where k_m comes from QPM. For perfect QPM the wavenumber mismatch becomes zero ($\Delta k = 0$). Denoting the electric field of 775 nm light as B and the electric field of 1550 nm light as A the coupled mode equations are expressed as:

$$\begin{aligned} \frac{dA}{dz} &= i\frac{\sqrt{\alpha_{SHG}}}{L}B(z)A(z)^* e^{i\Delta k z} \\ \frac{dB}{dz} &= i\frac{\sqrt{\alpha_{SHG}}}{L}A(z)^2 e^{-i\Delta k z} \end{aligned} \tag{5}$$

where α_{SHG} is the second harmonic generation (SHG) conversion efficiency in units of W^{-1} and L is the length of the waveguide. The equations are often stated using a normalized conversion efficiency in units of $W^{-1}m^{-2}$, with the normalized conversion efficiency being $\alpha_{SHG,norm} = \frac{\alpha_{SHG}}{L^2}$. We choose to use α_{SHG} in units of W^{-1} in order to make other equations simpler and since α_{SHG} is the measurable quantity that is stated on any SHG waveguide. You can read more about how the equations are derived in [2].

If we assume perfect QPM, we can rewrite equation 5 and solve them as follow:

$$\begin{aligned} \frac{dA}{dz} &= i\frac{\sqrt{\alpha_{SHG}}}{L}B(z)A(z)^* \\ \frac{dB}{dz} &= i\frac{\sqrt{\alpha_{SHG}}}{L}A(z)^2 \\ \frac{d^2B}{dz^2} &= 2i\frac{\sqrt{\alpha_{SHG}}}{L}\frac{dA}{dz}A = 2i^2\frac{\alpha_{SHG}}{L^2}B|A|^2 = -2\frac{\sqrt{\alpha_{SHG}}}{L}\frac{d|B|}{dz}B \end{aligned} \tag{6}$$

With no input SH ($|B(0)| = 0$), and with constant input light ($|A(0)| = const.$), we can calculate the power of the SHG light by solving equation 6 we get

$$|B(z)| = |A(0)|tanh(\sqrt{\alpha_{SHG}}|A(0)|\frac{z}{L}) \tag{7}$$

The power at the output of the waveguide is then:

$$P_{SHG} = P_{in}tanh^2(\sqrt{\alpha_{SHG}P_{in}}) \tag{8}$$

where $P_{in} = |A|^2$ is the input 1550 nm light and $P_{SHG} = |B|^2$ is the generated SHG light.

If we don't assume perfect QPM ($\Delta k \neq 0$), as in equation 8, but still assume no input SH ($|B(0)| = 0$), as well as no pump depletion ($A(z) = const.$), the solution to equation 5 is instead:

$$P_{SHG} \propto P_{in}sinc^2(\frac{\Delta k L}{2\pi}) \tag{9}$$

Note: ΔkL depends on the temperature, since temperature changes both the length and the refractive index of the waveguide.

3 Squeezing

To consider squeezing we return to the equations for a waveguide with FH and SH frequencies. However, we now rewrite the fields as quantum operators. This results in:

$$
\begin{aligned}
\frac{d\hat{A}(z)}{dz} &= i\frac{\sqrt{\alpha_{OPA}}}{L}\hat{B}(z)\hat{A}(z)^\dagger \\
\frac{d\hat{B}(z)}{dz} &= i\frac{\sqrt{\alpha_{OPA}}}{L}\hat{A}(z)^2
\end{aligned}
\tag{10}
$$

We now use a linearisation of operators to rewrite $\hat{A}(z) = \bar{A}(z) + \hat{a}(z)$ and $\hat{B}(z) = \bar{B}(z) + \hat{b}(z)$ where $\bar{A}$ indicates the average and $\hat{a}(z)$ a small deviation, in other words we have separated each field into its steady state component and fluctuating component where the fluctuating component is quantum mechanical in origin. Taking only the parts with no deviations we get:

$$
\begin{aligned}
\frac{d\bar{A}(z)}{dz} &= i\frac{\sqrt{\alpha_{OPA}}}{L}\bar{B}(z)\bar{A}(z)^* \\
\frac{d\bar{B}(z)}{dz} &= i\frac{\sqrt{\alpha_{OPA}}}{L}\bar{A}(z)^2
\end{aligned}
\tag{11}
$$

Here α_{OPA} is now the squeezing efficiency. Although this is the same the SHG efficiency (α_{SHG}), we choose to differentiate between the two quantities as experimentally the SHG and the OPA waveguide tend to have different conversion efficiencies. However, α_{OPA} is still typically measured by considering the SHG conversion efficiency as the latter can be measured easier experimentally.

If we consider the deviations to first order we have:

$$
\begin{aligned}
\frac{d\hat{a}(z)}{dz} &= i\frac{\sqrt{\alpha_{OPA}}}{L}(\hat{b}(z)\bar{A}(z)^* + \bar{B}(z)\hat{a}(z)^\dagger) \\
\frac{d\hat{b}(z)}{dz} &= i\frac{\sqrt{\alpha_{OPA}}}{L}2\hat{a}(z)\bar{A}(z)
\end{aligned}
\tag{12}
$$

We observe that first we generate the SH and then we squeeze the light in a separate waveguide. Hence $\bar{B}$ is approximated to be constant in the OPA. We also assume that we filter out the FH such that we only squeeze vacuum, this implies $\bar{A} = 0$. Using these assumptions equation 12 becomes:

$$
\frac{d\hat{a}(z)}{dz} = i\frac{\sqrt{\alpha_{OPA}}}{L}\bar{B}(z)\hat{a}(z)^\dagger \tag{13}
$$

Defining the amplitude, $\hat{X}$, and phase, $\hat{Y}$, quadrature as (sometimes these are defined as $\hat{X}_1$ and $\hat{X}_2$):

$$
\hat{X} = \hat{a} + \hat{a}^\dagger \tag{14}
$$

$$
\hat{Y} = i(\hat{a} - \hat{a}^\dagger) \tag{15}
$$

Equation 13 together with 14 gives:

$$
\frac{d\hat{X}}{dz} = \frac{d\hat{a}}{dz} + \frac{d\hat{a}^\dagger}{dz} = i\frac{\sqrt{\alpha_{OPA}}}{L}(\bar{B}(z)\hat{a}(z)^\dagger - \bar{B}(z)^*\hat{a}(z)) \tag{16}
$$

In order to get anti-squeezing in the X quadrature and squeezing in the Y quadrature, the phase of $\bar{B}$ is set to $-\frac{\pi}{2}$:

$$\frac{d\hat{X}}{dz} = \frac{\sqrt{\alpha_{OPA}}}{L}|\bar{B}|(\hat{a}+\hat{a}^{\dagger}) = \frac{\sqrt{\alpha_{OPA}}}{L}|\bar{B}|\hat{X} \tag{17}$$

Similarly we get for the phase quadrature:

$$\frac{d\hat{Y}}{dz} = -\frac{\sqrt{\alpha_{OPA}}}{L}|\bar{B}|\hat{Y} \tag{18}$$

Under the assumption that $|\bar{B}|$ is constant in the squeezer we get the solutions:

$$X(z) = X(0)exp(\sqrt{\alpha_{OPA}}|\bar{B}|\frac{z}{L}) \tag{19}$$

$$Y(z) = Y(0)exp(-\sqrt{\alpha_{OPA}}|\bar{B}|\frac{z}{L}) \tag{20}$$

Note : Variance for an operator $\hat{O}$ can be calculated by $\langle(\Delta\hat{O})^2\rangle = \langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2$

$$\begin{aligned}
\langle(\Delta X(z))^2\rangle &= \langle X(z)^2\rangle - \langle X(z)\rangle^2 \\
&= \langle\Big(X(0)\,e^{\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\Big)^2\rangle - \Big(\langle X(0)\,e^{\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\rangle\Big)^2 \\
&= e^{2\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\,\big(\langle X(0)^2\rangle - \langle X(0)\rangle^2\big) \\
&= e^{2\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\,\langle(\Delta X(0))^2\rangle
\end{aligned}$$

The Y-quadrature can be calculated similarly :

$$\begin{aligned}
\langle(\Delta Y(z))^2\rangle &= \langle Y(z)^2\rangle - \langle Y(z)\rangle^2 \\
&= \langle\Big(Y(0)\,e^{-\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\Big)^2\rangle - \Big(\langle Y(0)\,e^{-\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\rangle\Big)^2 \\
&= e^{-2\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\,\big(\langle Y(0)^2\rangle - \langle Y(0)\rangle^2\big) \\
&= e^{-2\sqrt{\alpha_{OPA}}|\bar{B}|z/L}\,\langle(\Delta Y(0))^2\rangle
\end{aligned}$$

In order to simplify the equations we write $V(X) = \langle(\Delta X(z))^2\rangle$, $V(X(0)) = \langle(\Delta X(0))^2\rangle$; $V(X) = \langle(\Delta X(z))^2\rangle$, $V(X(0)) = \langle(\Delta X(0))^2\rangle$. This now simplifies the variance equations listed above to the following :

$$V(X) = V(X(0))e^{2\sqrt{\alpha_{OPA}}|\bar{B}|z/L} \tag{21}$$

$$V(Y) = V(Y(0))e^{-2\sqrt{\alpha_{OPA}}|\bar{B}|z/L} \tag{22}$$

The squeezing and anti-squeezing levels at the output after the waveguide becomes:

$$V(X)/V(X(0)) = exp(2\sqrt{\alpha_{OPA}P_{SHG}}) \tag{23}$$

$$V(Y)/V(Y(0)) = exp(-2\sqrt{\alpha_{OPA}P_{SHG}}) \tag{24}$$

Where we have used $P_{SHG} = |B|^2$ and normalized to vacuum noise $V(X(0))$ and $V(Y(0))$.

4 Balanced Homodyne Detection

We detect the squeezing of light through the use of Balanced Homodyne Detection (BHD). Homodyne detection is used since the power of the variance signal is too low to detected directly. The process consists of interfering the variance signal with a strong local oscillator (LO) field which enhances the signal to be measured. For a schematic of this setup, refer to the notes from the introductory lecture to the lab.

BHD is phase-sensitive and it consists of the signal and LO interfering on a beamsplitter (BS) with a phase difference θ and measuring the photocurrent difference. If the BS is 50/50 and the variance signal consists of squeezed vacuum the photocurrent difference will be:

$$i_- \approx \bar{\beta}(Xcos\theta + Ysin\theta) \tag{25}$$

Where $\bar{\beta}$ is the expectation value of the LO field.

The variance of the photocurrent i_- is given by :

$$V = \langle (\Delta i_-)^2 \rangle = \langle i_-^2 \rangle - \langle i_- \rangle^2 \tag{26}$$

For vacuum noise: $\langle X \rangle = \langle Y \rangle = 0 \implies \langle i_- \rangle = 0$
Now the variance simplifies to : $V = \langle i_-^2 \rangle$

Substituting for i_- from 25 we get the variance to be

$$V = \langle \left[\bar{\beta}\left(X\cos\theta + Y\sin\theta\right)\right]^2 \rangle \tag{27}$$

which can now be simplified to be

$$V = \bar{\beta}^2 \langle (X\cos\theta + Y\sin\theta)^2 \rangle \tag{28}$$

Expanding only the terms inside the expectation values :

$$(X\cos\theta + Y\sin\theta)^2 = (X)^2\cos^2\theta + (Y)^2\sin^2\theta + (XY + YX)\cos\theta\sin\theta \tag{29}$$

$$V = \bar{\beta}^2\left[\langle (X)^2 \rangle \cos^2\theta + \langle (Y)^2 \rangle \sin^2\theta + \langle XY + YX \rangle \cos\theta\sin\theta\right] \tag{30}$$

since $\langle XY + YX \rangle = \langle 2i(a^2 - {a^\dagger}^2 \rangle = 0$ for vacuum, this gives:

$$V = \bar{\beta}^2\left[\langle (X)^2 \rangle \cos^2\theta + \langle (Y)^2 \rangle \sin^2\theta\right] \tag{31}$$

We can now re-define our X and Y quadratures respectively as :

$$\begin{aligned} V_+ &= \langle (\Delta X)^2 \rangle = \langle (X)^2 \rangle \\ V_- &= \langle (\Delta Y)^2 \rangle = \langle (Y)^2 \rangle \end{aligned} \tag{32}$$

Now, we can use equation 32 in 31 to obtain a familiar looking equation

$$V = V_+ cos^2\theta + V_- sin^2\theta \tag{33}$$

Thus by varying the phase of the LO (θ) we can observe the variance in either the X or Y quadrature. This also means that if we have phase noise (small deviations in θ from the optimal value) the measured squeezing will be lower since the anti-squeezing quadrature is larger. The above equations are for the lossless case.

5 Incorporation of losses

Optical loss is detrimental to squeezed states. Some examples are incoupling (η_{in}) to the OPA, outcoupling from the OPA (η_{out}), propagation efficiency in the squeezer (η_{prop}), and readout efficiency (η_{PD}) in the homodyne detector.

$$V_{out} = \eta_i V_{in} + 1 - \eta_i \tag{34}$$

Where η_i is the efficiency that is under consideration. If we want to consider multiple losses at the same time, the term η_i can be modified to include all of them as $\eta_i = \eta_{PD}\eta_{prop}\eta_{out}$.

Note that the incoupling efficiency (η_{in}) will not reduce squeezing in the same way, since it occurs before the squeezed light is generated. However, it will reduce the input SHG power by $P_{SHG} \rightarrow \eta_{in} P_{SHG}$, which will reduce the total squeezing as well.

References

[1] Robert W Boyd. *Nonlinear Optics 4e*. Nikki Levy, 2020.

[2] Malvin Carl Teich Bahaa E. A. Saleh. *Fundamentals of Photonics*. John Wiley and Sons, Ltd, 1991. ISBN: 9780471213741.