Quantum Connections in Sweden-16
Summer School

# Less is more:

## The power of vacuum quantum fluctuations

Qing-Dong Jiang

# Zero-point Energy

Planck Blackbody Radiation

$$E = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1} \cong k_B T - \frac{1}{2}\hbar\omega$$

Einstein, Stern 1913’

- Einstein and Stern concluded that “the existence of zero-point energy $\hbar\omega/2$ is probable”

# Zero-point Energy

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Einstein, Stern 1913'

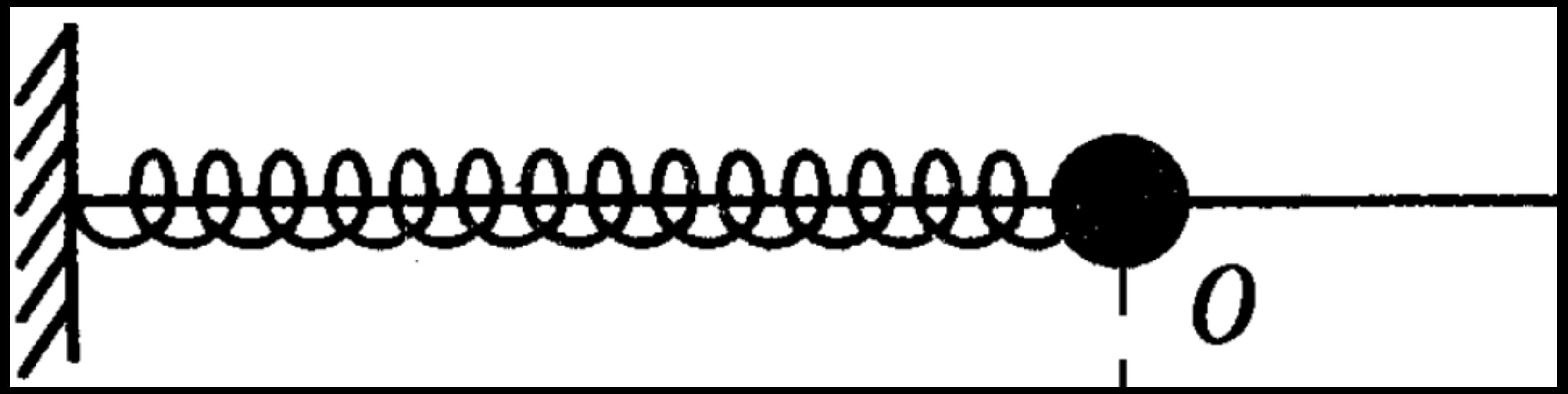
- Einstein and Stern concluded that "the existence of zero-point energy $\hbar\omega/2$ is probable"
- A bit later, however, Einstein wrote to Ehrenfest that zero-point energy is "dead as a doornail"

# Zero-point Energy

$$H_{\mathrm{HO}} = \frac{1}{2}\left(\frac{p^2}{m} + m\Omega^2 x^2\right)$$
$$= \frac{m}{2}(\dot{x}^2 + \Omega^2 x^2)$$

Ground state root-mean-square

$$\Delta x = (\hbar/2\Omega m)^{1/2}$$



# Zero-point Energy

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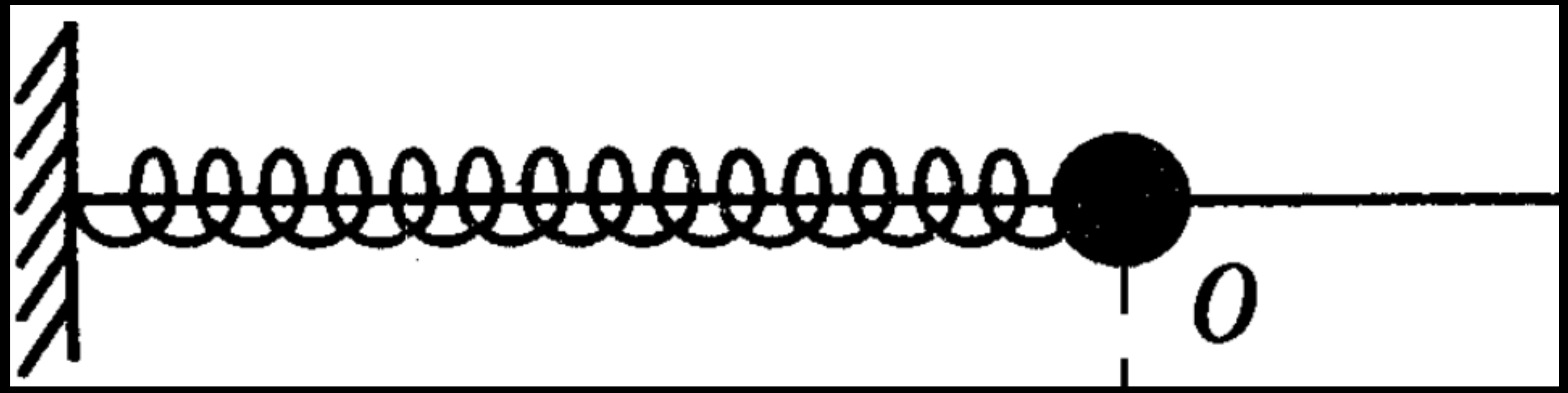



$$E_N = (N + \frac{1}{2})\hbar\Omega$$

**First experimental evidence by Mulliken 1924'** Nature 114, 349 (1924)

$B^{10}O^{16}$

$$\nu_A(v', v'') = \nu_e + \omega_e'\left(v' + \frac{1}{2}\right) - \omega_e''\left(v'' + \frac{1}{2}\right)$$

$B^{11}O^{16}$

$$\nu_B(v', v'') = \nu_e + \rho\omega_e'\left(v' + \frac{1}{2}\right) - \rho\omega_e''\left(v'' + \frac{1}{2}\right)$$

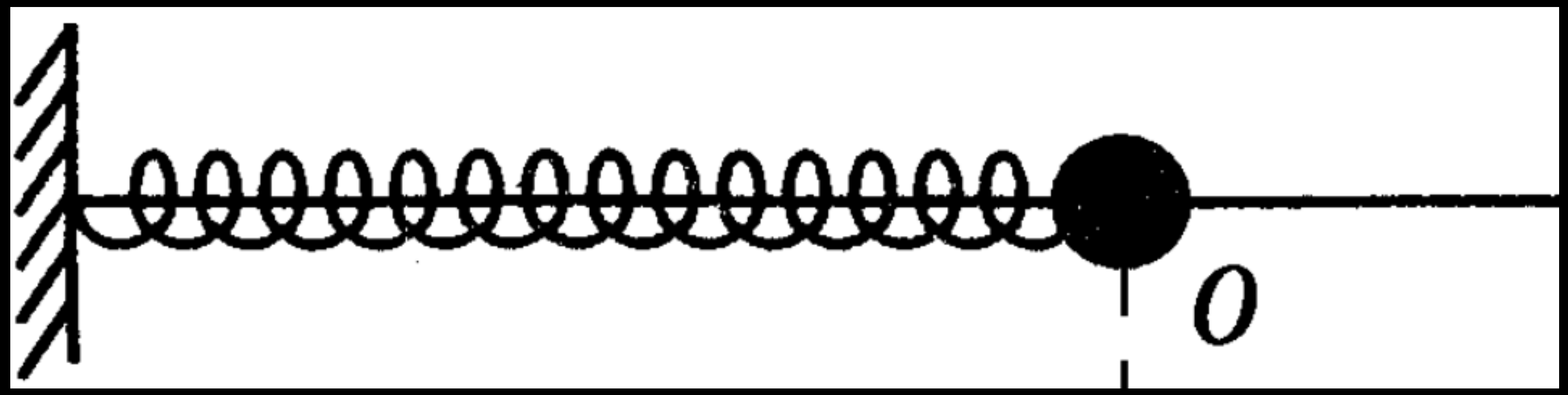
$$\rho = \sqrt{m_A/m_B}$$

# Vacuum Quantum Fluctuations

$$H_{\mathrm{HO}} = \frac{1}{2}\left(\frac{p^2}{m} + m\Omega^2 x^2\right) = \frac{m}{2}(\dot{x}^2 + \Omega^2 x^2)$$

Ground state root-mean-square

$$\Delta x = (\hbar/2\Omega m)^{1/2}$$



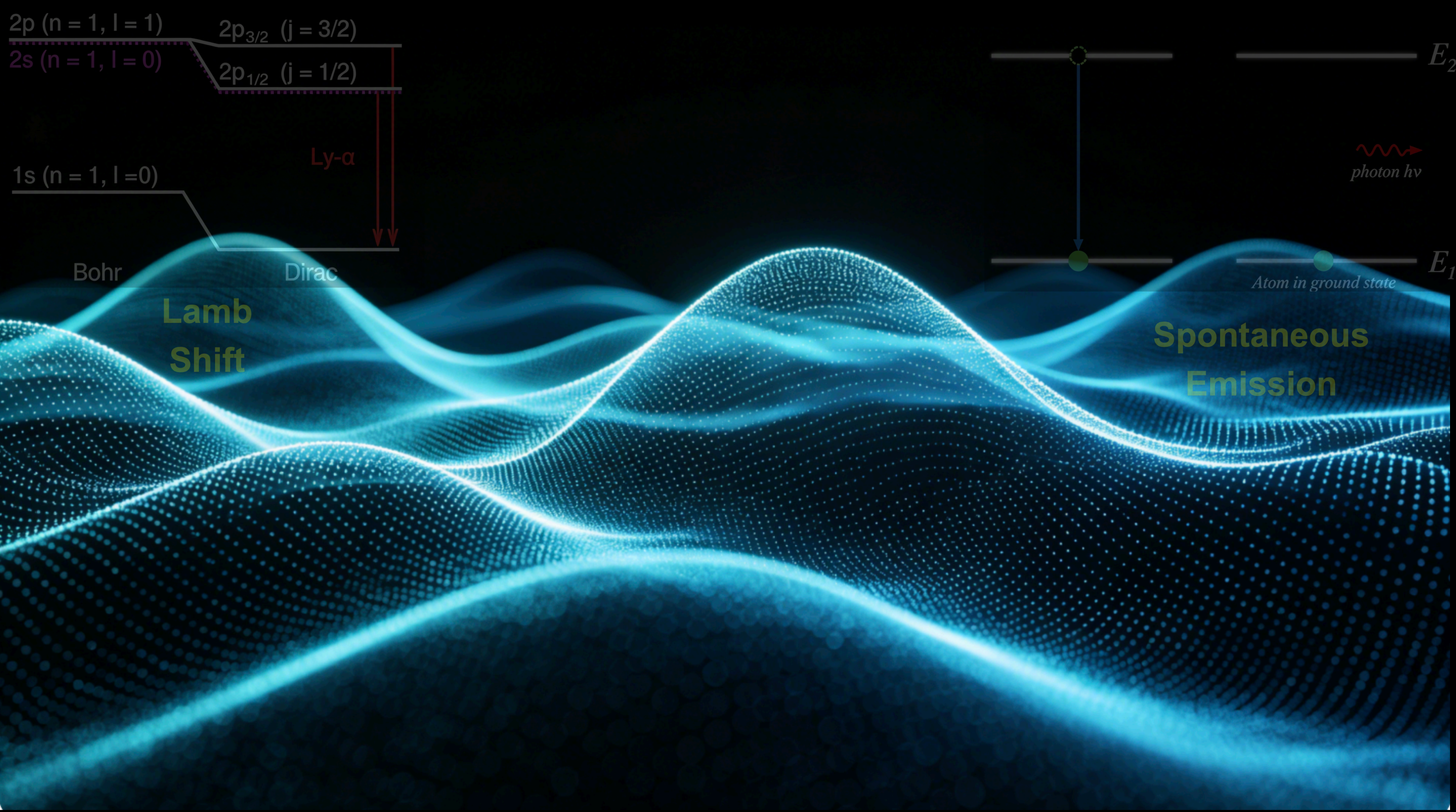

$$H_{\mathrm{RF}} = \frac{\varepsilon_0 V}{2}(E^2 + c^2 B^2) = \frac{\varepsilon_0 V}{2}(\dot{A}^2 + c^2 |\mathbf{k} \times \mathbf{A}|^2)$$

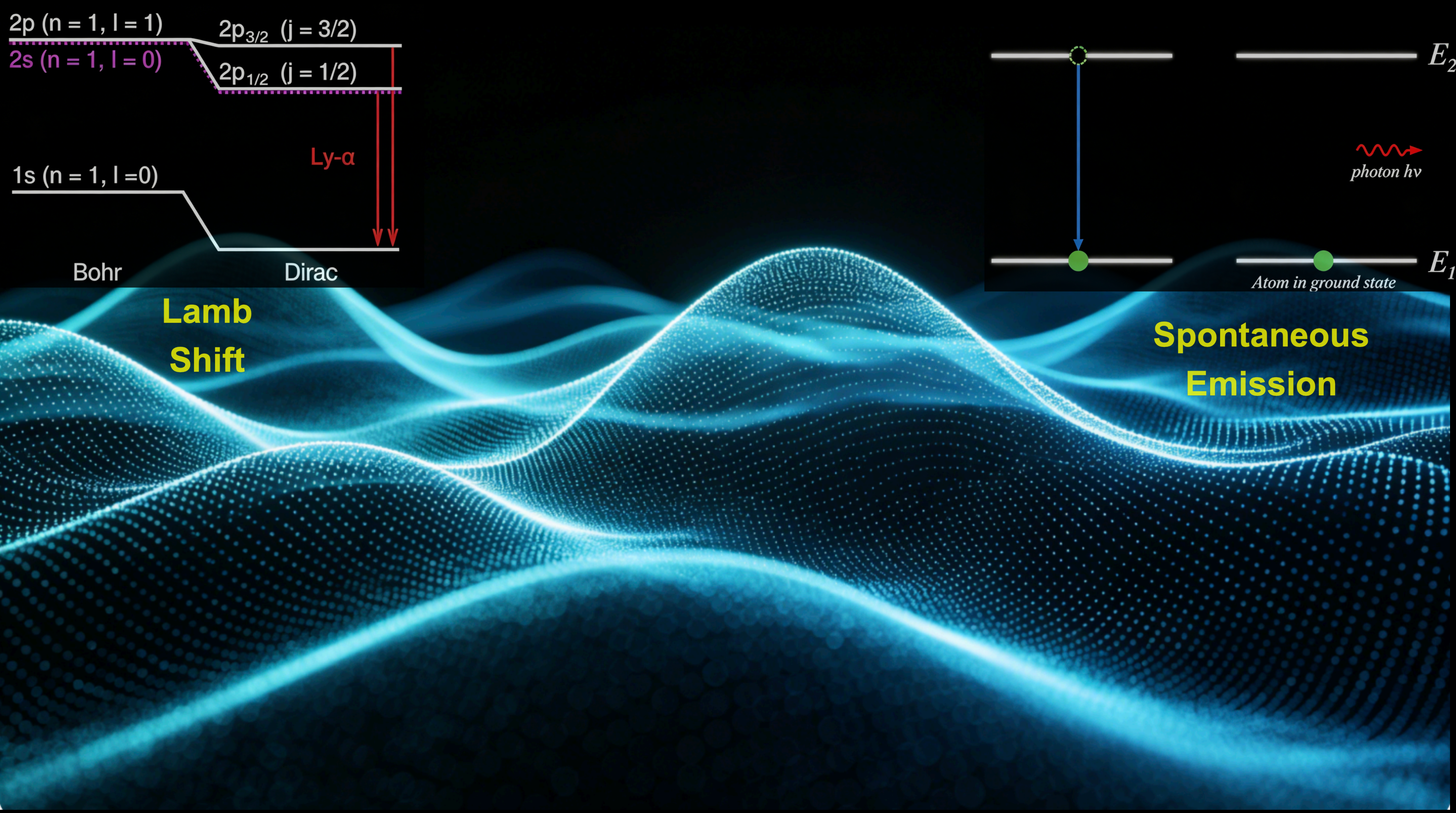
$$m \rightarrow \varepsilon_0 V \qquad \Omega \rightarrow c\mathrm{k}$$

$$\Delta A = (\hbar/2\Omega\varepsilon_0 V)^{1/2}$$

Benea-Chelmus et al., Nature, 568, 202 (2019)

Riek et al., Science, 350, 420 (2015)


2p (n = 1, l = 1)
2s (n = 1, l = 0)
2p3/2 (j = 3/2)
2p1/2 (j = 1/2)
Ly-α
1s (n = 1, l =0)
Bohr
Dirac
Lamb Shift
E2
E1
photon hν
Atom in ground state
Spontaneous Emission


2p (n = 1, l = 1)
2s (n = 1, l = 0)
$2p_{3/2}$ (j = 3/2)
$2p_{1/2}$ (j = 1/2)
Ly-α
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$E_1$
photon hv
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## Lecture 1: Renormalization and Casimir Physics

## Lecture 2: Casimir Spectra, Torque, Friction

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## Lecture 1: Renormalization and Casimir Physics

## Lecture 2: Casimir Spectra, Torque, Friction

## Lecture 3: Quantum atmosphere

## Lecture 4: Cavity Quantum Materials

## The Goal:

1. Have an Overview of Casimir Physics

2. Learn new development of cavity materials

3. Learn to calculate Casimir effects

4. Learn to calculate Cavity quantum materials

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# Lecture 1: Renormalization and Casimir Physics

Qing-Dong Jiang

Tsung-Dao Lee Institute, Shanghai Jiao Tong University

# Dimensional Analysis

## Standard Units

## Gaussian Units

**Reference Frame**

**On Absolute Units, I: Choices**

Frank Wilczek

# Dimensional Analysis

**Reference Frame**

**On Absolute Units, I: Choices**

Frank Wilczek

**Natural Units:** $\hbar = c = 1$

$$E = mc^2 = pc$$

$$[L] = [\hbar]/[p] \quad [T] = [c]/[L]$$

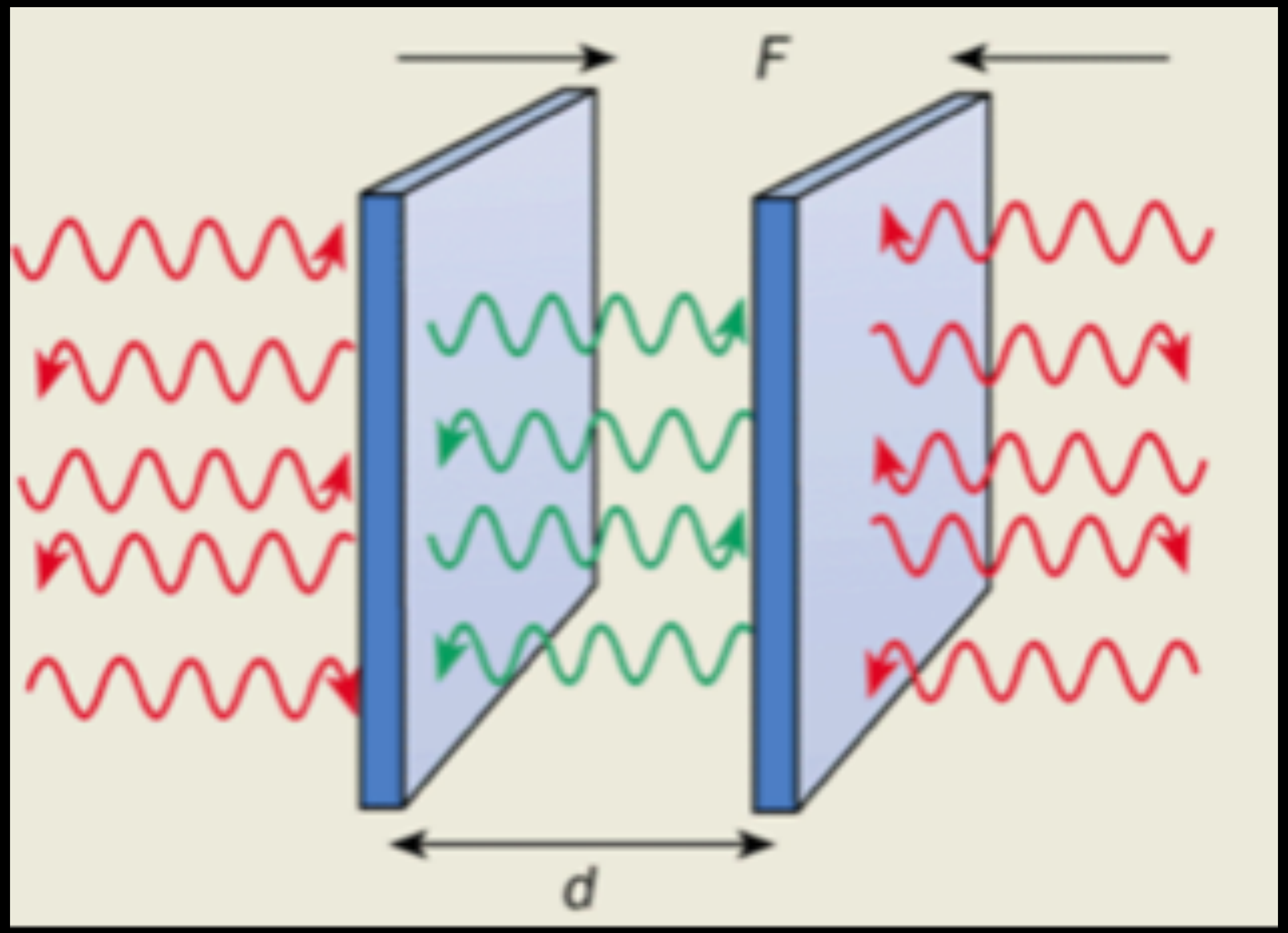
$$[E] = [M] = [p] = 1/[L] = [T]^{-1}$$

Time passed $3 \times 10^8 \; m$

Energy $(200 \; nm)^{-1}$

# I. What is Casimir force?

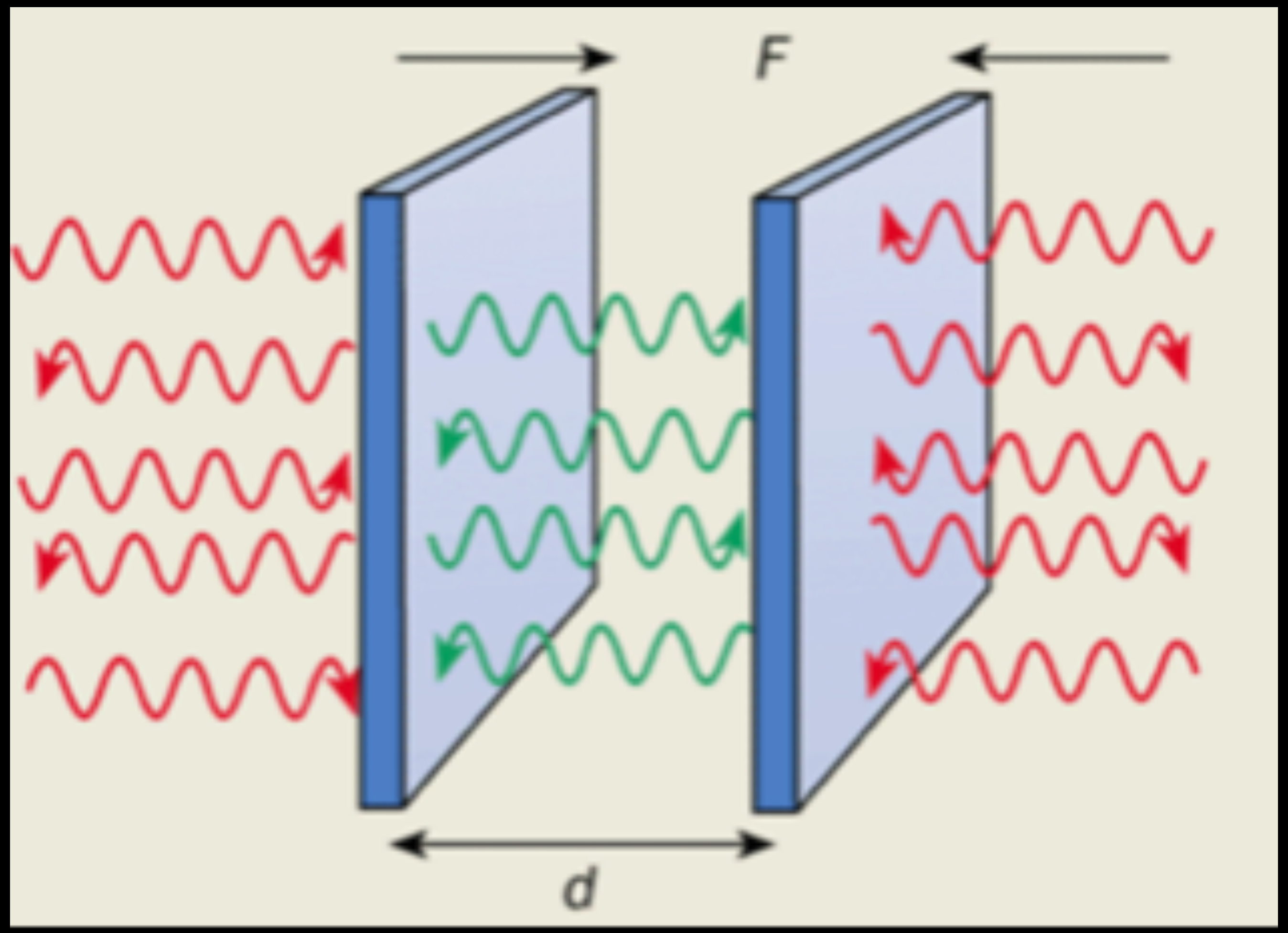
$$F = -\frac{\pi^2 \hbar c}{240} \frac{A}{d^n}$$




Casimir 1948

# I. What is Casimir force?

$$F = -\frac{\pi^2 \hbar c}{240} \frac{A}{d^n}$$

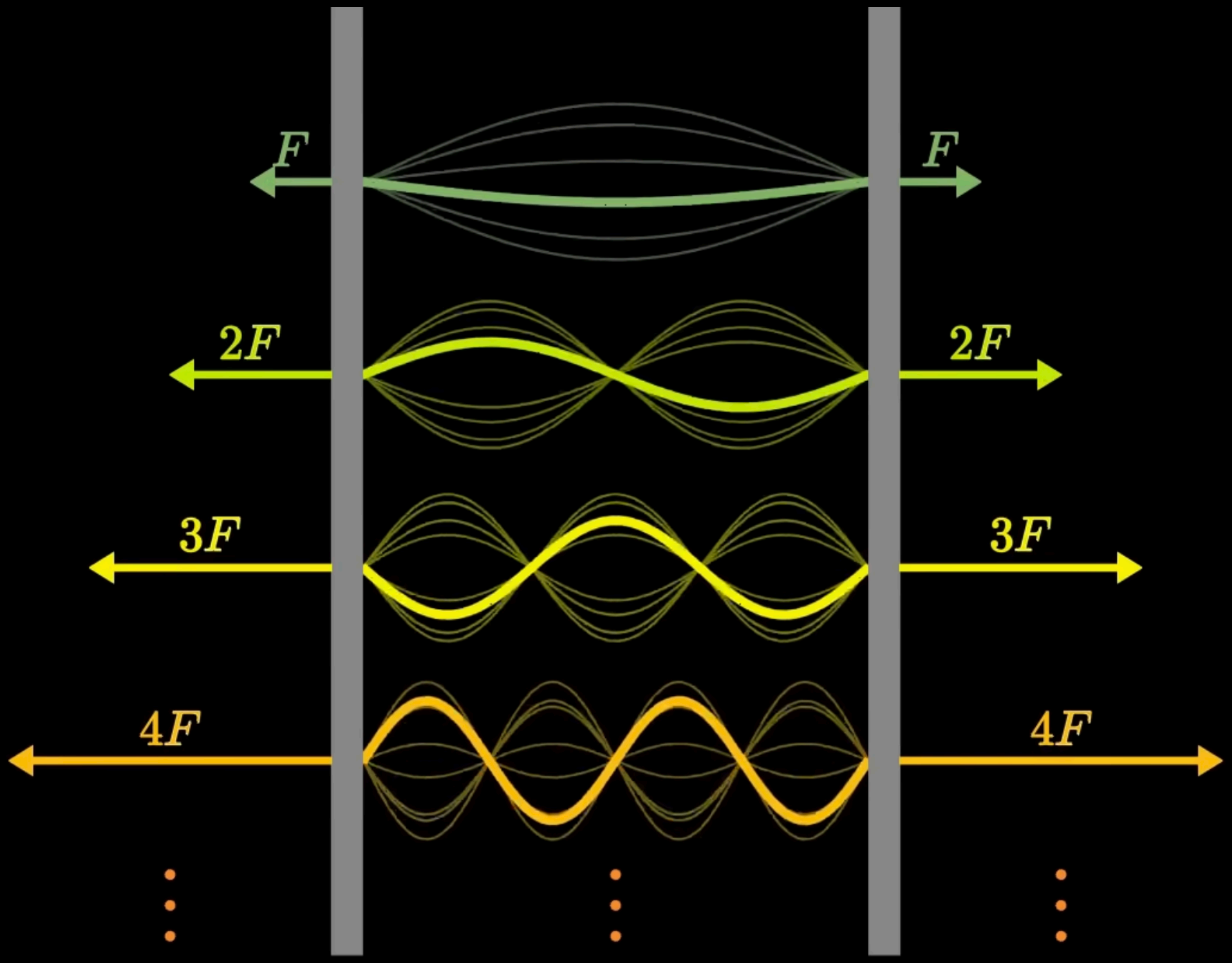



Casimir 1948

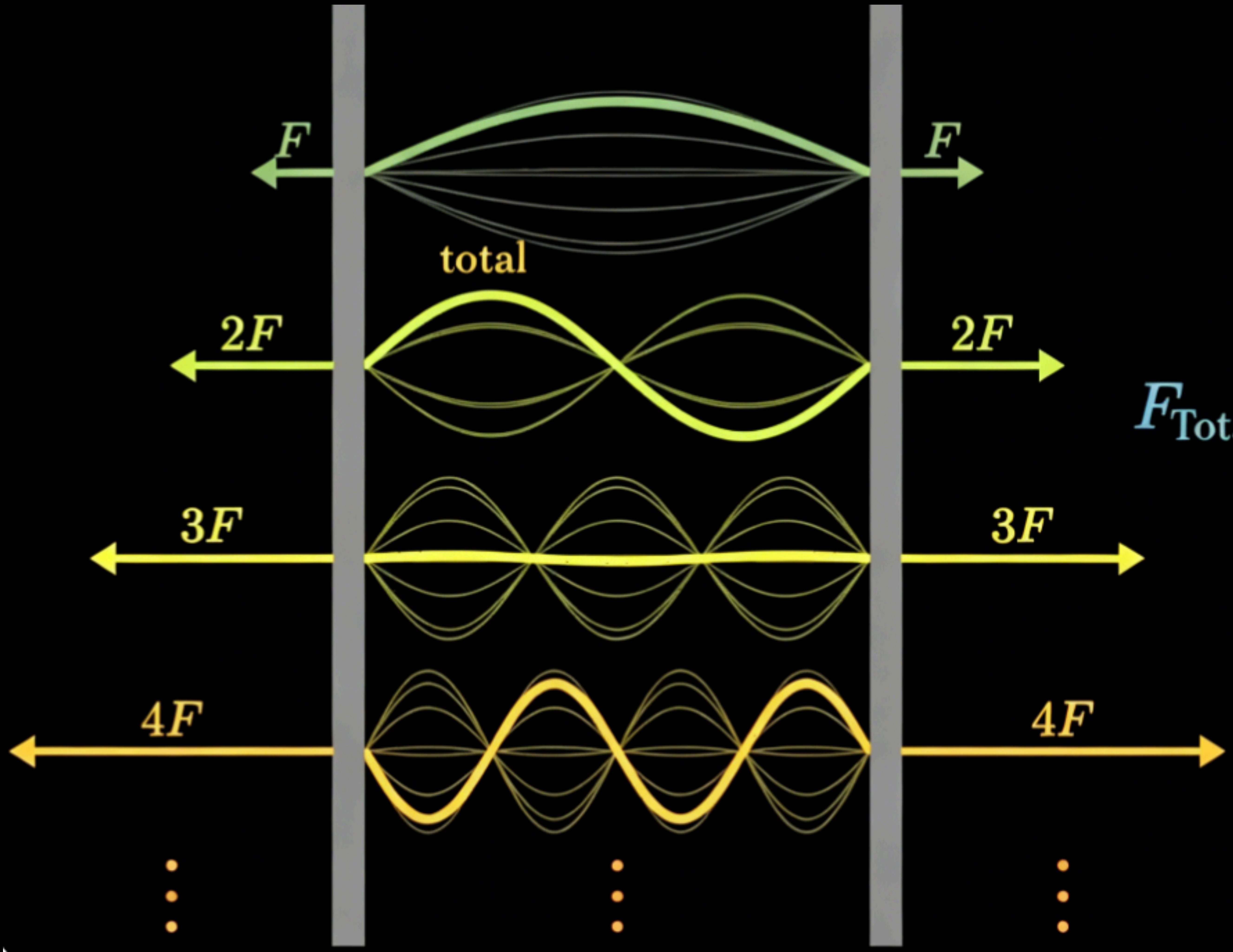
Two questions: Why attractive?

What is n? Dimensional analysis $n = 4$

# How do we understand? A paradox



# How do we understand? A paradox



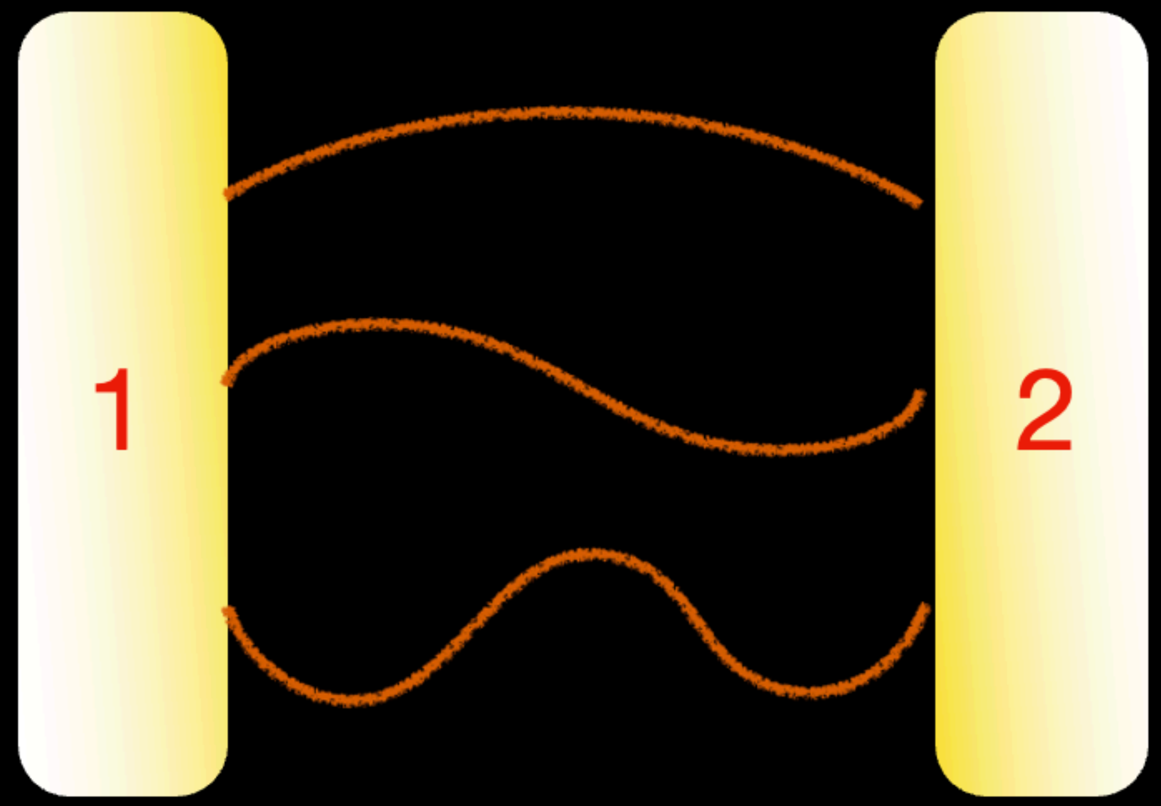

$$F_{\text{Total}} = F + 2F + 3F + 4F + \cdots$$

$$= F(1 + 2 + 3 + 4 + \cdots)$$

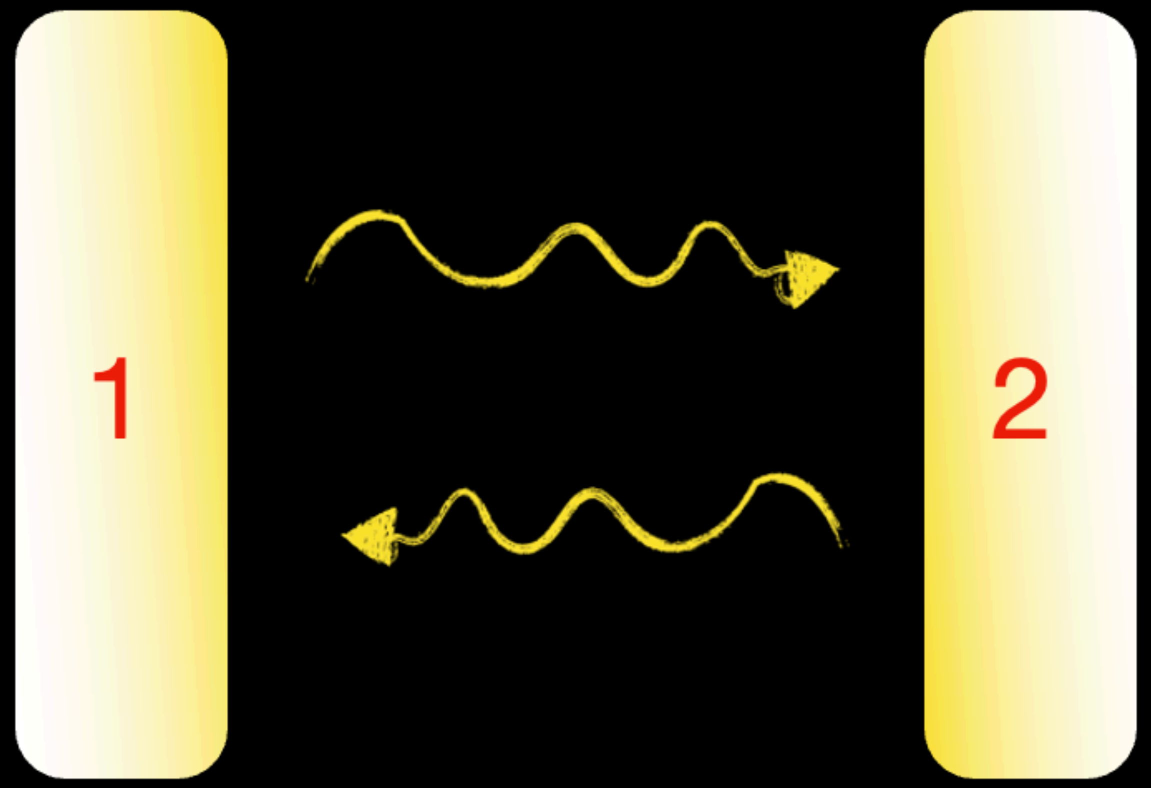
$+\infty$ Or $-\dfrac{1}{12}$

# Casimir forces: Lifshitz formula

Counting modes (for simple geometry)

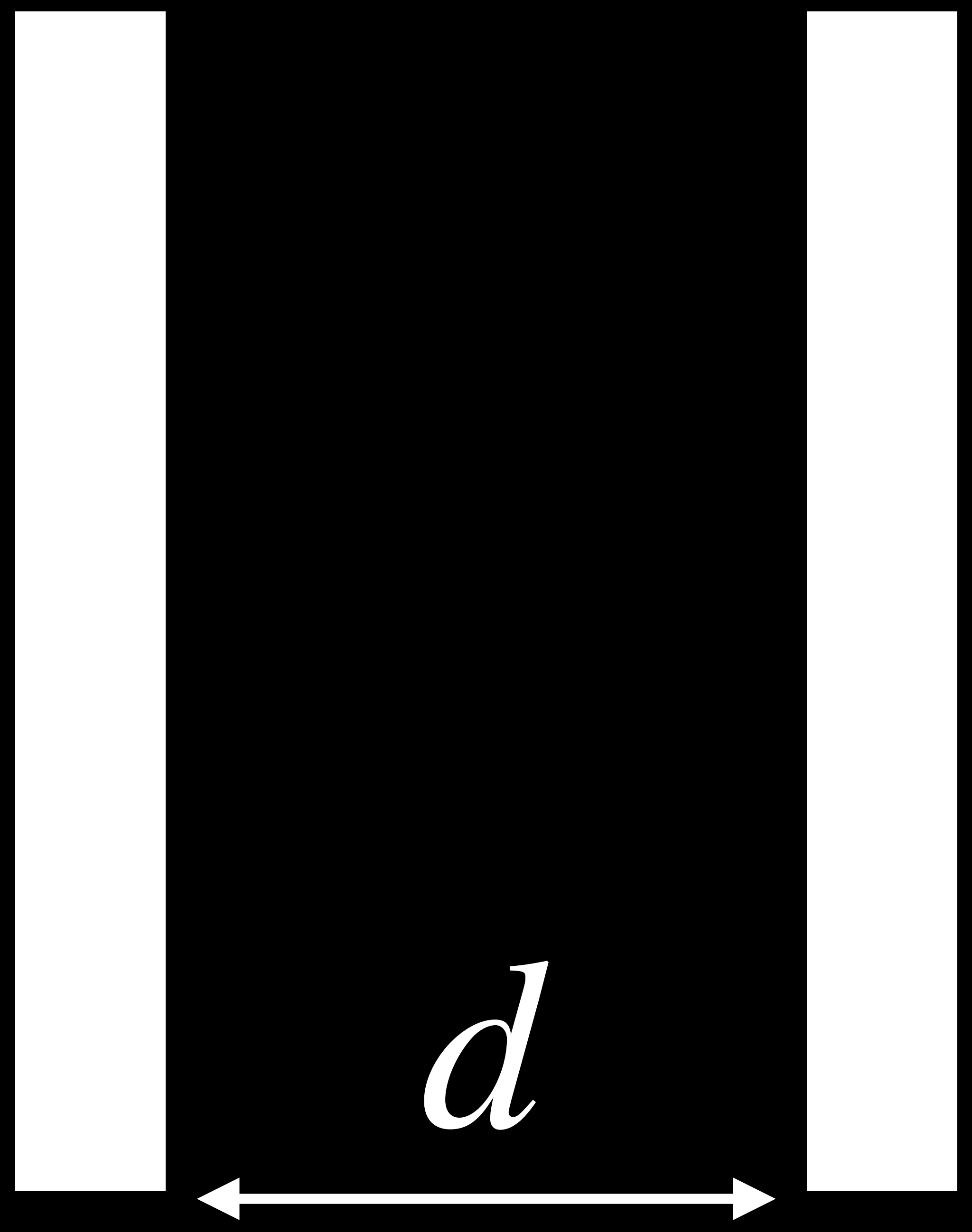



Quantum field theory (Lifshitz 1958')



# Counting modes: 1 d case

$$E_T = E(d) + E(L-d)$$

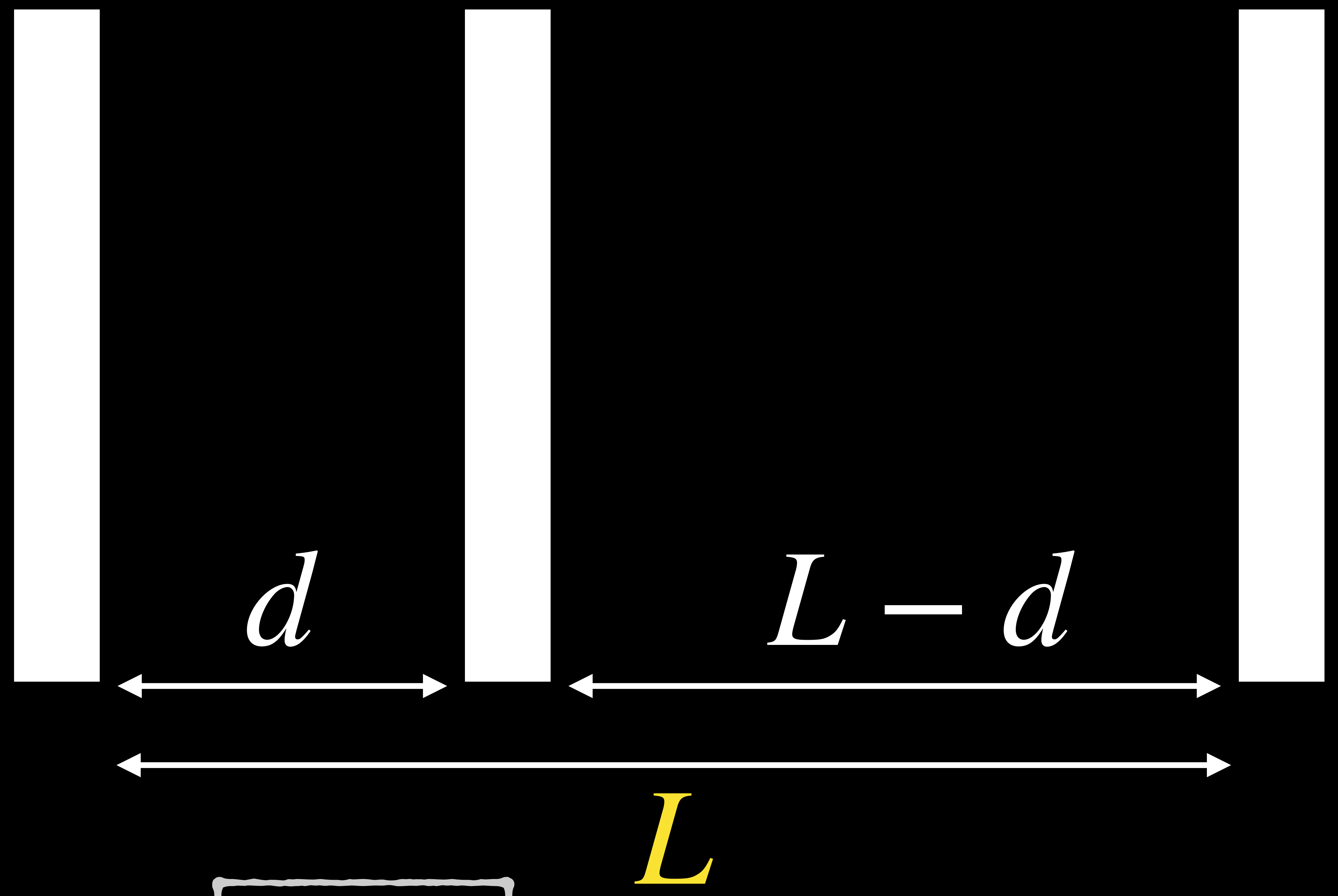



$$E(d) = \frac{\hbar}{2}\sum_n \omega_n = \frac{\pi\hbar c}{2}\sum_n \frac{n}{d}$$

$$E(d) = \frac{\pi\hbar c}{2}\sum_n \frac{n}{d}e^{-\alpha n/d} = \frac{\pi\hbar c}{2d}\frac{e^{-\alpha/d}}{(1-e^{-\alpha/d})^2} = \frac{\pi\hbar c}{2d}\left[\frac{d^2}{\alpha^2} - \frac{1}{12} + \frac{1}{240}\left(\frac{\alpha}{d}\right)^2 + \dots\right]$$

# Counting modes: 1 d case

$$E_T = E(d) + E(L - d)$$

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# Counting modes: 1 d case

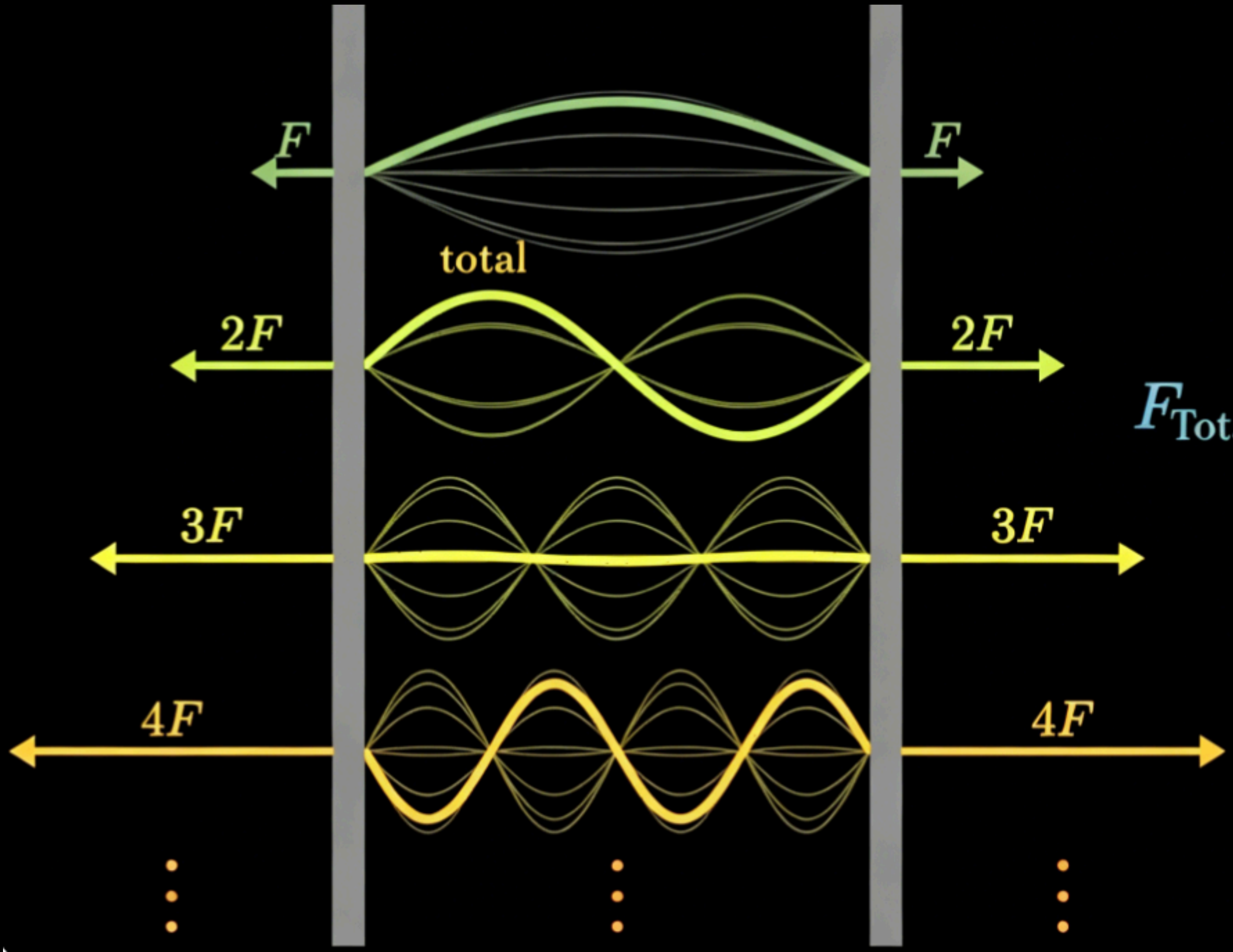
$$\alpha \to 0$$

$$F = -\left[E'(d) - E'(L-d)\right] = -\frac{\pi \hbar c}{24}\frac{1}{d^2}$$

$$\Big\downarrow \quad dim = 3$$

$$F = -\frac{\pi^2 \hbar c}{240}\frac{A}{d^4}$$

# Paradox




$$F_{\text{Total}} = F + 2F + 3F + 4F + \cdots$$

$$= F(1 + 2 + 3 + 4 + \cdots)$$

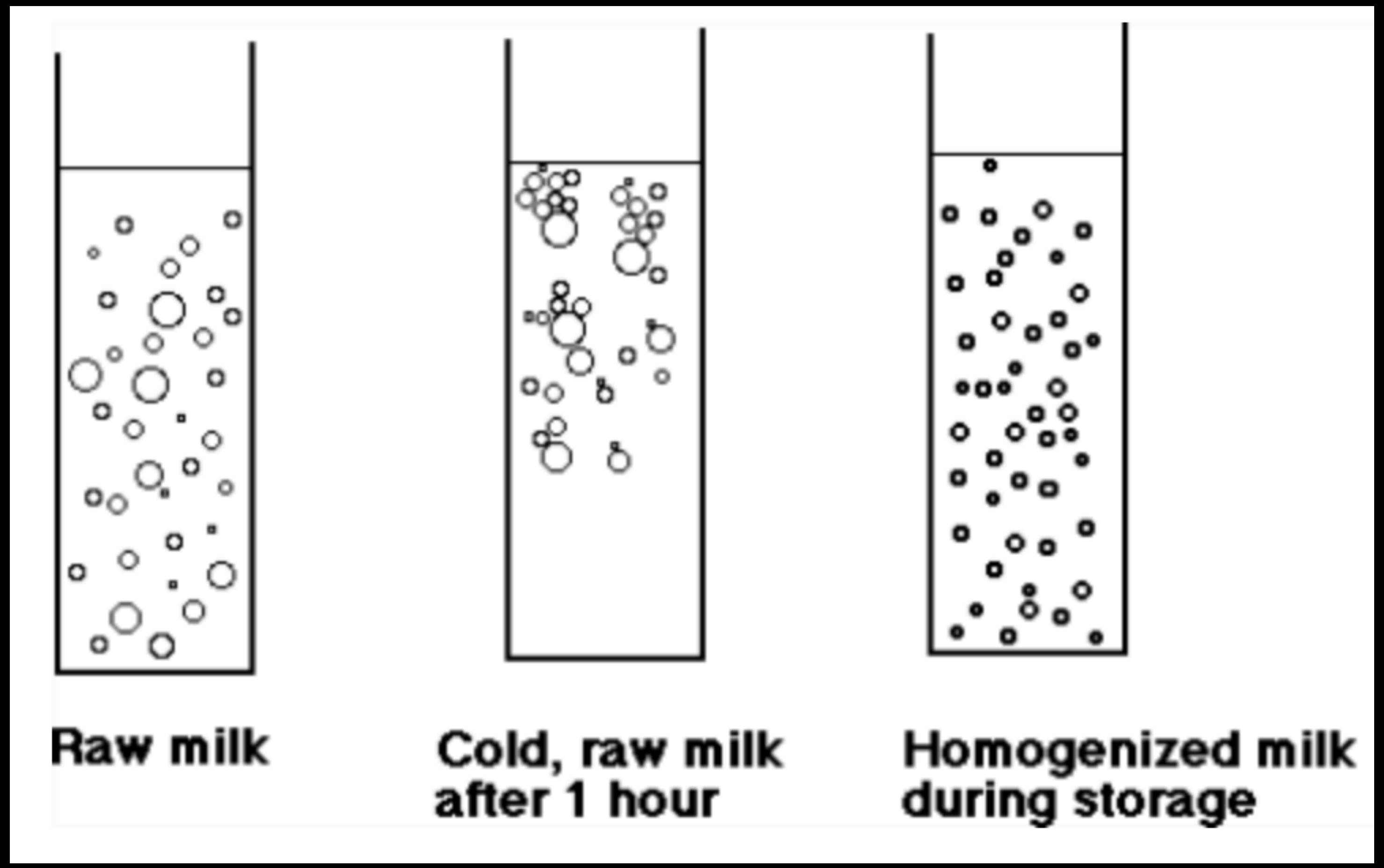
$$-\frac{1}{12}$$

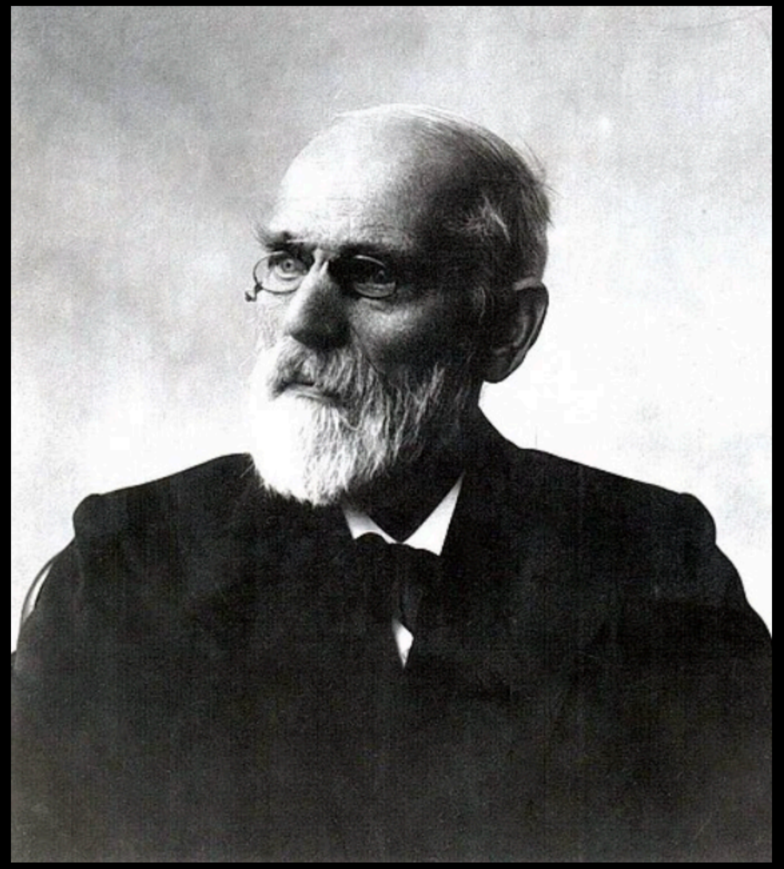
$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + \ldots = -\frac{1}{12}$$

$$1+2+3+4+5+6+7+8+\ldots=-\frac{1}{12}+\infty$$

# What led Casimir to his formula?

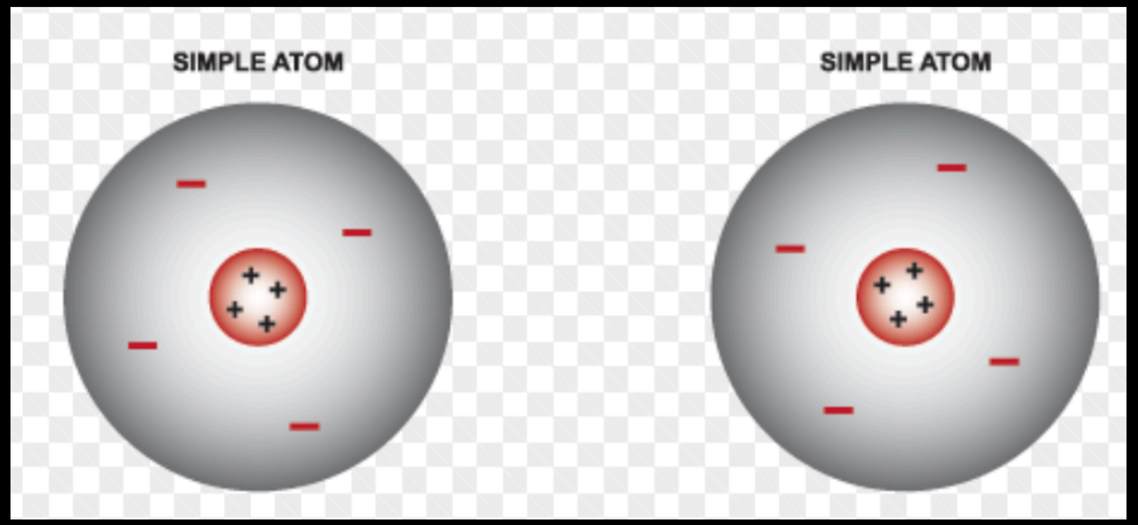
Stability of colloids: J. van der Waals (36 yrs old)



van der Waals

van der Waals force:




Type I: both with permanent electric dipole

Type II: one with permanent electric dipole

Type III: Neither with permanent electric dipole

# van der Waals force

Orientational force: $\delta E_I = -\frac{C_{Keesom}}{r^6}$

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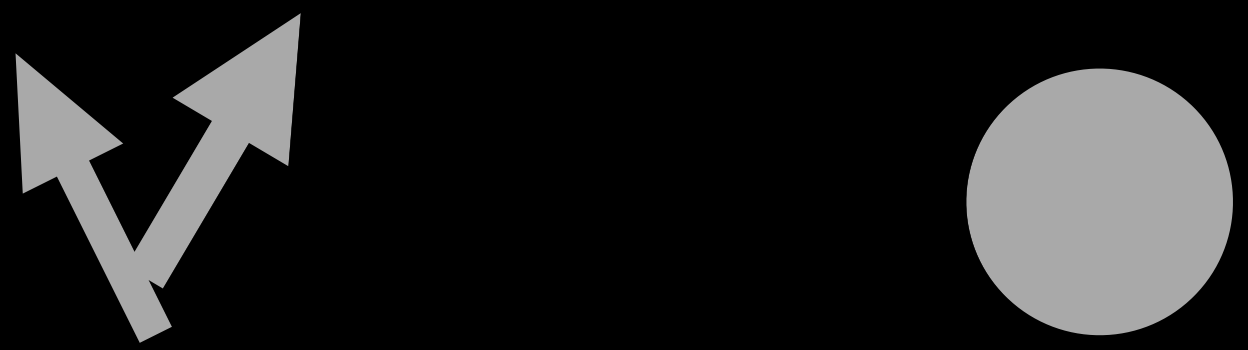
Orientational force: $$\delta E_I = -\frac{C_{Keesom}}{r^6}$$

Induction force: $$\delta E_{II} = -\frac{C_{Debye}}{r^6}$$



# van der Waals force

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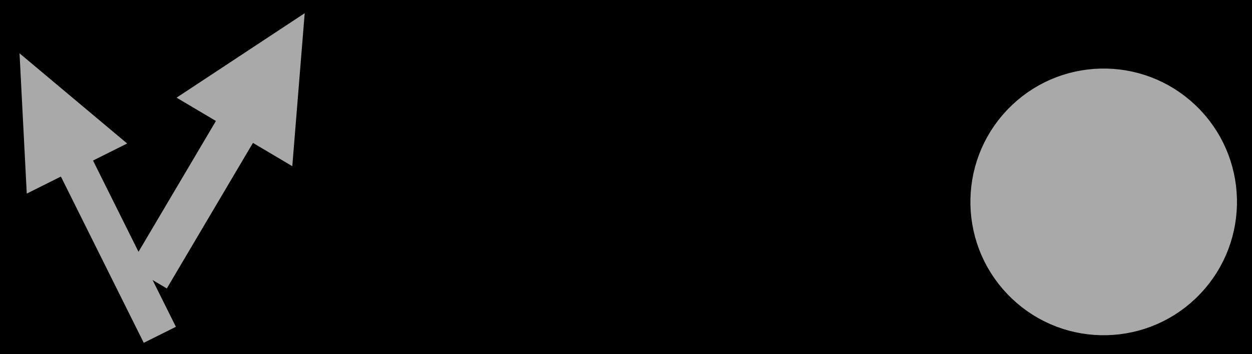
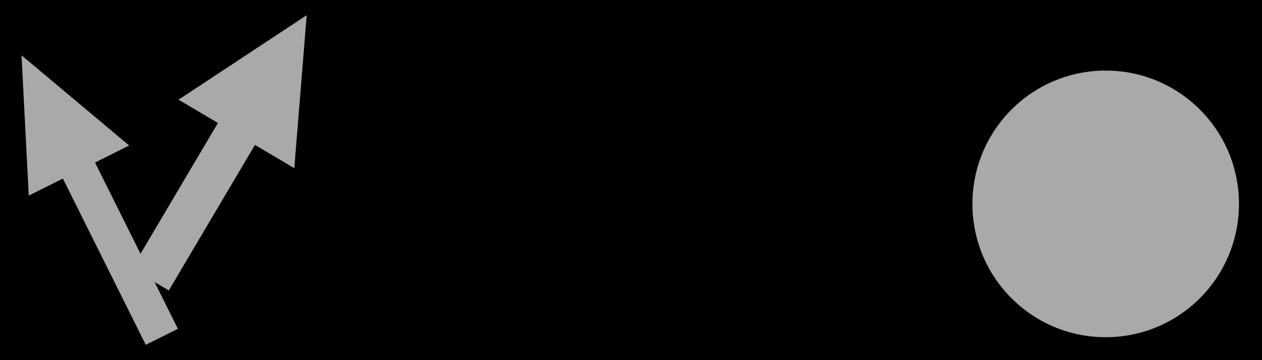


Dispersion force: $$\delta E_{III} = -\frac{C_{London}}{r^6}$$

# van der Waals force: Which is Bigger?

Orientational force: $$\delta E_I = -\frac{C_{Keesom}}{r^6}$$

Induction force: $$\delta E_{II} = -\frac{C_{Debye}}{r^6}$$



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# van der Waals force

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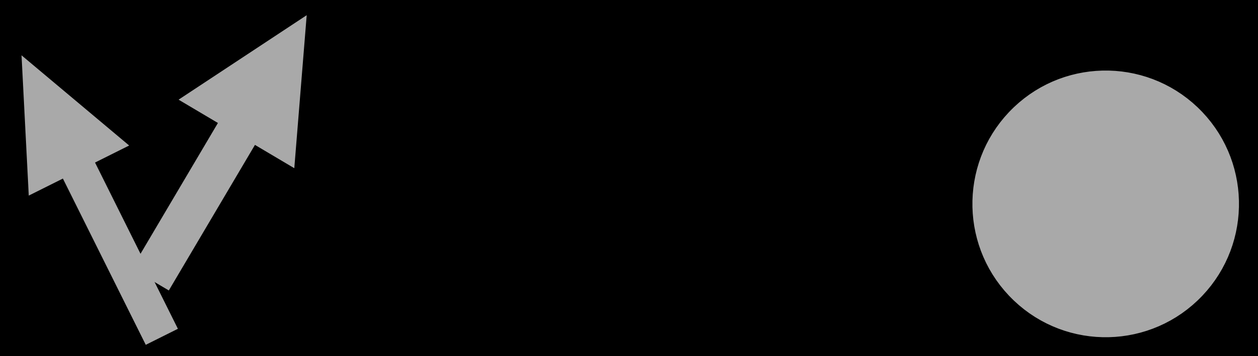


Dispersion force: $\delta E_{III} = -\frac{C_{London}}{r^6}$

# van der Waals force

**Table 6.3** Induction, Orientation, and Dispersion Free Energy Contributions to the Total Van der Waals Energy in a Vacuum for Various Pairs of Molecules at 293K

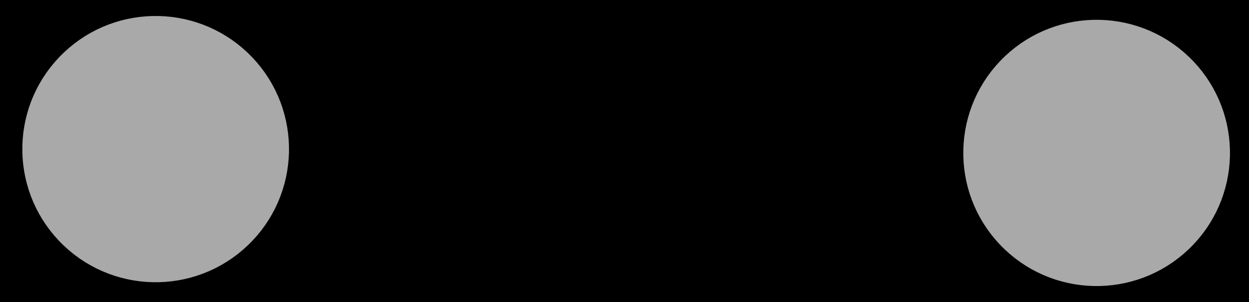
| Similar Molecules | | | | Van der Waals Energy Coefficients $C$ ($10^{-79}$ J m$^6$) | | | Total VDW Energy $C_{VDW}$ | | |
|---|---|---|---|---|---|---|---|---|---|
| Interacting Molecules | Electronic Polarizability $\frac{\alpha_0}{4\pi\varepsilon_0}$ ($10^{-30}$m$^3$) | Permanent Dipole Moment $u$ (D) [a] | Ionization Potential $I = h\nu_I$ (eV) [b] | $C_{ind}$ $\frac{2u^2\alpha_0}{(4\pi\varepsilon_0)^2}$ | $C_{orient}$ $\frac{u^4}{3kT(4\pi\varepsilon_0)^2}$ | $C_{disp}$ $\frac{3\alpha_0^2 h\nu_I}{4(4\pi\varepsilon_0)^2}$ | Theoretical Eq. (6.17) | From Gas Law Eq. (6.14) | Dispersion Energy Contribution to Total (Theoretical) (%) |
| Ne–Ne | 0.39 | 0 | 21.6 | 0 | 0 | 4 | 4 | 4 | 100 |
| $CH_4$–$CH_4$ | 2.60 | 0 | 12.6 | 0 | 0 | 102 | 102 | 101 | 100 |
| HCl–HCl | 2.63 | 1.08 | 12.7 | 6 | 11 | 106 | 123 | 157 | 86 |
| HBr–HBr | 3.61 | 0.78 | 11.6 | 4 | 3 | 182 | 189 | 207 | 96 |
| HI–HI | 5.44 | 0.38 | 10.4 | 2 | 0.2 | 370 | 372 | 350 | 99 |
| $CH_3Cl$–$CH_3Cl$ | 4.56 | 1.87 | 11.3 | 32 | 101 | 282 | 415 | 509 | 68 |
| $NH_3$–$NH_3$ | 2.26 | 1.47 | 10.2 | 10 | 38 | 63 | 111 | 162 | 57 |
| $H_2O$–$H_2O$ | 1.48 | 1.85 | 12.6 | 10 | 96 | 33 | 139 | 175 | 24 |
| **Dissimilar Molecules** | | | | $\frac{u_1^2\alpha_{02}+u_2^2\alpha_{01}}{(4\pi\varepsilon_0)^2}$ | $\frac{u_1^2u_2^2}{3kT(4\pi\varepsilon_0)^2}$ | $\frac{3\alpha_{01}\alpha_{02}h\nu_1\nu_2}{2(4\pi\varepsilon_0)^2(\nu_1+\nu_2)}$ | | | |
| Ne–$CH_4$ | | | | 0 | 0 | 19 | 19[c] | — | 100 |
| HCl–HI | | | | 7 | 1 | 197 | 205 | — | 96 |
| $H_2O$–Ne | | | | 1 | 0 | 11 | 12 | — | 92 |
| $H_2O$–$CH_4$ | | | | 9 | 0 | 58 | 67 | — | 87 |

[a] 1 D = 3.336 × $10^{-30}$ Cm.
[b] 1 eV = 1.602 × $10^{-19}$ J.
[c] This approximate value may be compared with the ab initio calculation by Fowler et al., (1989) that gives 23 × $10^{-79}$ J m$^6$.

hydrophobic effect

# van der Waals force

Dispersion force: $$\delta E_{III} = -\frac{C_{London}}{r^6}$$



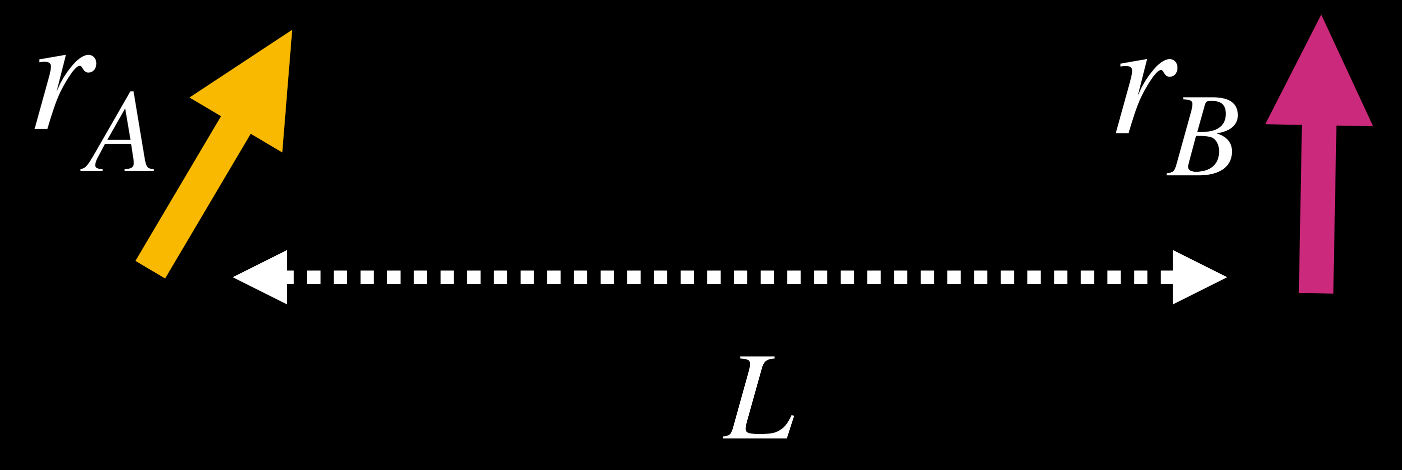
**Why it is always attractive?**

**Why it is called dispersion force?**

**Second order perturbation!**

# Van der Waals force

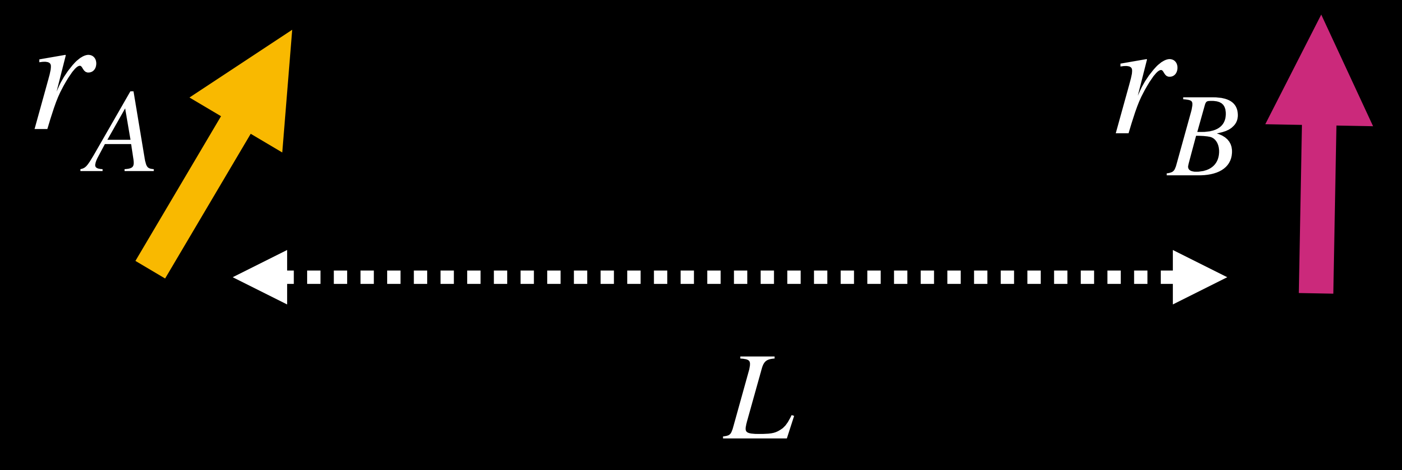
$$\hat{H} = \hat{H}_A + \hat{H}_B + \delta\hat{V}_{int}$$




$$\hat{H}_0 = \hat{H}_A + \hat{H}_B \qquad \delta\hat{V}_{int} = \frac{\hat{\vec{r}}_A \cdot \hat{\vec{r}}_B - 3(\hat{\vec{r}}_A \cdot \hat{L})(\hat{\vec{r}}_B \cdot \hat{L})}{L^3}$$

# Van der Waals force

$$\hat{H} = \hat{H}_A + \hat{H}_B + \delta\hat{V}_{int}$$




$$\hat{H}_0 = \hat{H}_A + \hat{H}_B \qquad \delta\hat{V}_{int} = \frac{\vec{\hat{r}}_A \cdot \vec{\hat{r}}_B - 3(\vec{\hat{r}}_A \cdot \hat{L})(\vec{\hat{r}}_B \cdot \hat{L})}{L^3}$$

$$E_0 = E_{A0} + E_{B0}$$

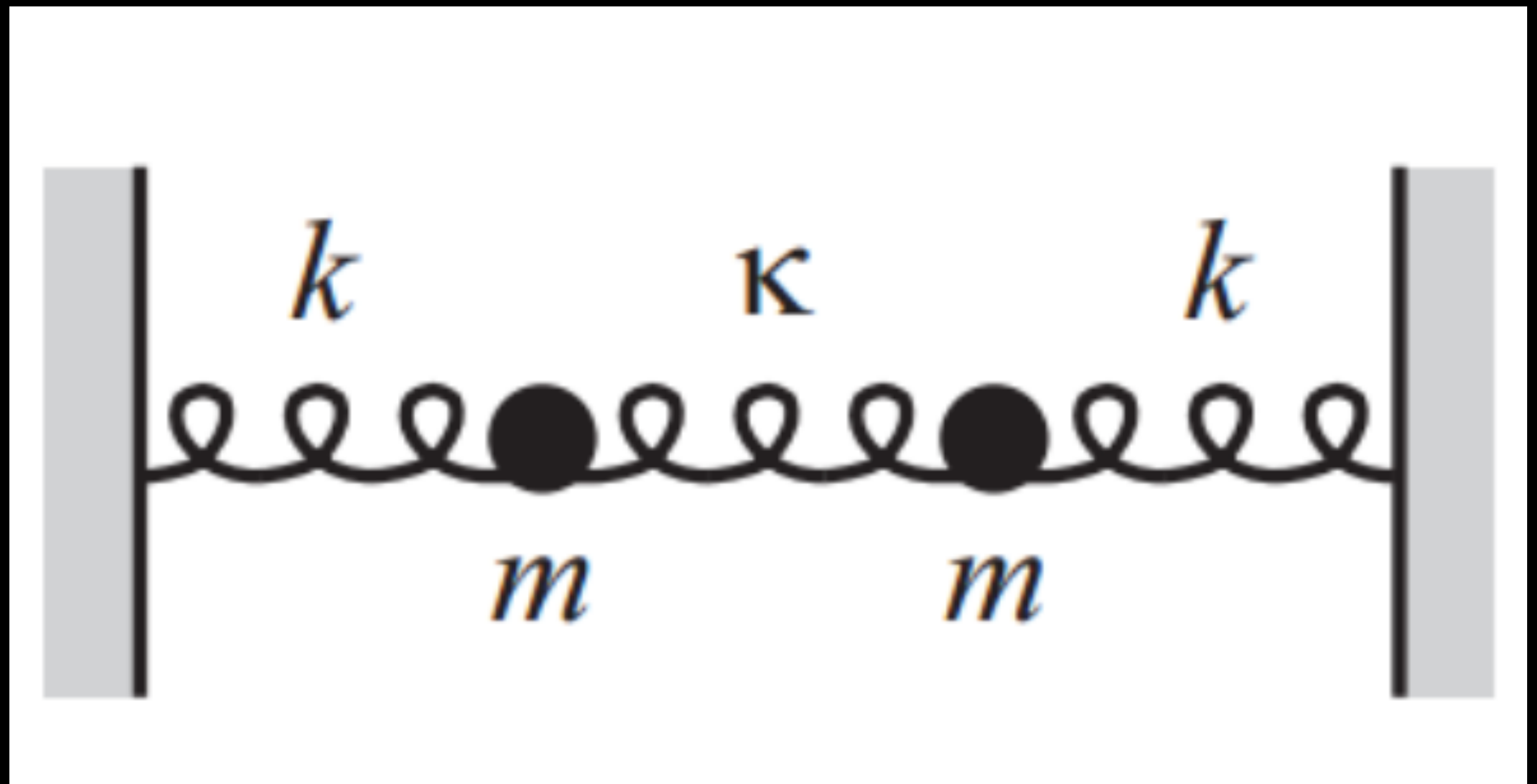
$$|G\rangle = |g_A\rangle \otimes |g_B\rangle$$

$$\delta E_0^{(1)} = \langle G|\delta\hat{V}_{int}|G\rangle = 0$$

$$\delta E_0^{(2)} = -\sum_k \frac{|\langle k|\delta\hat{V}_{int}|G\rangle|^2}{E_k - E_0} < 0$$

# Coupled harmonic oscillator

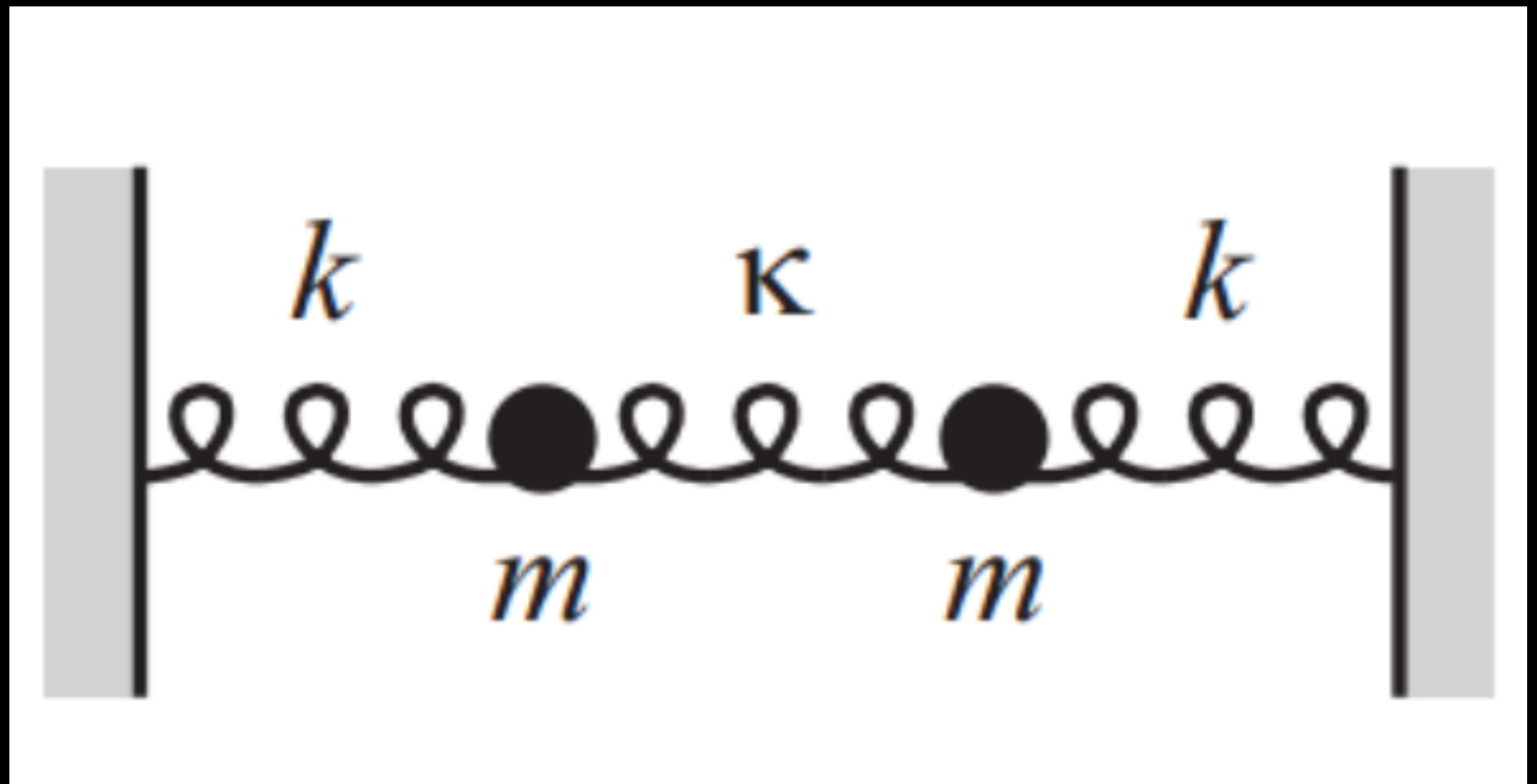
$$\hat{H} = \frac{p_A^2}{2m} + \frac{1}{2}m\omega_0 x_A^2 + \frac{p_B^2}{2m} + \frac{1}{2}m\omega_0 x_B^2 + \kappa \hat{x}_A \hat{x}_B$$




$$\kappa \propto 1/L^3$$

$$\omega_\pm = \sqrt{\omega_0^2 \pm \kappa^2}$$

# Coupled harmonic oscillator



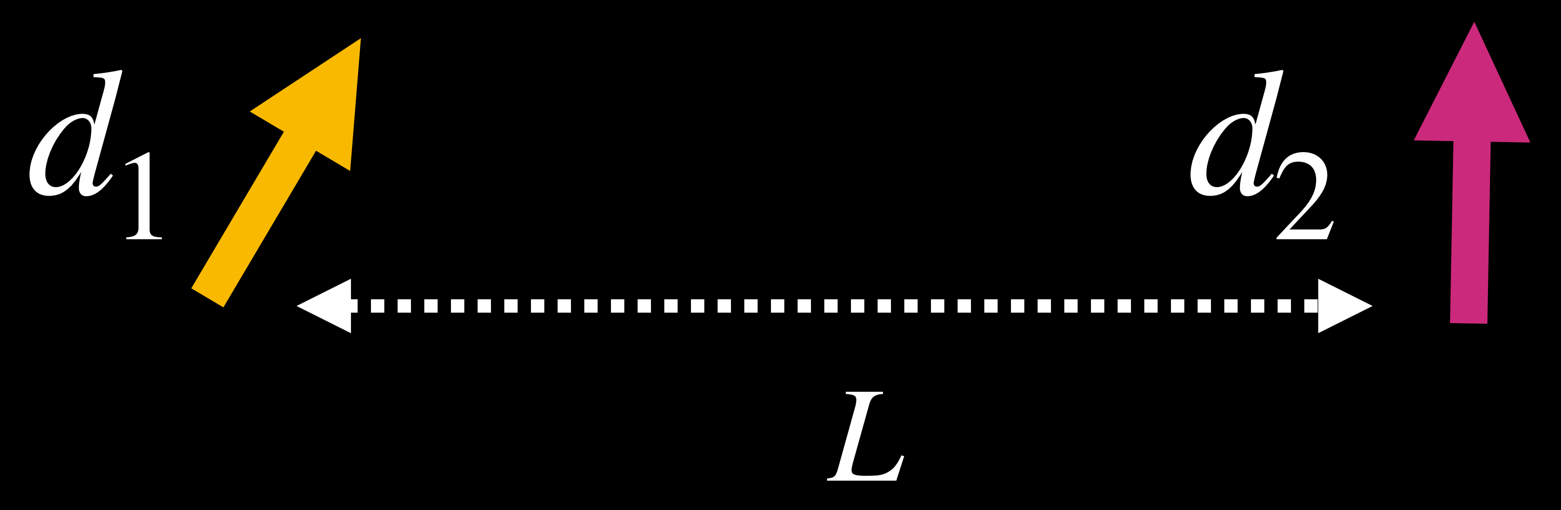

$$\hat{H} = \frac{p_A^2}{2m} + \frac{1}{2}m\omega_0 x_A^2 + \frac{p_B^2}{2m} + \frac{1}{2}m\omega_0 x_B^2 + \kappa \hat{x}_A \hat{x}_B$$

$$\kappa \propto 1/L^3$$

$$\omega_\pm = \sqrt{\omega_0^2 \pm \kappa^2}$$

$$\delta E_G = \hbar(\omega_+ + \omega_- - 2\omega_0) \propto -\kappa^2 \sim -\frac{1}{L^6}$$

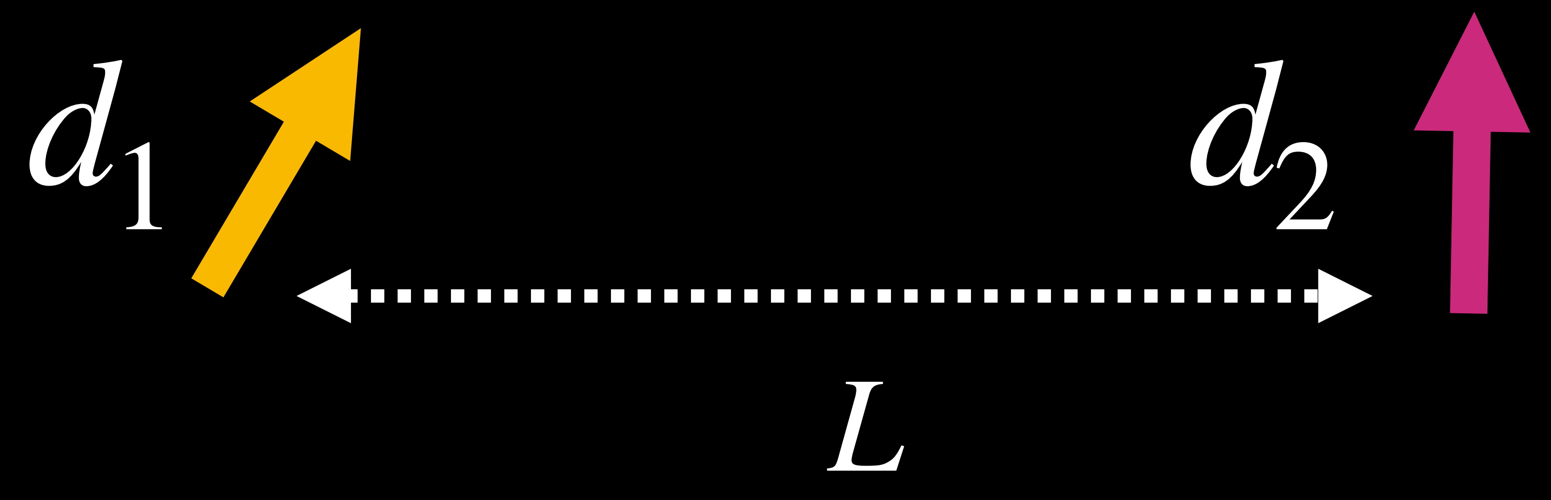
# QED approach: field theory




$H_{int} \sim d_2 \, G_{21} \, d_1$

$d_2 \propto E_2 \, \alpha_2 \propto d_1 G_{12} \alpha_2$

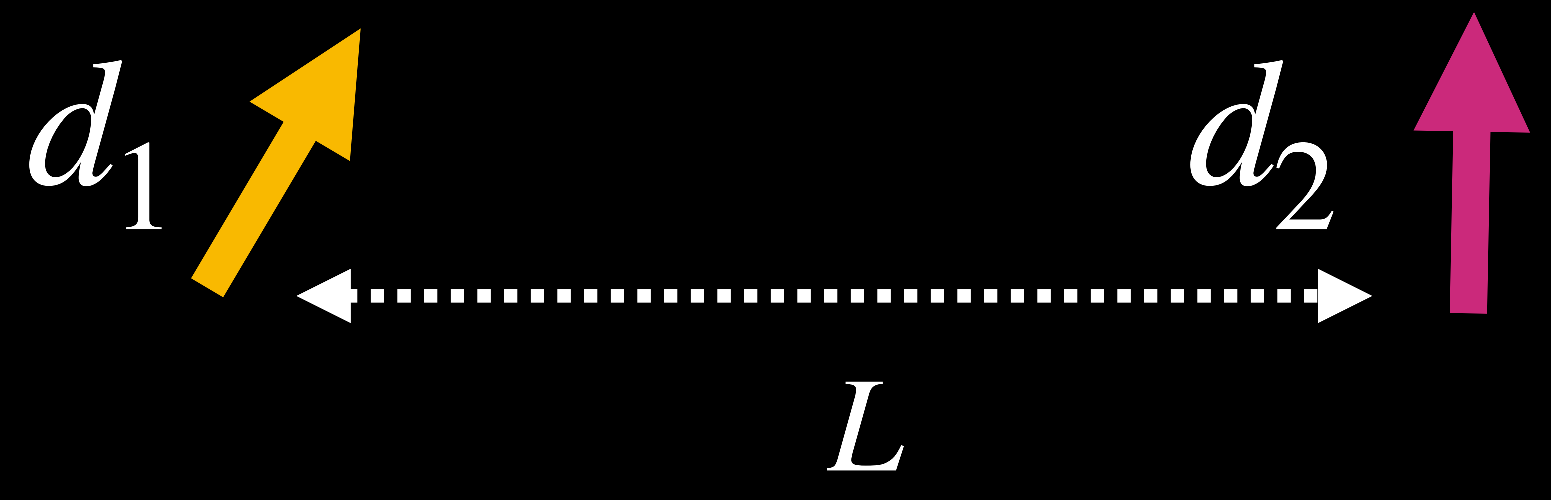
# QED approach: field theory




$$H_{int} \sim d_2\, G_{21}\, d_1 \qquad d_2 \propto E_2\, \alpha_2 \propto d_1 G_{12} \alpha_2$$

$$\delta E \sim \int d\omega \,\langle d_2\, G_{21}\, d_1 \rangle = \int d\omega \,\langle d_1 G_{12} \alpha_2 G_{21}\, d_1 \rangle = \int d\omega\, \alpha_1\, G_{12}\, \alpha_2\, G_{21}$$

# QED approach: field theory

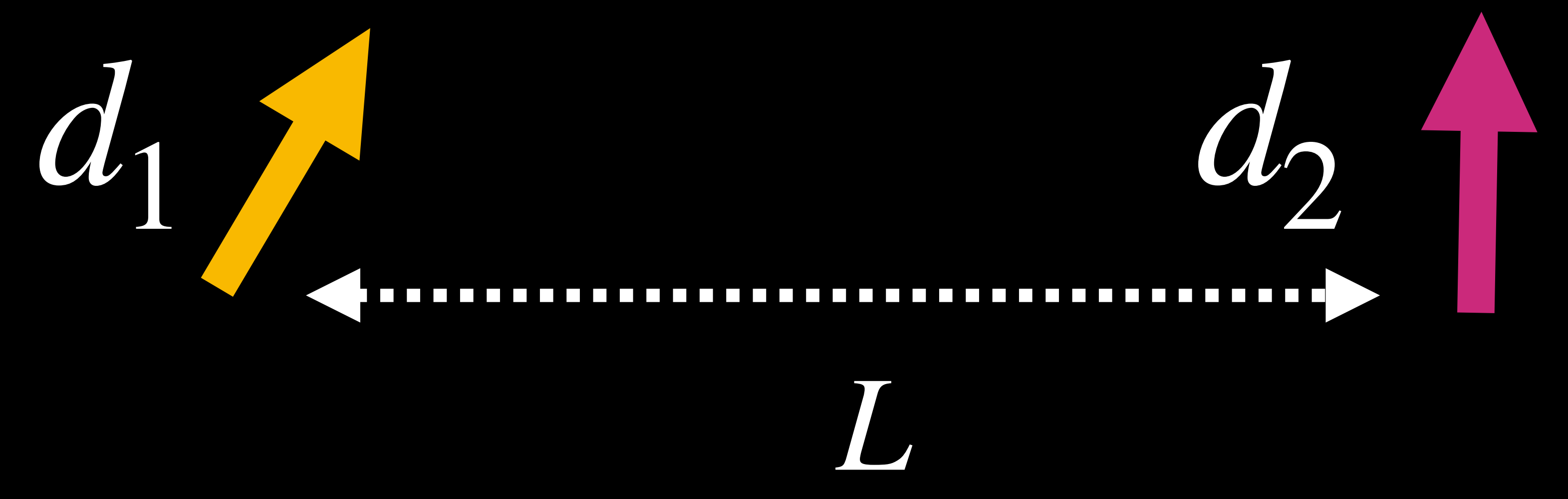



$$H_{int} \sim d_2\, G_{21}\, d_1 \qquad d_2 \propto E_2\, \alpha_2 \propto d_1 G_{12} \alpha_2$$

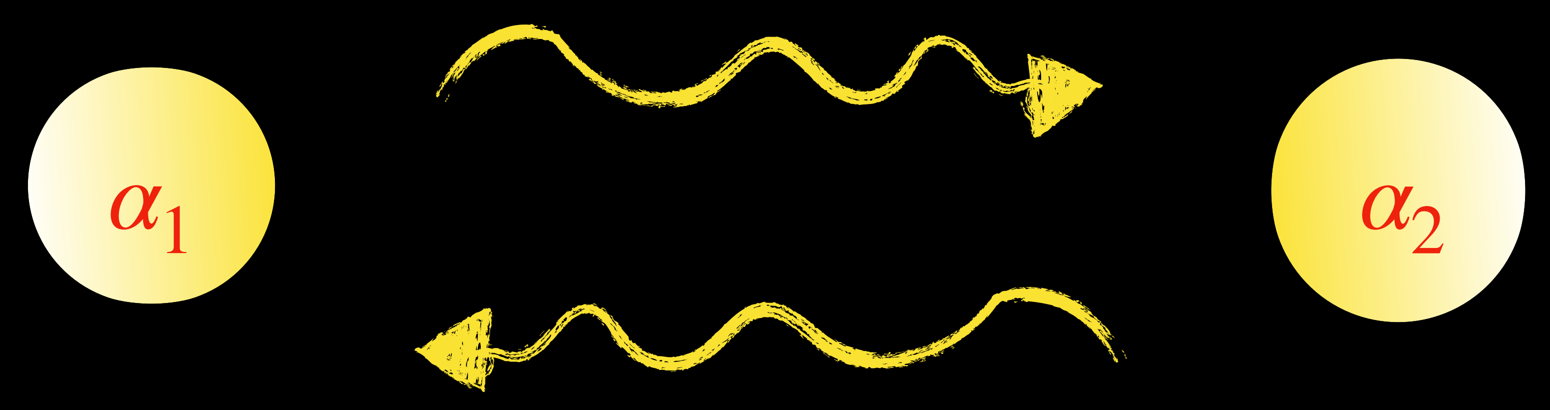
$$\delta E \sim \int d\omega\, \langle d_2\, G_{21}\, d_1 \rangle = \int d\omega\, \langle d_1 G_{12} \alpha_2 G_{21}\, d_1 \rangle = \int d\omega\, \alpha_1\, G_{12}\, \alpha_2\, G_{21}$$

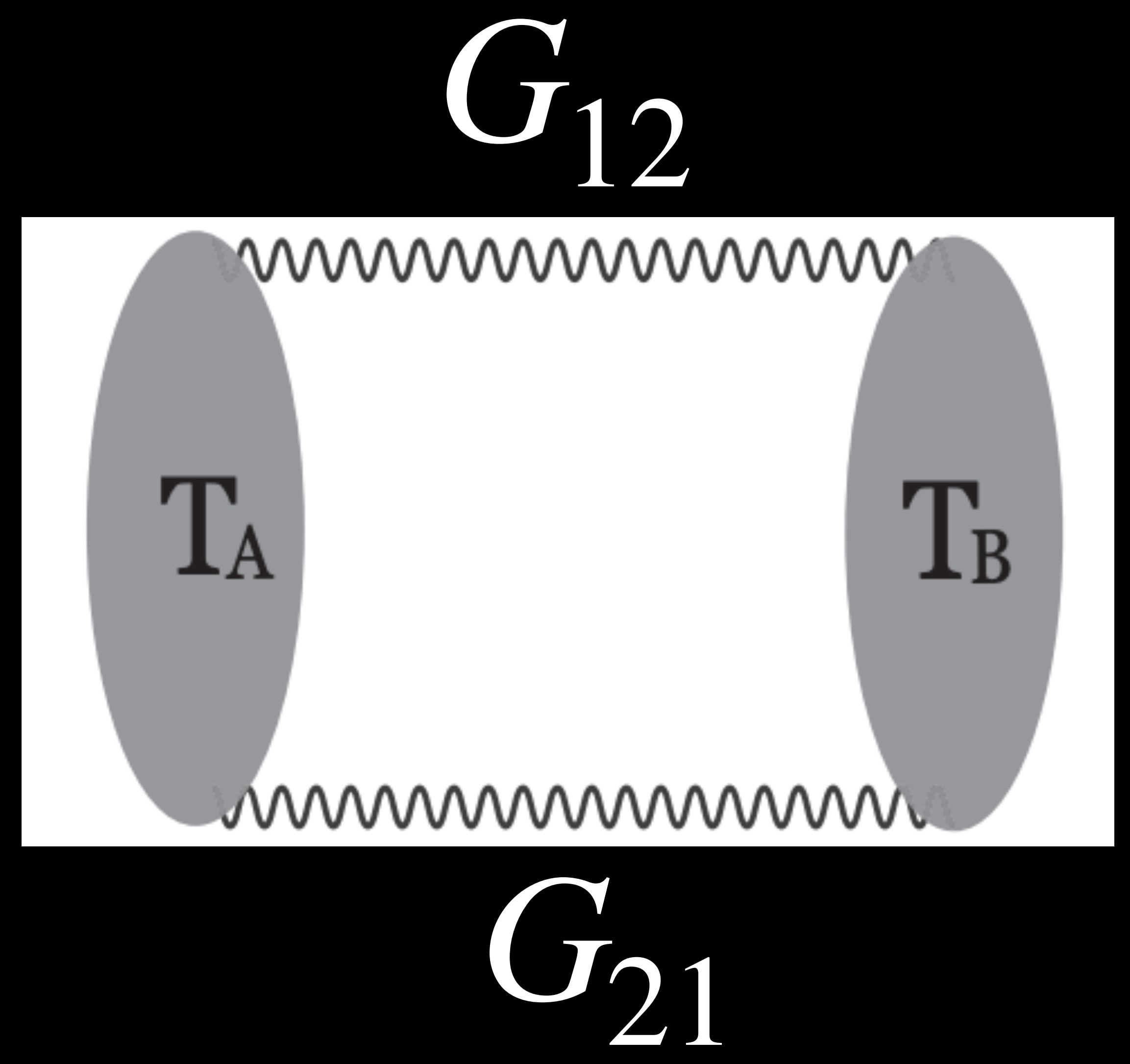
$$\sim \frac{\alpha_1(\overline{\omega})\overline{\alpha}_2(\overline{\omega})\overline{\omega}}{L^6} \text{ (short range)} \qquad \sim \frac{\alpha_1(0)\alpha_2(0)}{L^7} \text{ (long range)}$$

# QED approach: Feynman diagram



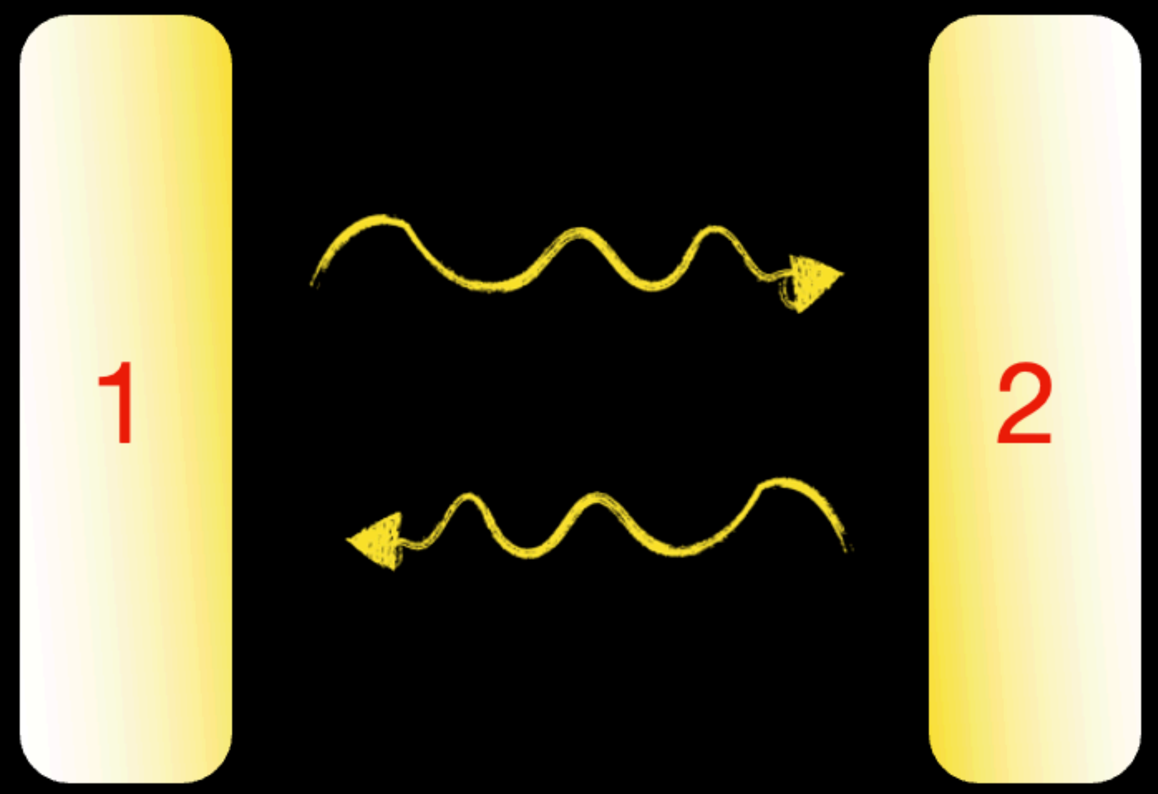

$$\delta E \sim \int d\omega \, \alpha_1 \, G_{12} \, \alpha_2 \, G_{21}$$



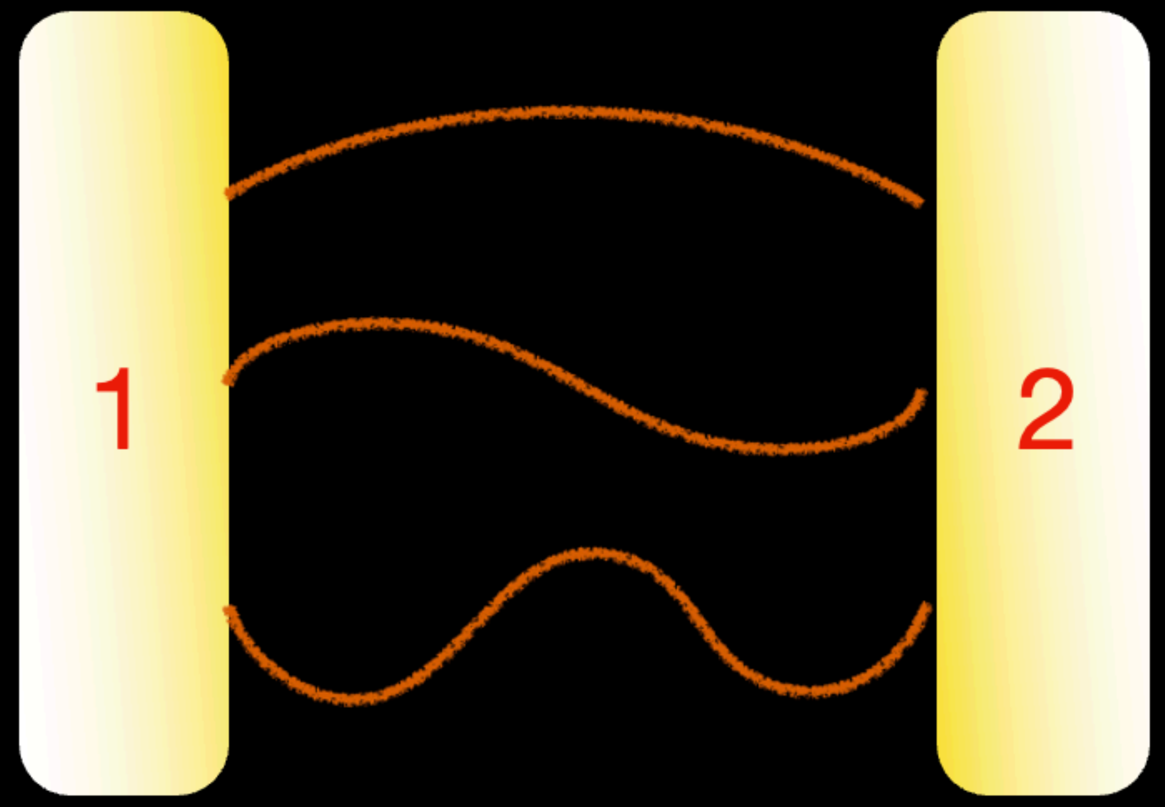



# Casimir forces: Lifshitz formula

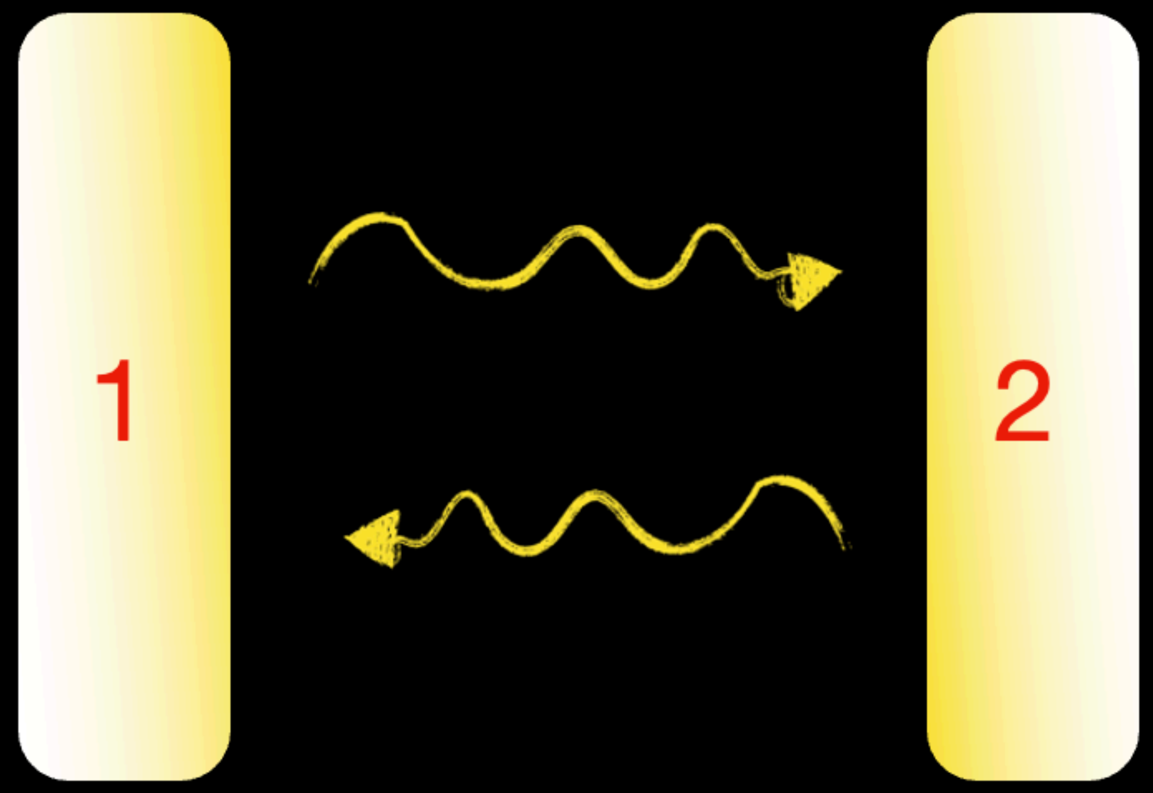
Quantum field theory (Lifshitz 1958')




Counting modes (for simple geometry)

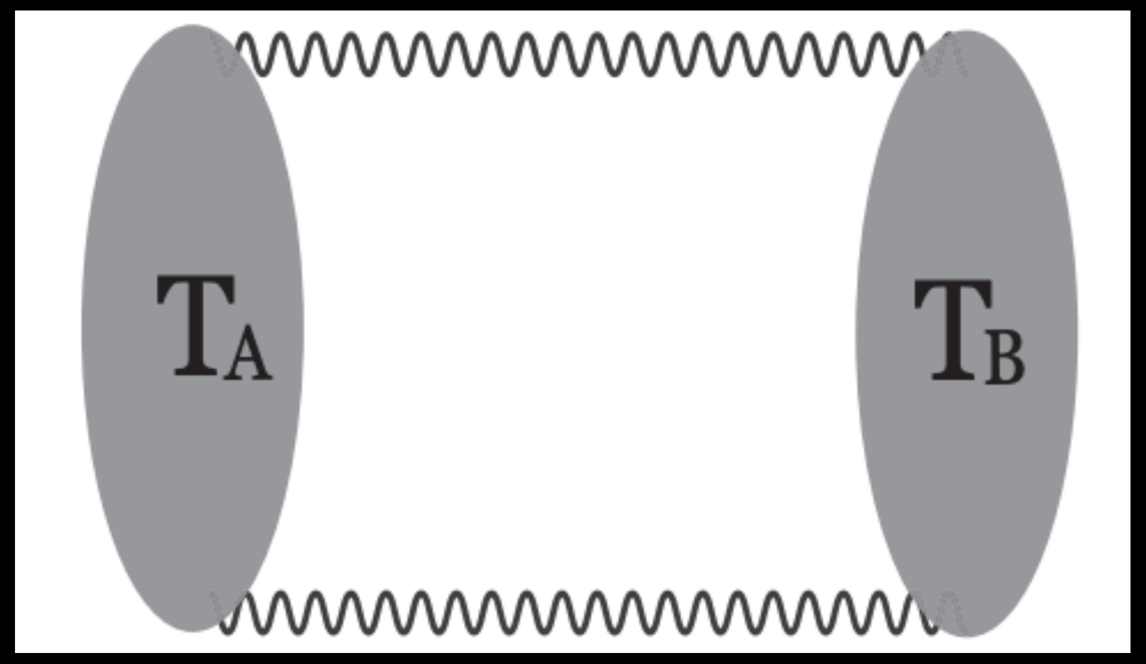


# Lifshitz formula for perfect metals




Quantum field theory (Lifshitz 1958')

$$E_c = \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \, \mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$



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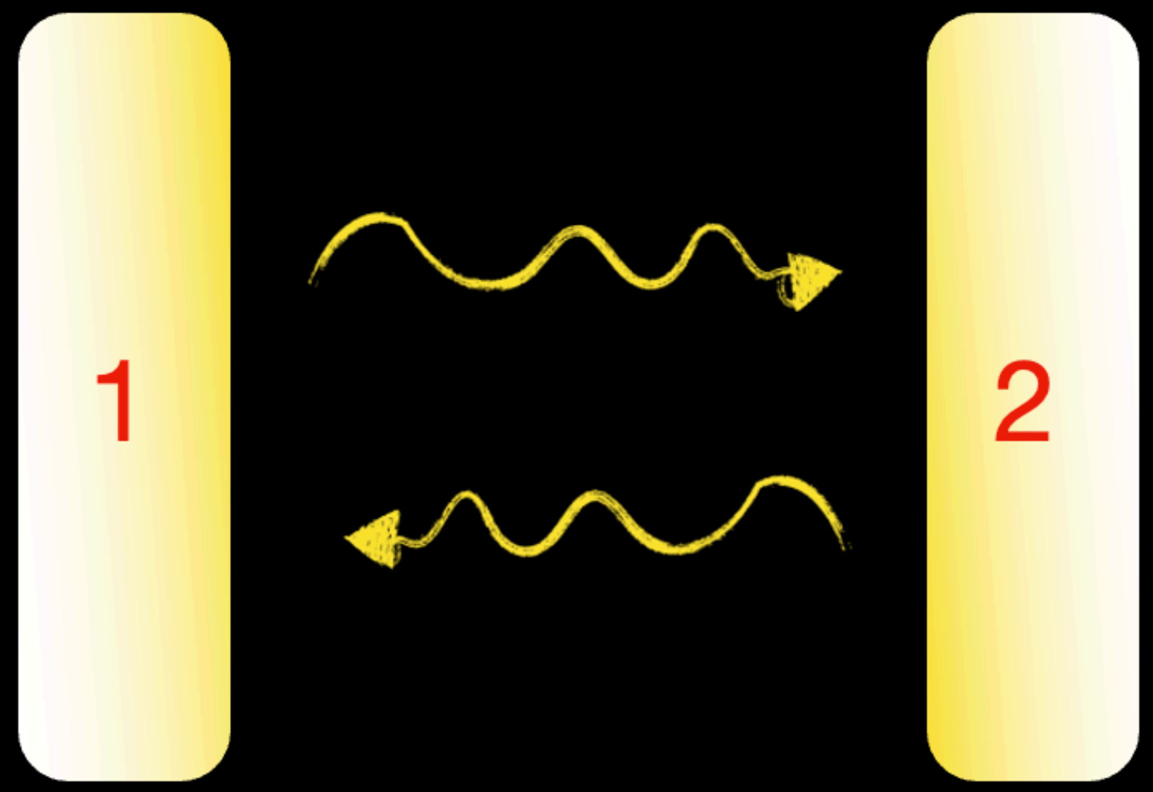


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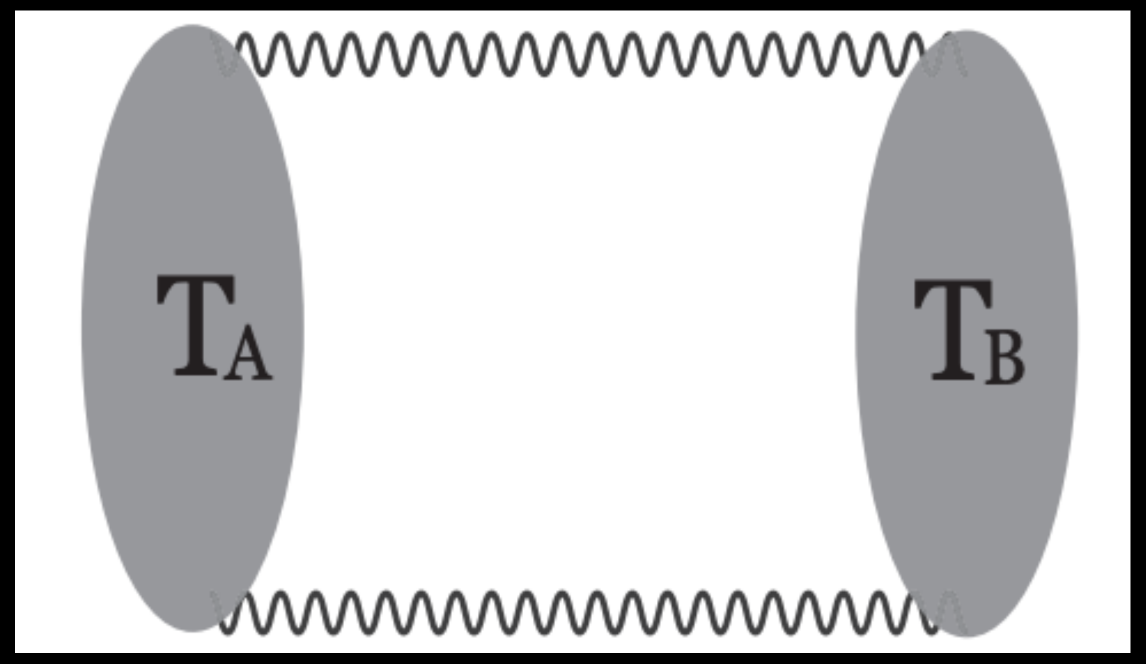
$$= \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \, \mathrm{Log}\left(1 - e^{-2k_z L}\right)$$



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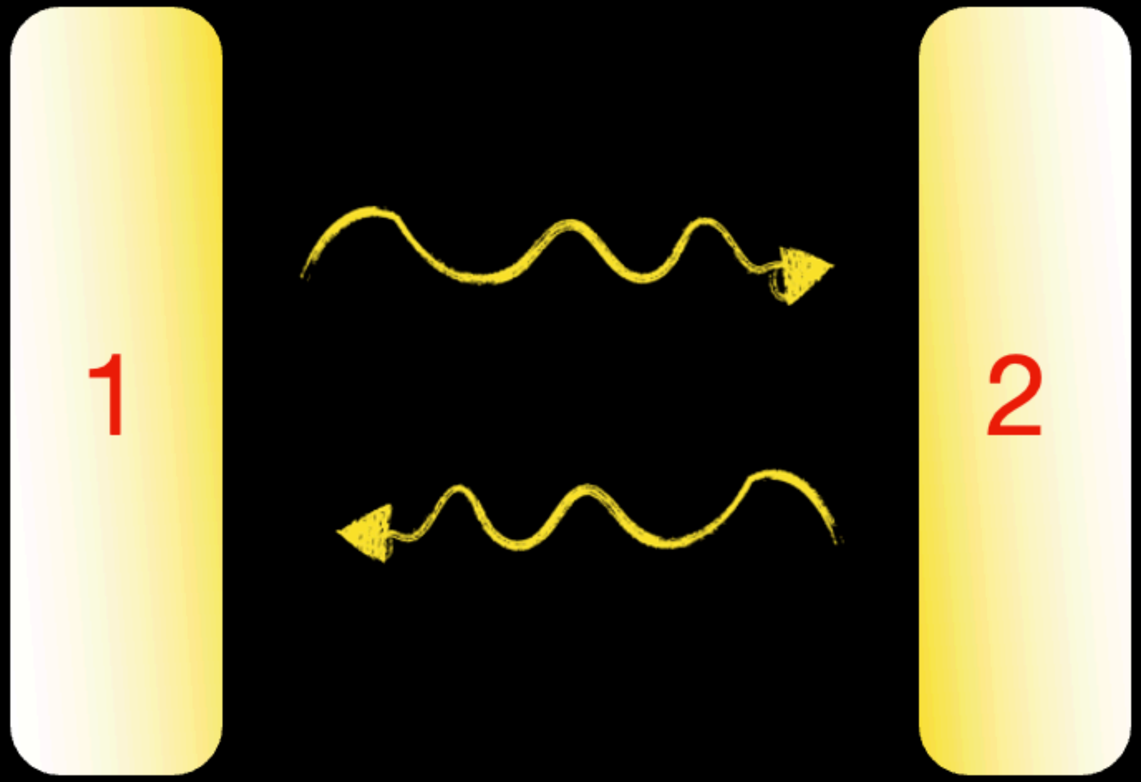
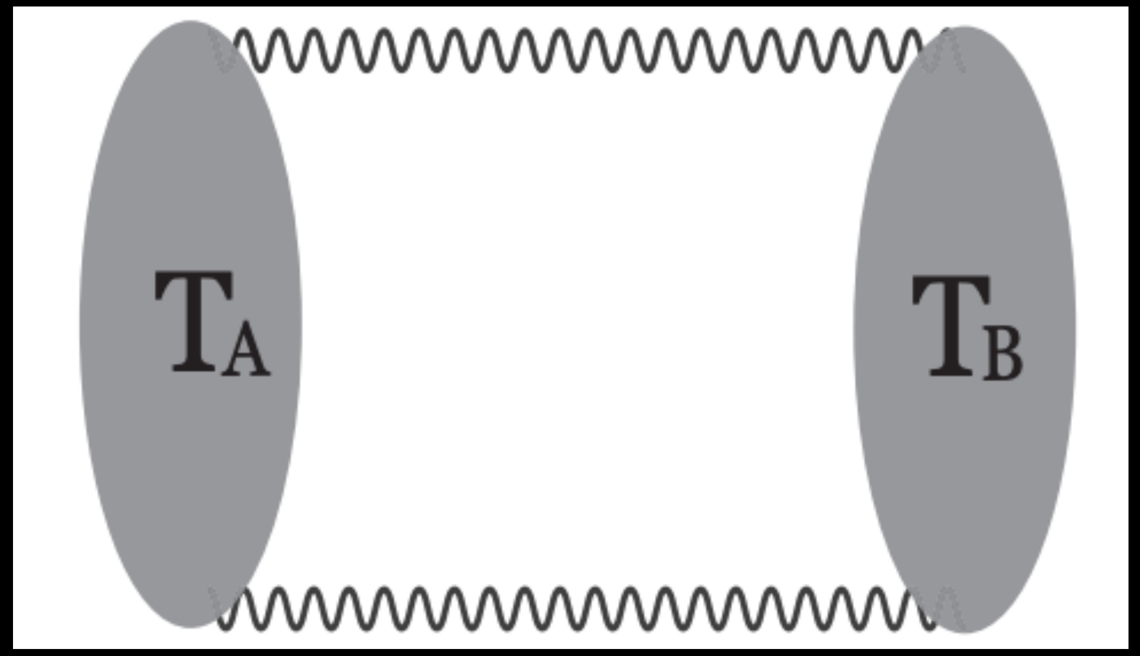



$$E_c = \frac{\hbar A}{(2\pi)^3}\int_0^\infty d\xi \int dk_x dk_y \,\mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

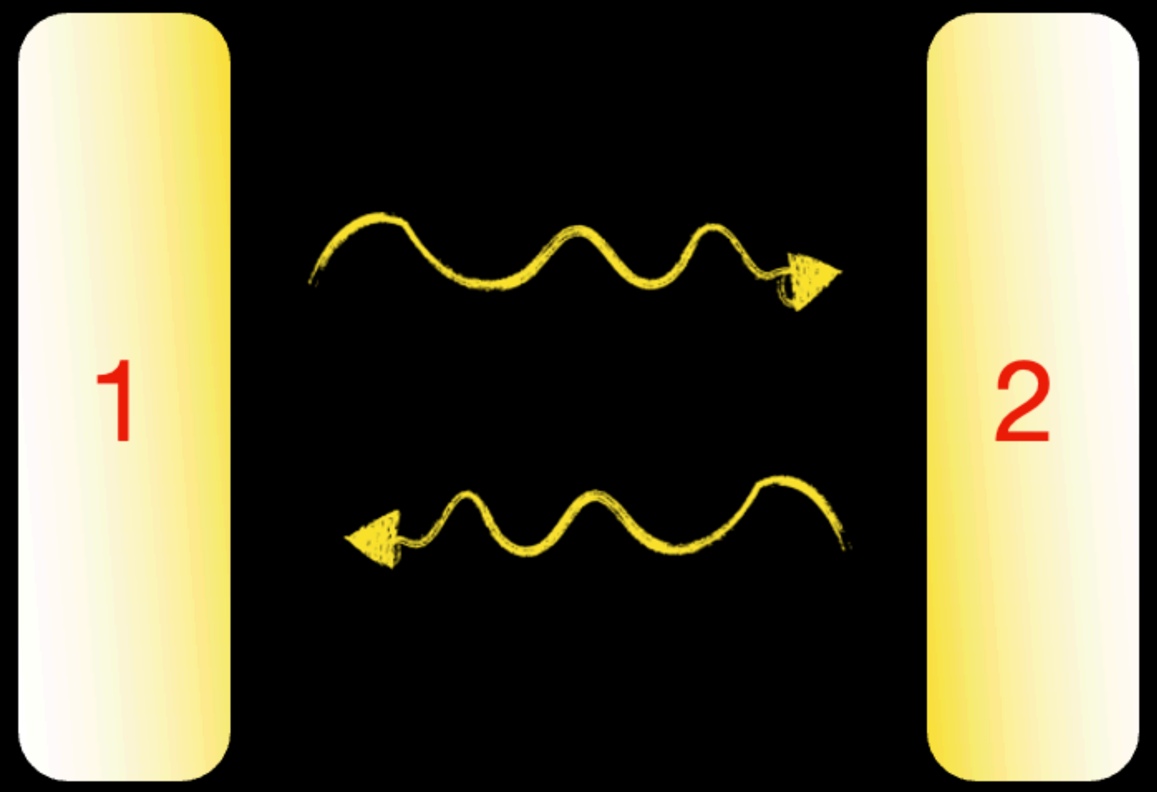
$$= \frac{\hbar A}{(2\pi)^3}\int_0^\infty d\xi \int dk_x dk_y \,\mathrm{Log}\left(1 - e^{-2k_z L}\right)$$

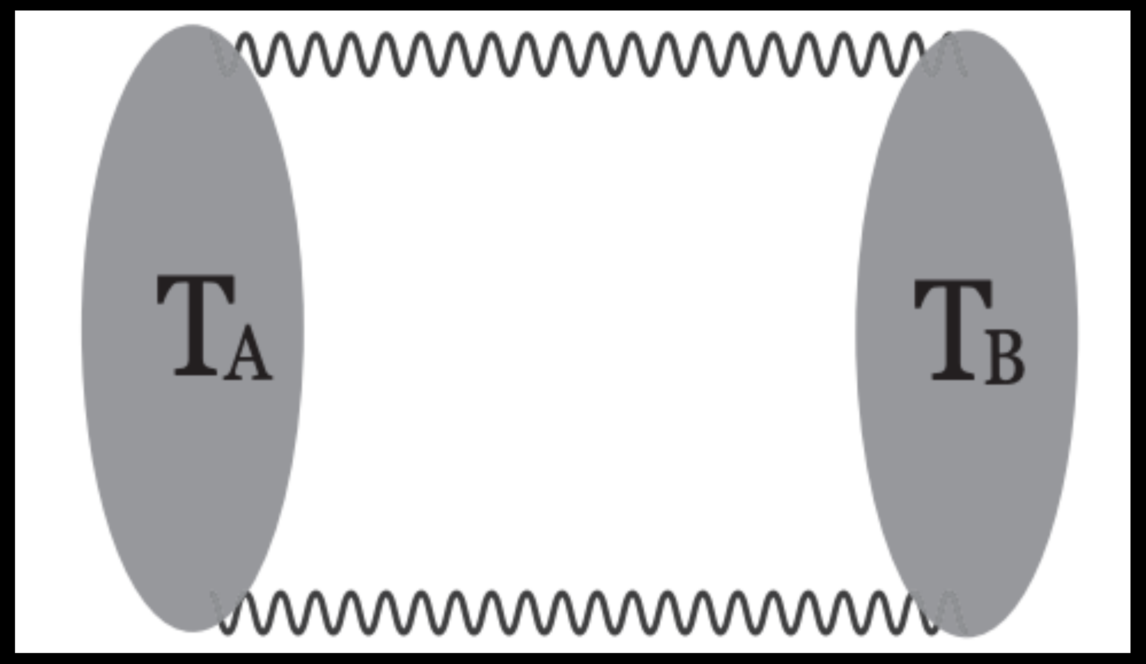
$$\frac{\pi^4}{90L^3}$$

$$= \frac{\hbar A}{(2\pi)^3}\int_0^\infty d\xi \int dk_x dk_y \,\mathrm{Log}\left(1 - e^{-2\sqrt{\xi^2 + k_x^2 + k_y^2}L}\right) = -\frac{\pi^2 \hbar c A}{1440 L^3}$$

# Lifshitz formula for perfect metals

Quantum field theory (Lifshitz 1958')

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$\times\, 2$ polarizations

# Bigger or Smaller

$$E_c = \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \, \mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

Reflection matrix

Metal > Insulator

Three examples

# Bigger or Smaller

$$E_c = \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \, \mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

Reflection matrix

Photon Propagator

Metal > Insulator

Two examples

# Attraction or Repulsion

$$E_c = \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \ \mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

$ABAB^*$

# Attraction or Repulsion

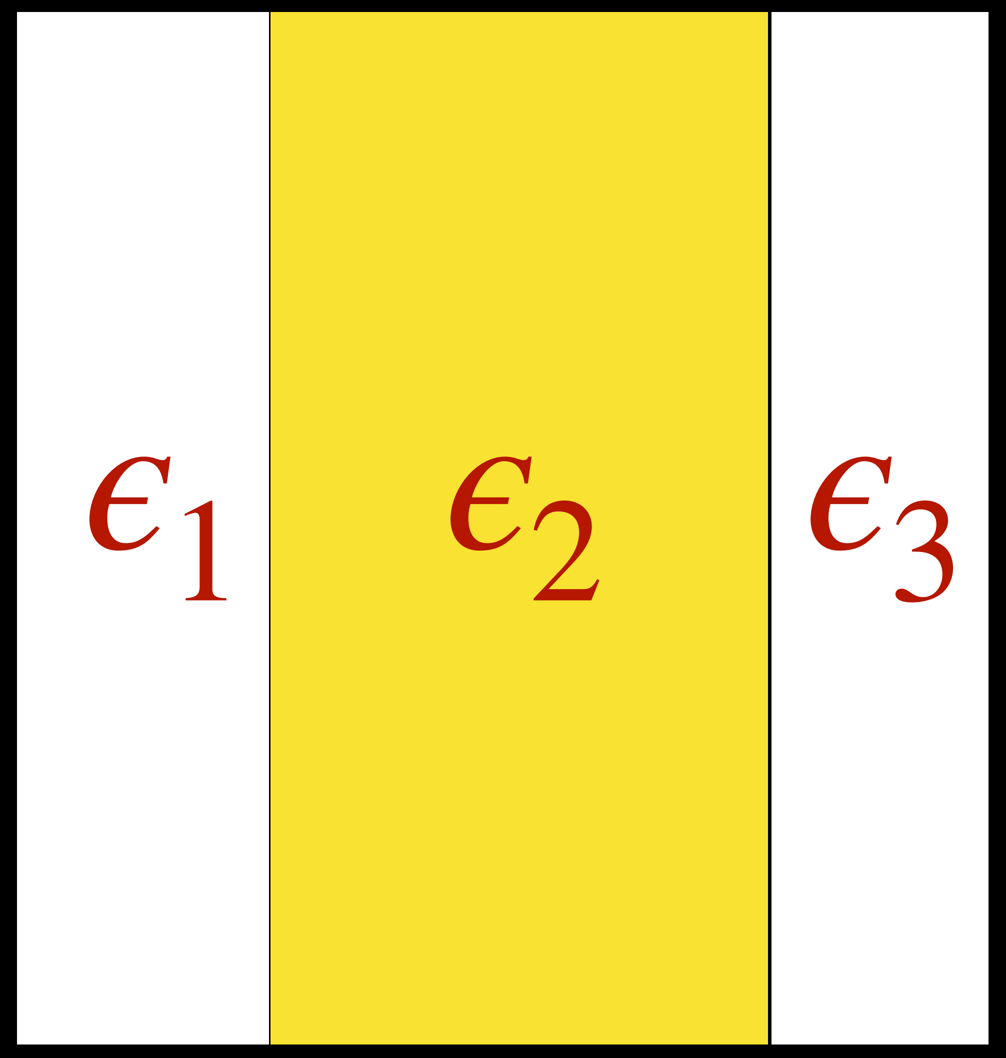
$$E_c = \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \, \mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

*ABAB**

$$F_c \propto -\int \frac{(\epsilon_A(i\xi) - \epsilon_C(i\xi))(\epsilon_B(i\xi) - \epsilon_C(i\xi))}{(\epsilon_A(i\xi) + \epsilon_C(i\xi))(\epsilon_B(i\xi) + \epsilon_C(i\xi))} d\xi$$

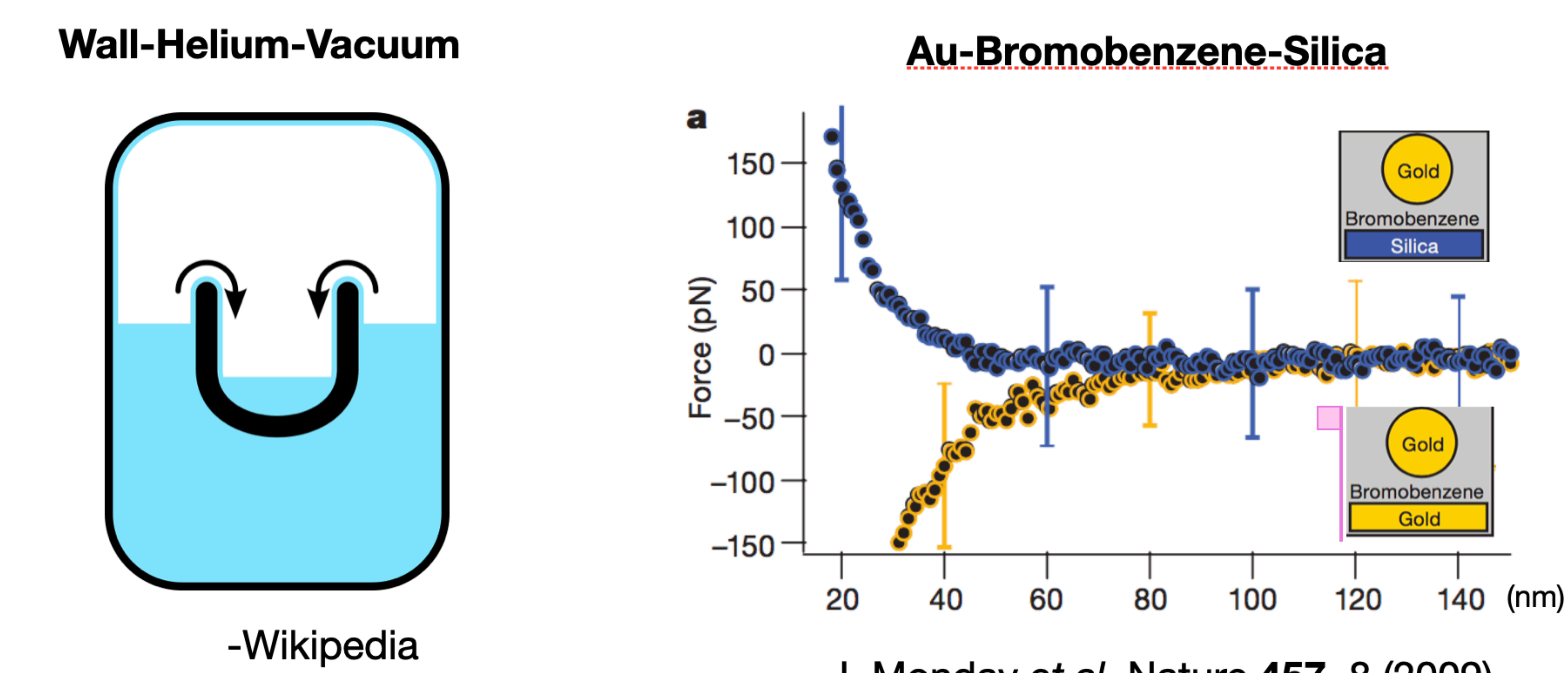
Lifshitz 1957'

$$F_c \propto (\epsilon_2 - \epsilon_1)(\epsilon_2 - \epsilon_3)$$



# Attraction or Repulsion

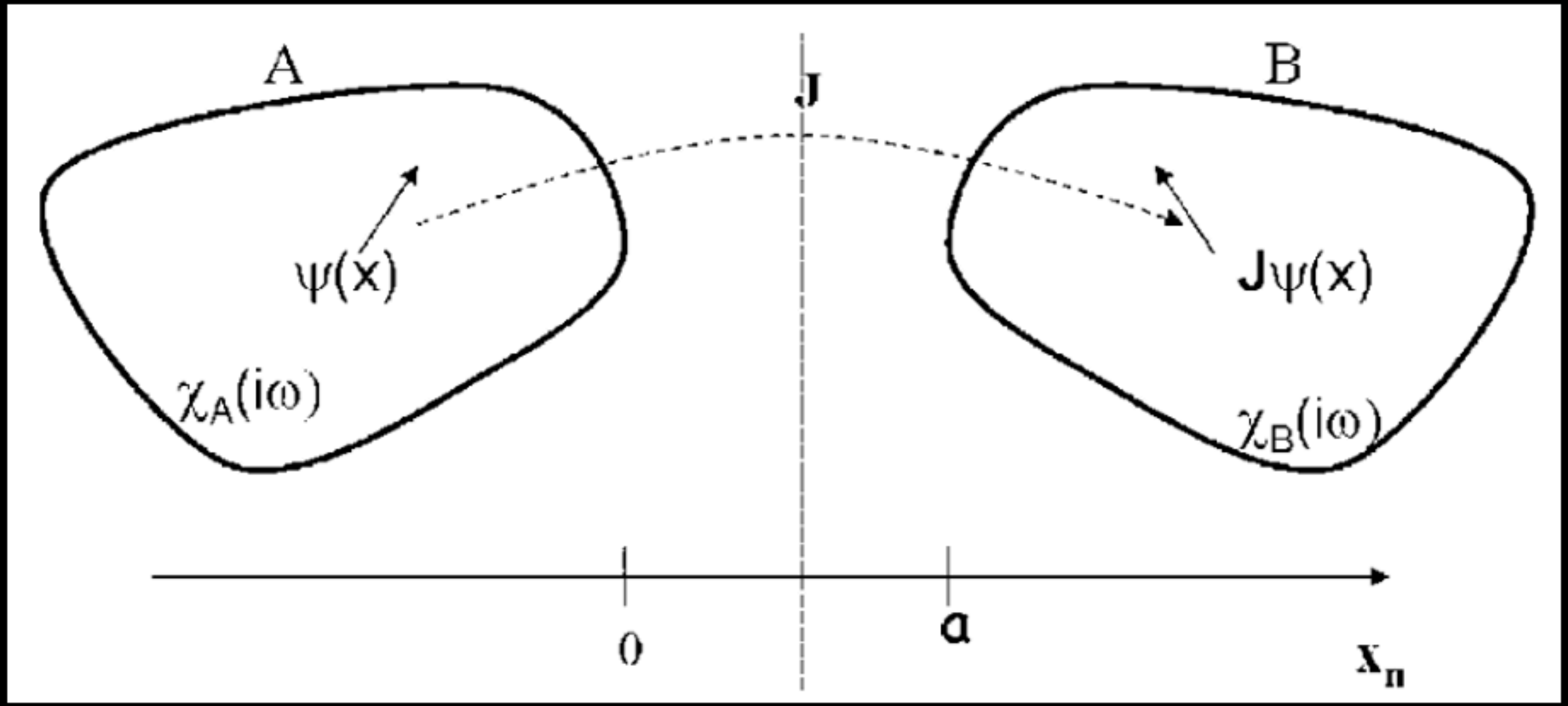
$F_c(\omega) \propto (\epsilon_2 - \epsilon_1)(\epsilon_2 - \epsilon_3)$ $\epsilon(\omega)!!$ Whole spectrum Integral




J. Monday *et al.* Nature **457,** 8 (2009)

# No-go theorem: opposite attracts

**No-go theorem:**
Always attractive if there is parity symmetry




Keneth & Klich
Phys. Rev. Lett. **97** 160401 (2006)

$$E_c = -\int_0^\infty \frac{d\xi}{2\pi} \mathrm{Tr}\left\{ \mathrm{T}_1 \mathrm{G}_{12} \mathrm{T}_2 \mathrm{G}_{21} \right\} \quad \propto -\Gamma^\dagger \Gamma < 0$$

# How to make repulsive? Two ways!

Attractive Casimir forces make things stick together
- bad for nano devices

$$E_c = \frac{\hbar A}{(2\pi)^3}\int_0^\infty d\xi \int dk_x dk_y \,\mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

Reflection matrix

Photon Propagator

# Engineering R matrices

$$E_c = \frac{\hbar A}{(2\pi)^3} \int_0^\infty d\xi \int dk_x dk_y \, \mathrm{Log}\left(1 - R_1 U_{12} R_2 U_{21}\right)$$

quantum Hall systems — Phys. Rev. Lett. 109, 236806 (2012)

topological insulators — Phys. Rev. Lett. 106, 020403 (2011)

Chern insulators — Phys. Rev. Lett. 112, 056804 (2014)

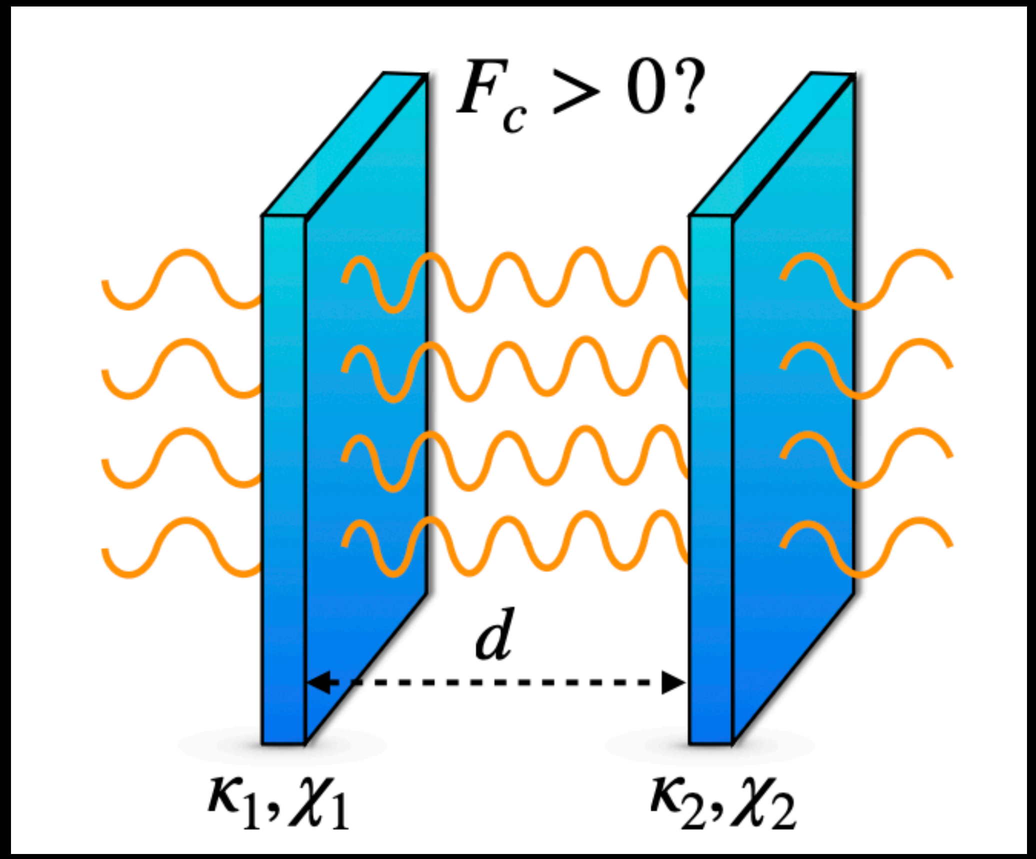
metamaterials — Phys. Rev. Lett. 100, 183602 (2008)

Weyl semimetals — Phys. Rev. B 91, 235115 (2015)

…

# Engineering R matrices

$$\mathbf{D} = \epsilon\mathbf{E} + (\chi - i\kappa)\sqrt{\epsilon_0\mu_0}\mathbf{H},$$
$$\mathbf{B} = \mu\mathbf{H} + (\chi + i\kappa)\sqrt{\epsilon_0\mu_0}\mathbf{E}.$$



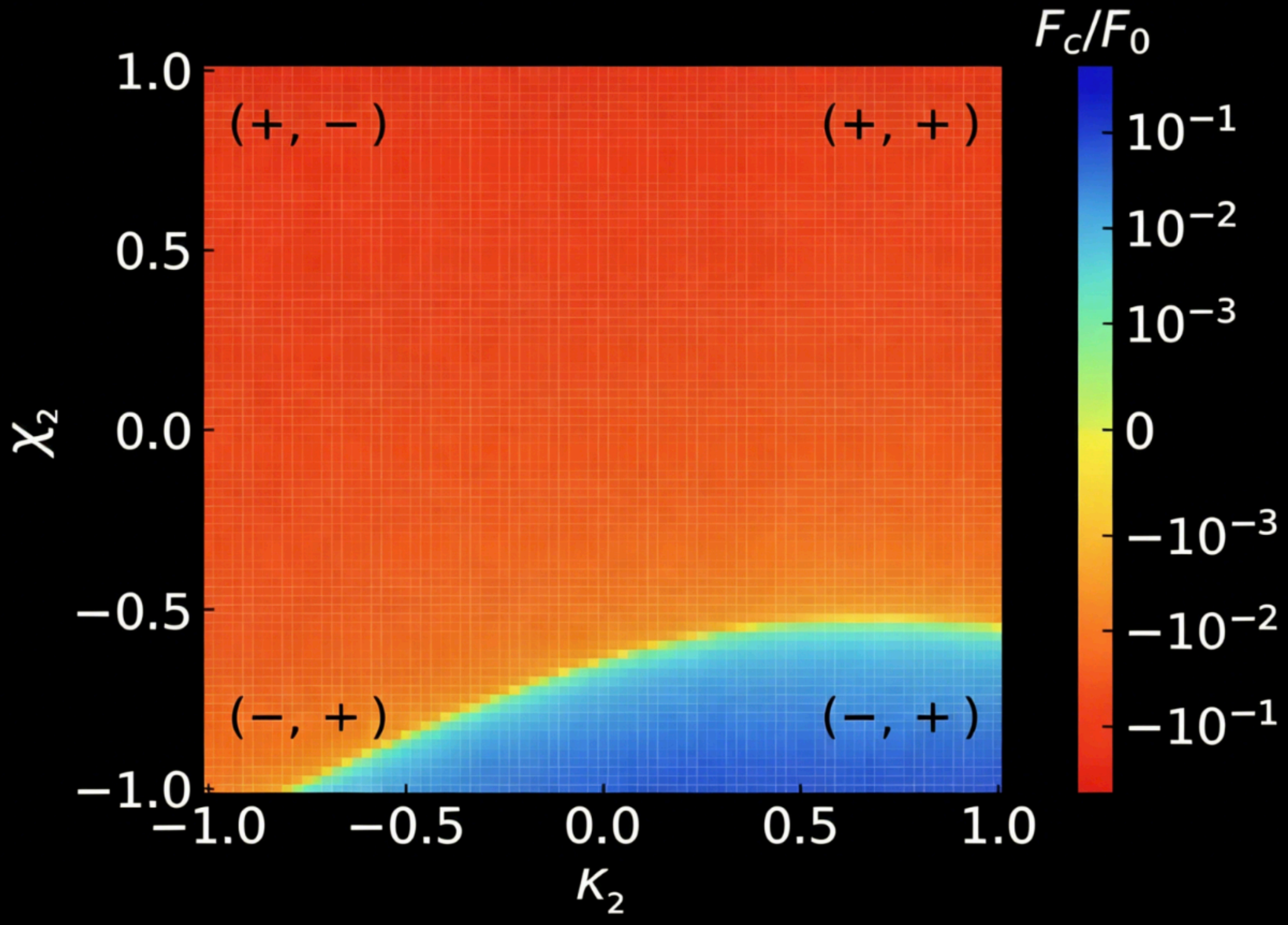

| Parameter | Symbol | Time reversal $T$ | Spatial inversion $P$ | Typical systems |
|---|---|---|---|---|
| Permittivity | $\varepsilon$ | + | + | Dielectrics |
| Permeability | $\mu$ | + | + | Magnetic materials |
| Non-reciprocal parameter | $\chi$ | − | − | Topological insulators, Tellegen metamaterials |
| Chirality parameter | $\kappa$ | + | − | Chiral molecules, chiral metamaterials |

# Engineering R matrices

$$\mathbf{D} = \epsilon\mathbf{E} + (\chi - i\kappa)\sqrt{\epsilon_0\mu_0}\mathbf{H},$$

$$\mathbf{B} = \mu\mathbf{H} + (\chi + i\kappa)\sqrt{\epsilon_0\mu_0}\mathbf{E}.$$

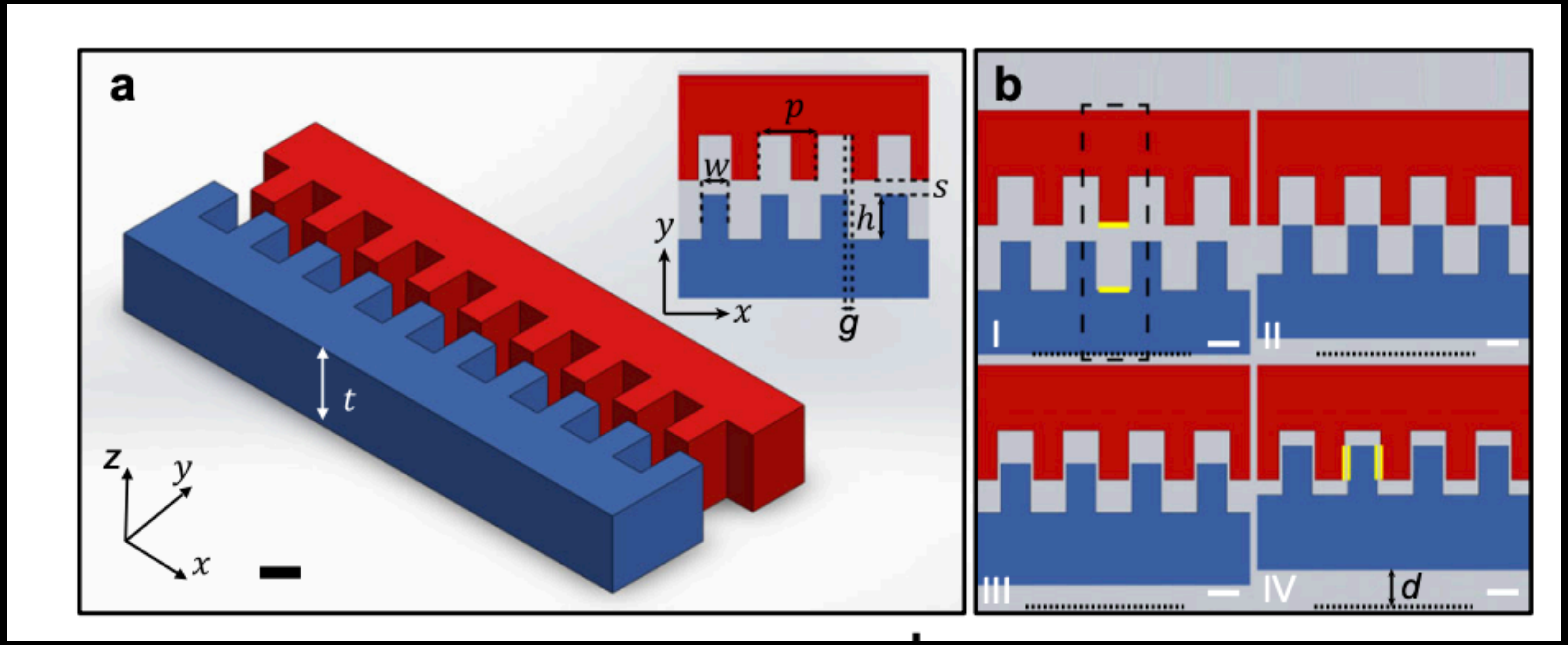
| Parameter | Symbol | Time reversal $T$ | Spatial |
|---|---|---|---|
| Permittivity | $\varepsilon$ | + | + |
| Permeability | $\mu$ | + | + |
| Non-reciprocal parameter | $\chi$ | − | − |
| Chirality parameter | $\kappa$ | + | − |



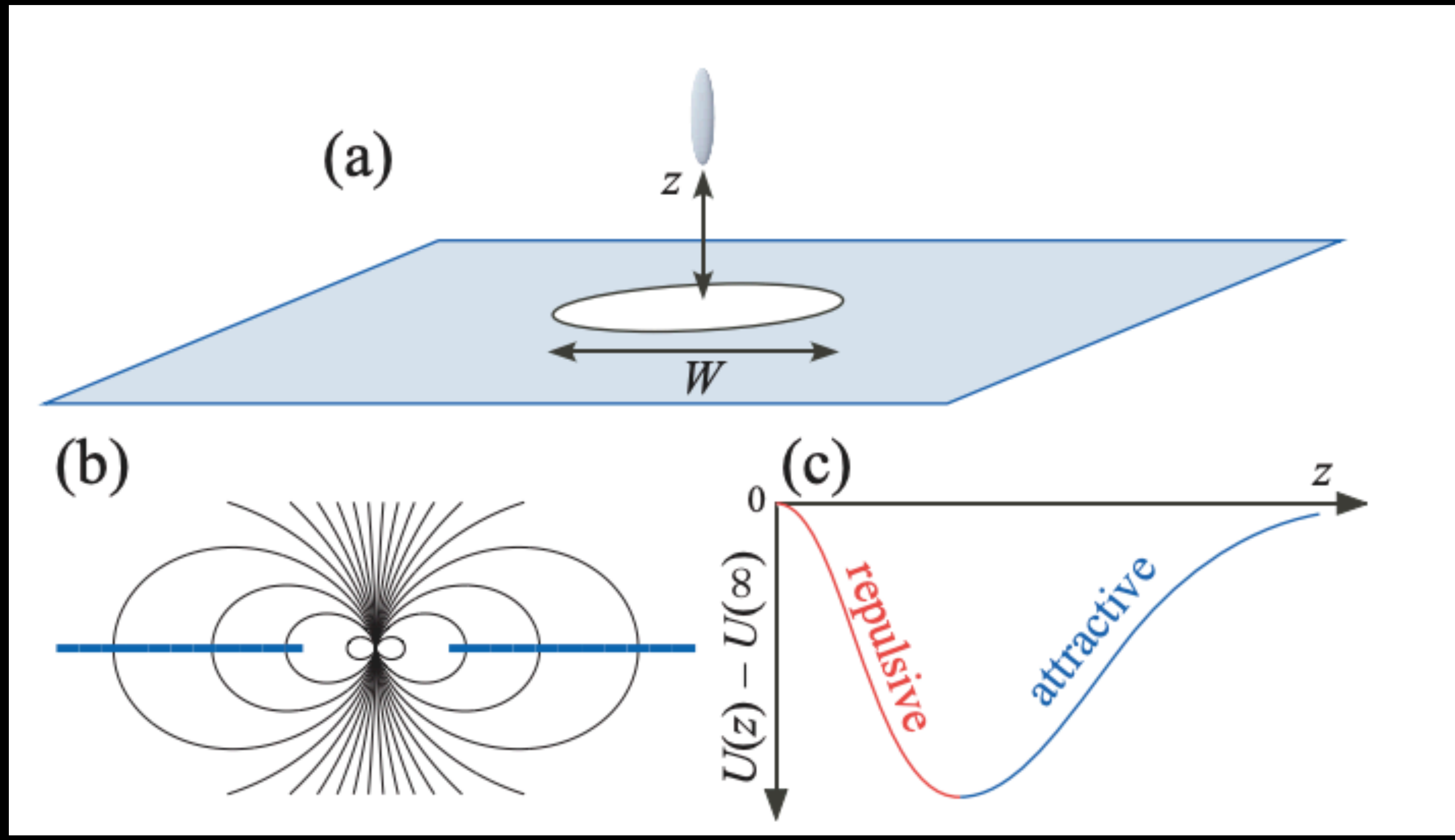

Zixuan Dai, QDJ, Journal of Physics: Condensed Matter 37, 405301 (2025)

# Engineering R matrices

- Geometry: nanoscale grating, or dripping hole…



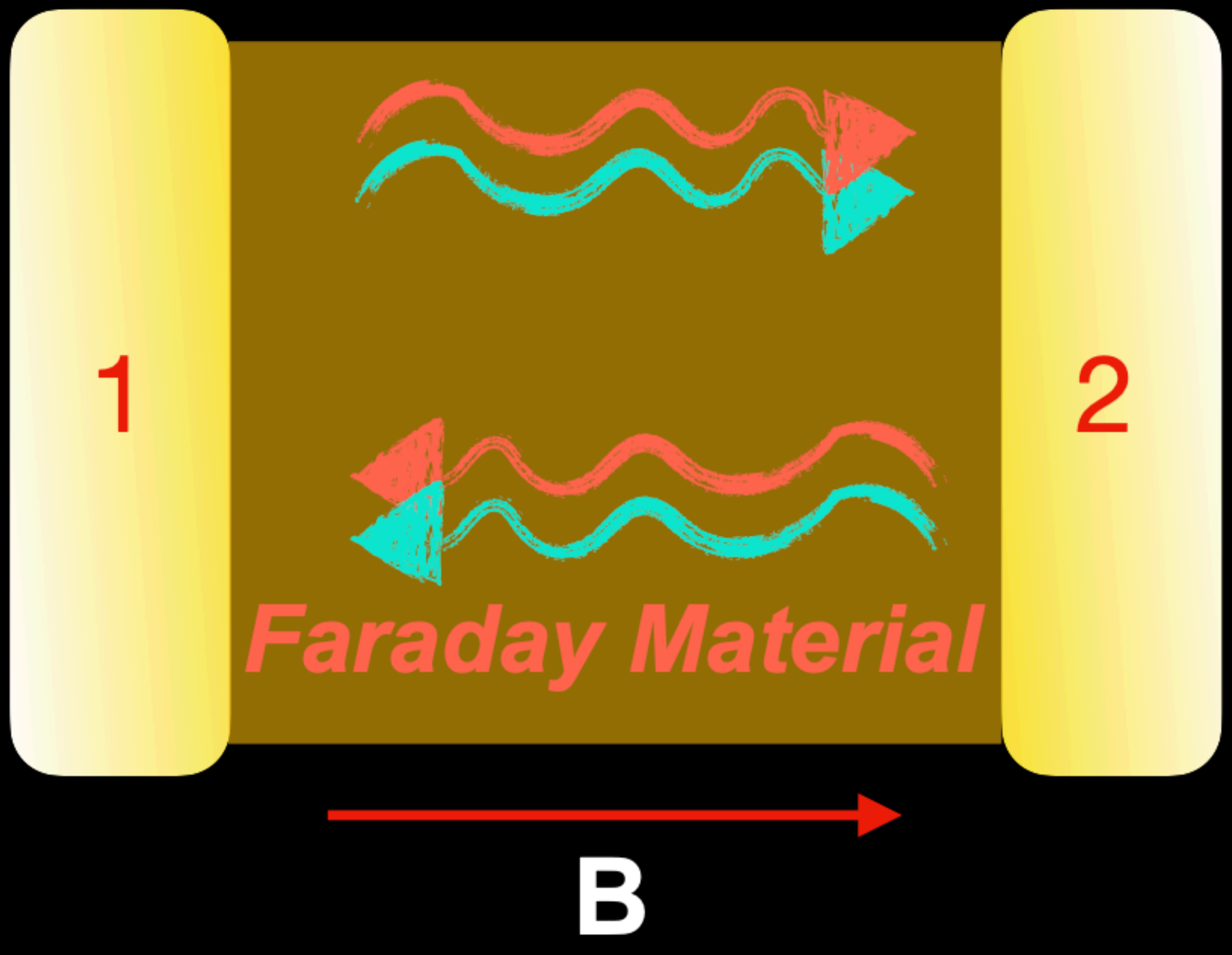

H. B. Chan group, Nat. Comm. 2021

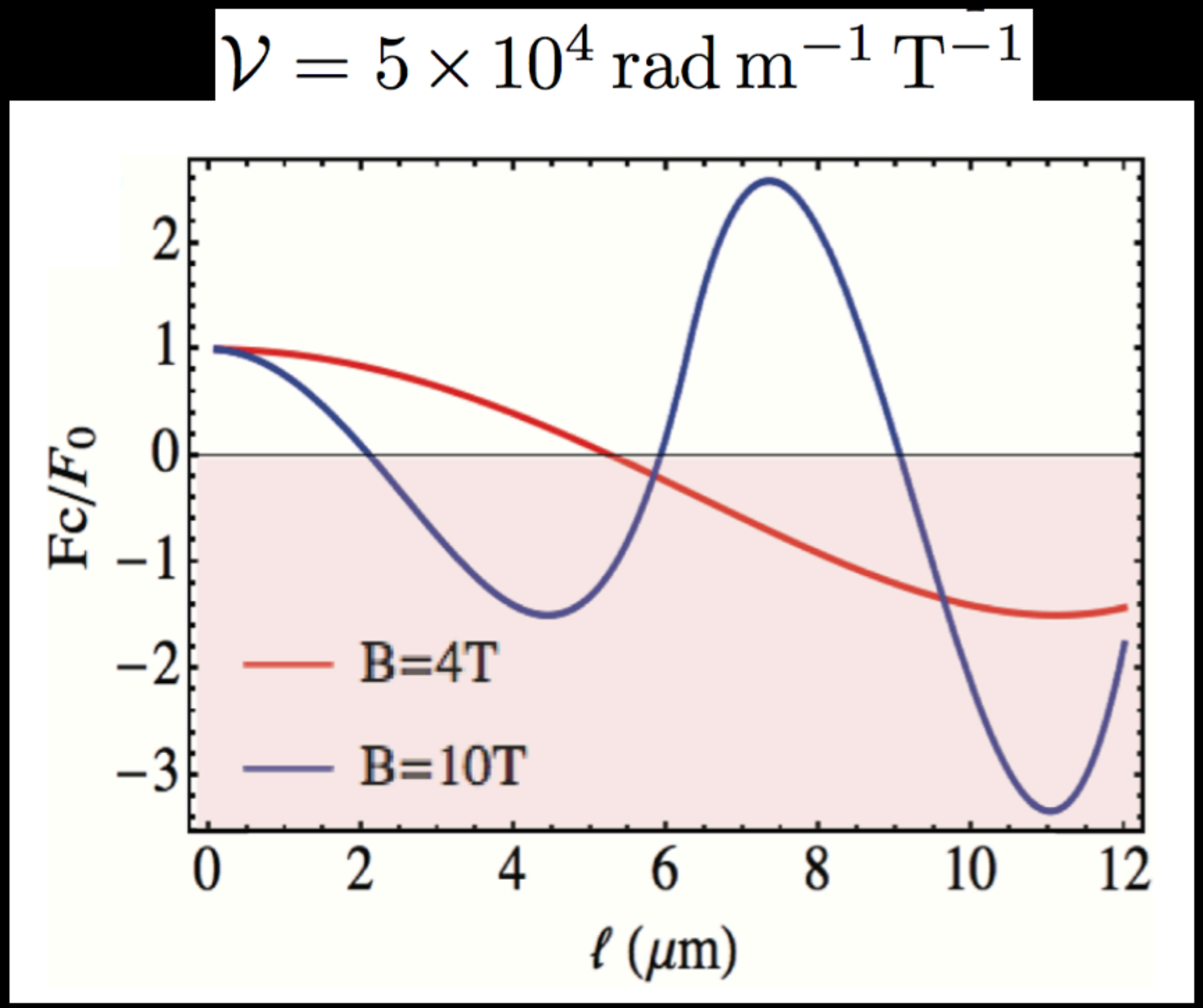



Michael Levin et al. PRL 2010

# Engineering U matrices

## Exp. Setup

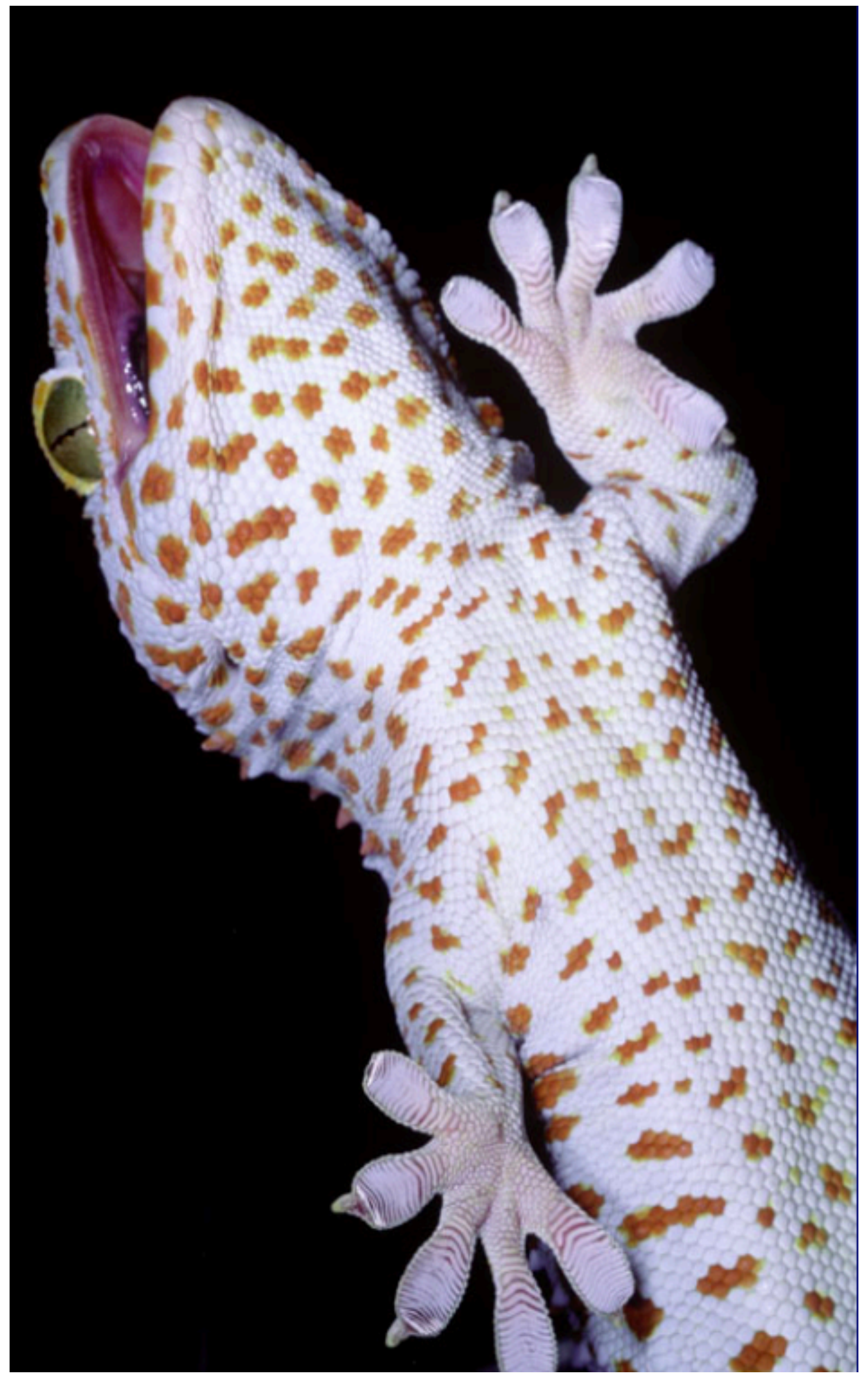


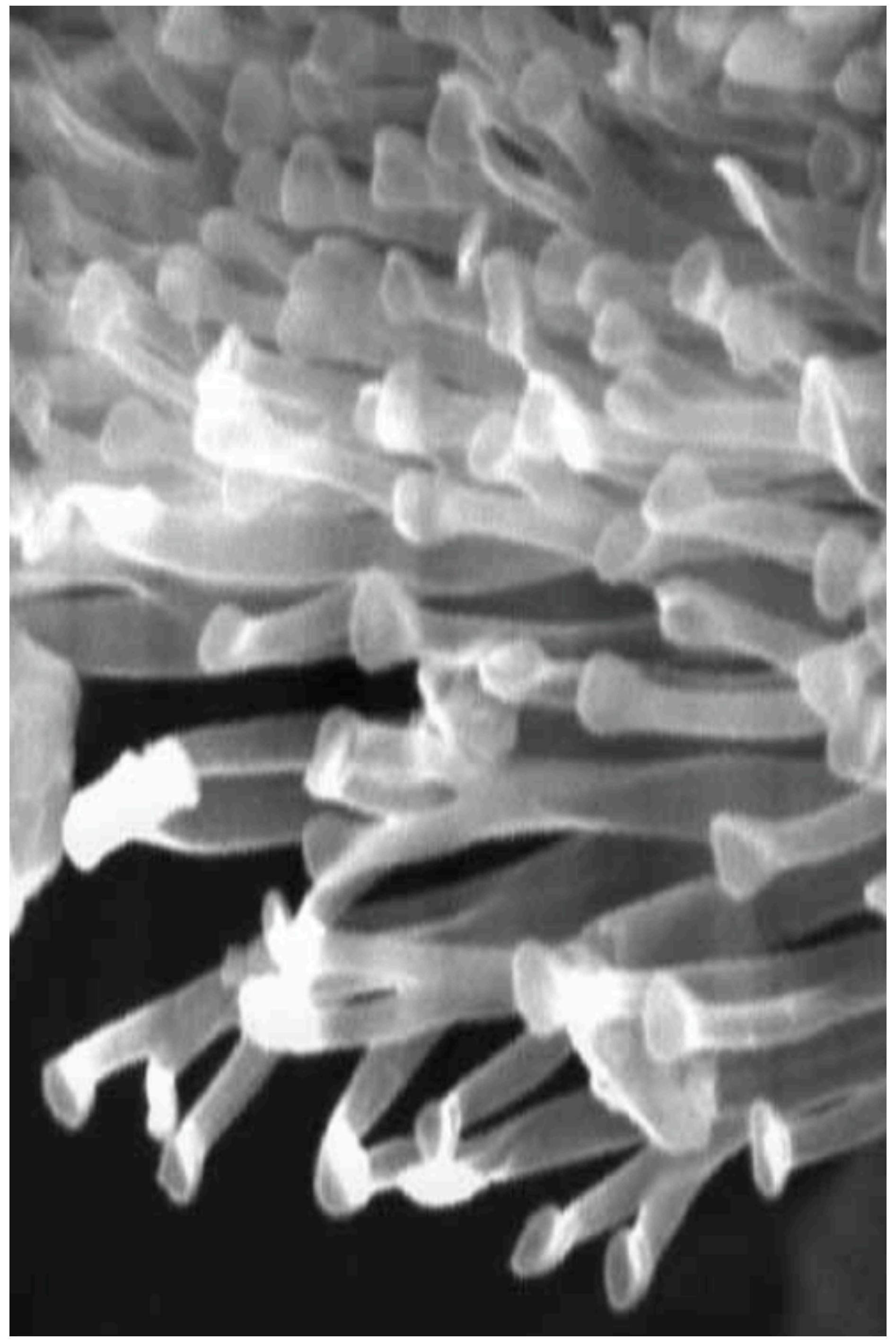




Loophole: $G_{12} \neq G_{21}^{\dagger}$

QDJ and Frank Wilczek, Phys. Rev. B 99, 125403 (2019)
(Editors' Suggestion & Featured in Physics)

# Pouring Chiral Liquid to Slip Down a Gecko?





K. Autumn, W.-P. Chang, R. Fearing, T. Hsieh, T. Kenny, L. Liang, W. Zesch, R.J. Full. Nature 2000. Adhesive force of a single gecko foot-hair.

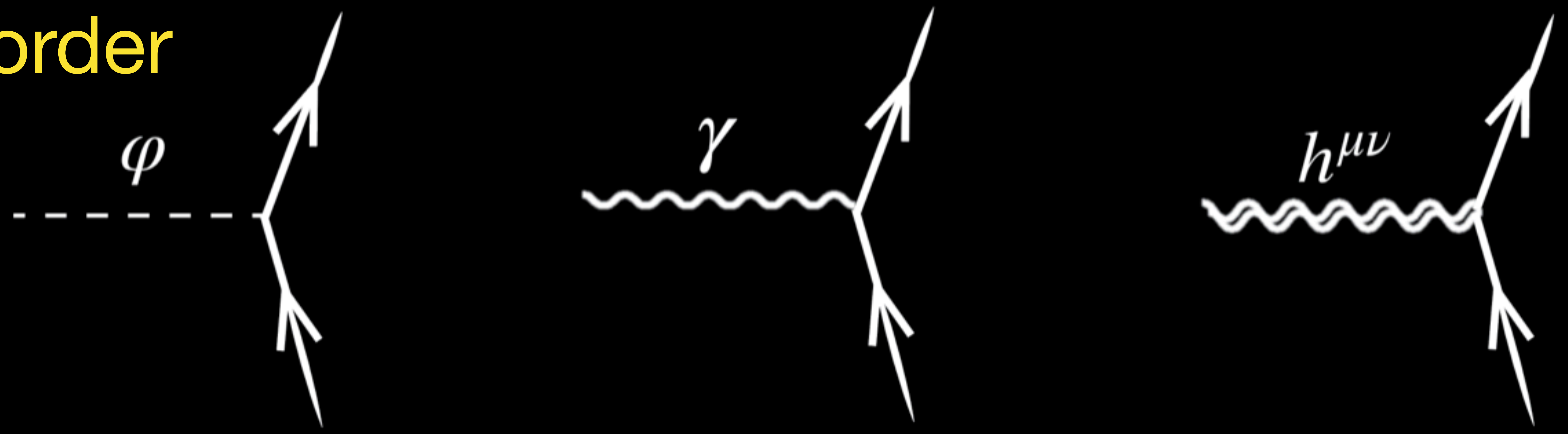
How does Gecko manage to walk on vertical smooth waals?

Suction? (Salamander). Capillary adhesion? (Small frogs). Interlocking? (Cockroach)

**It's van der Waals interactions!**

# Summary: Forces Mediated by Bosons

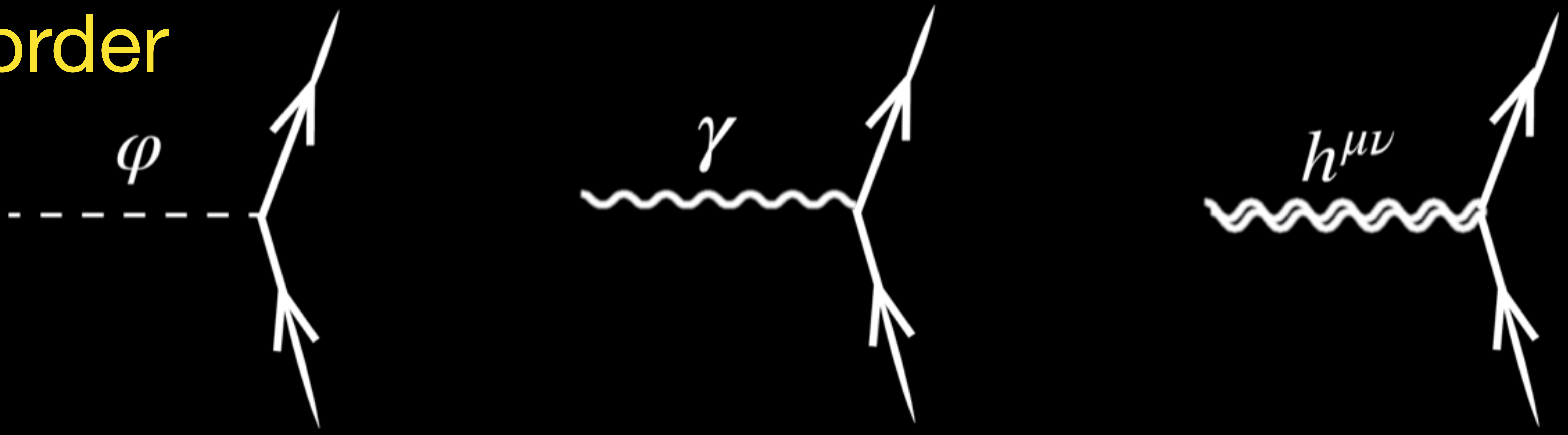
First order




Yukawa potential Electromagnetic forces Gravitational forces

# Summary: Forces Mediated by Bosons

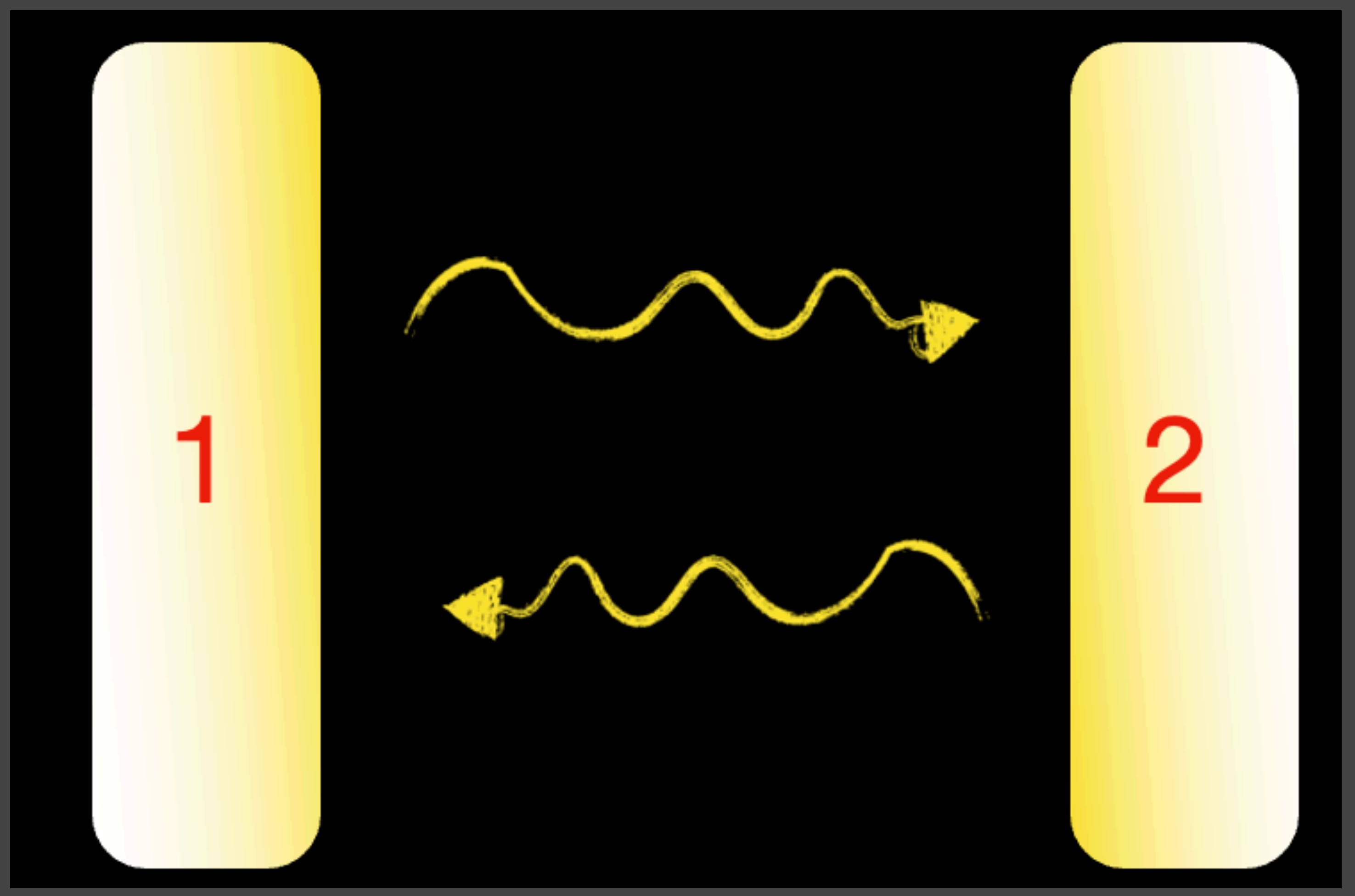
## First order




**Yukawa potential** **Electromagnetic forces** **Gravitational forces**

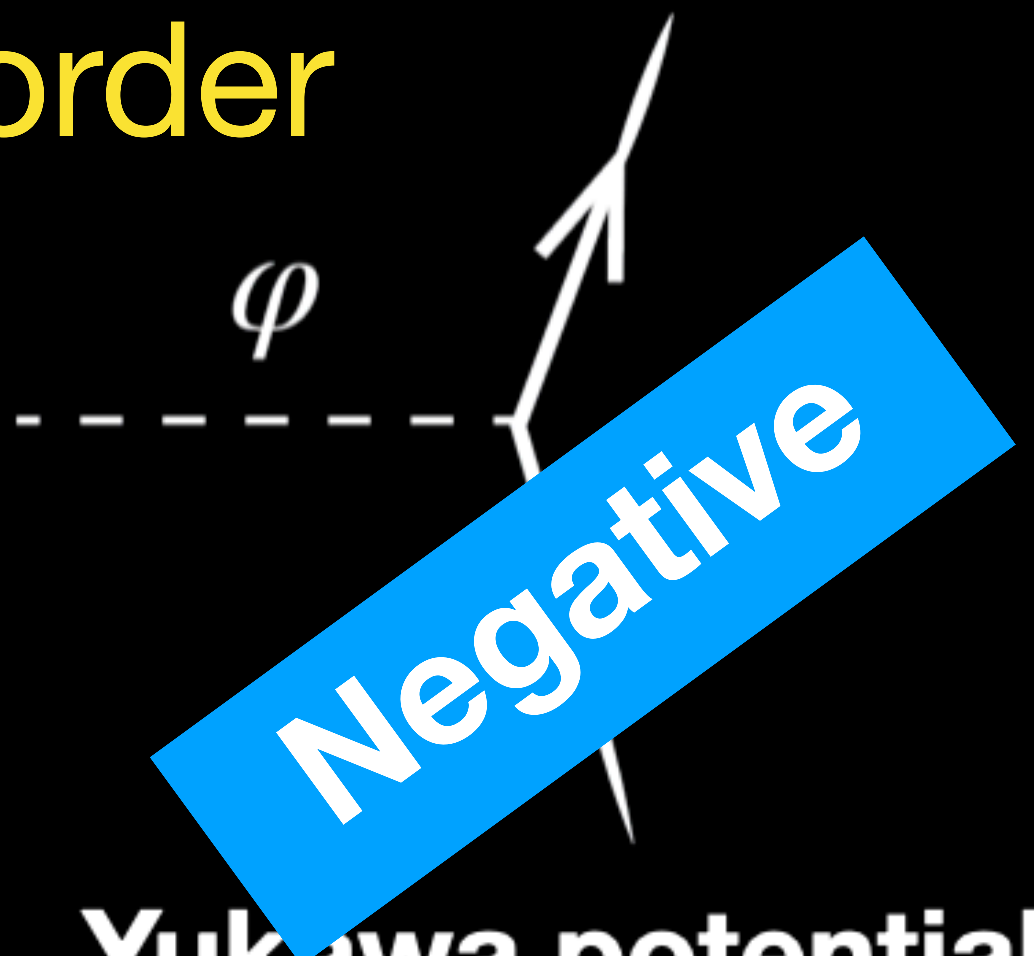
## Second order

Phonons, Magnons…

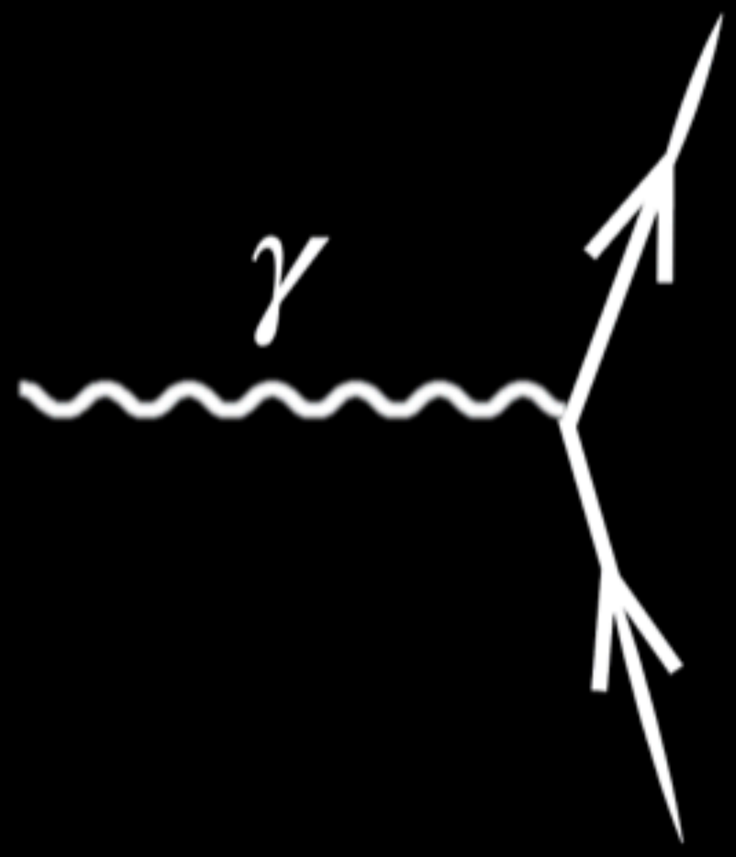


# Forces mediated by bosons

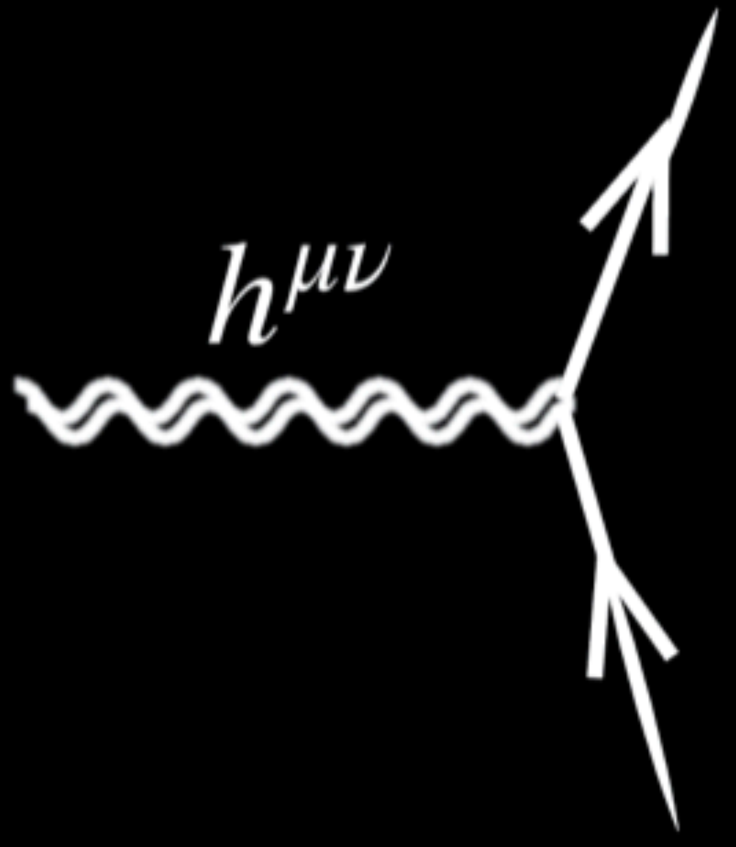
First order




Yukawa potential



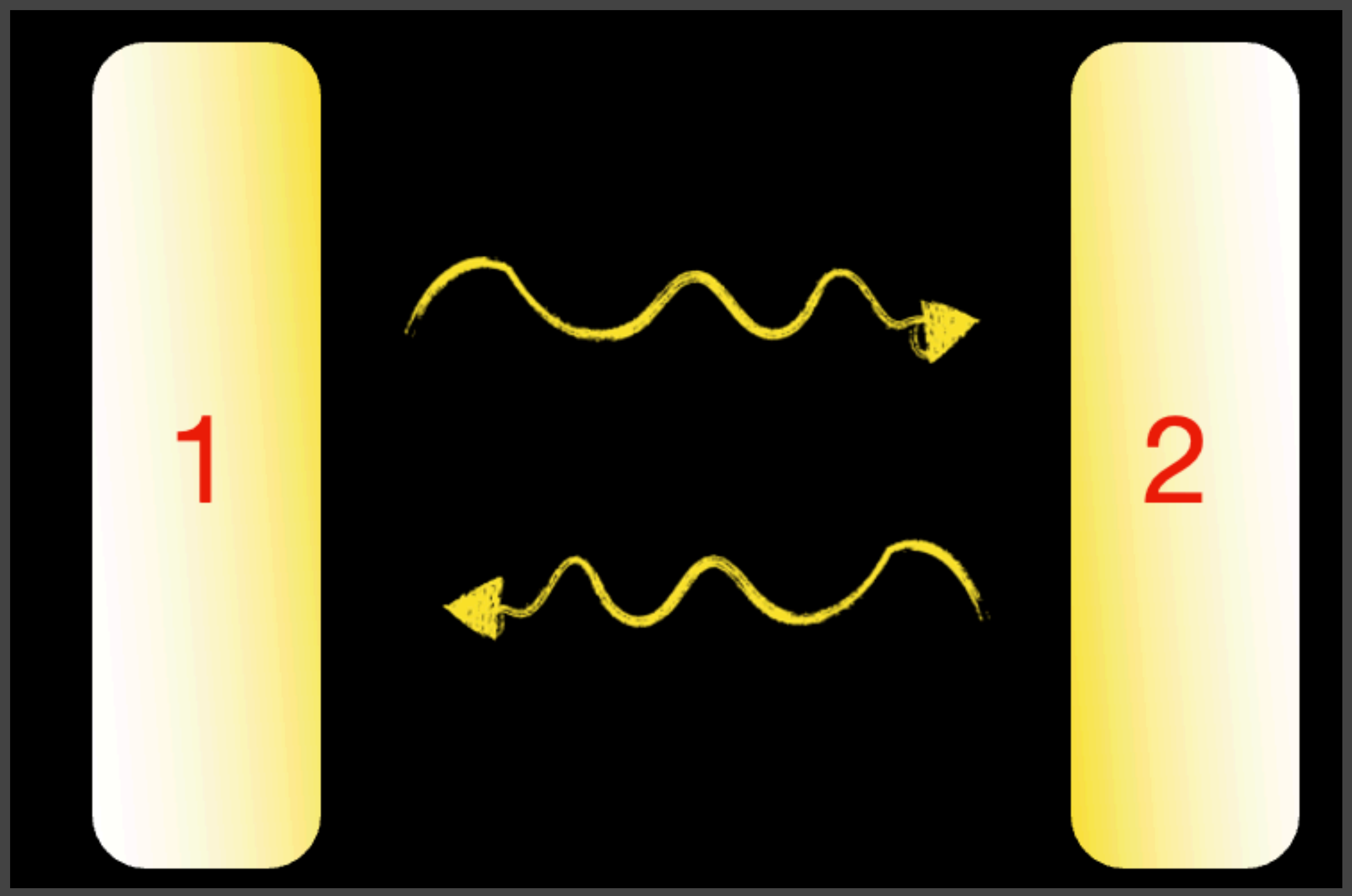

Electromagnetic forces




Gravitational forces

Second order

Phonons, Magnons,
Gravitons…



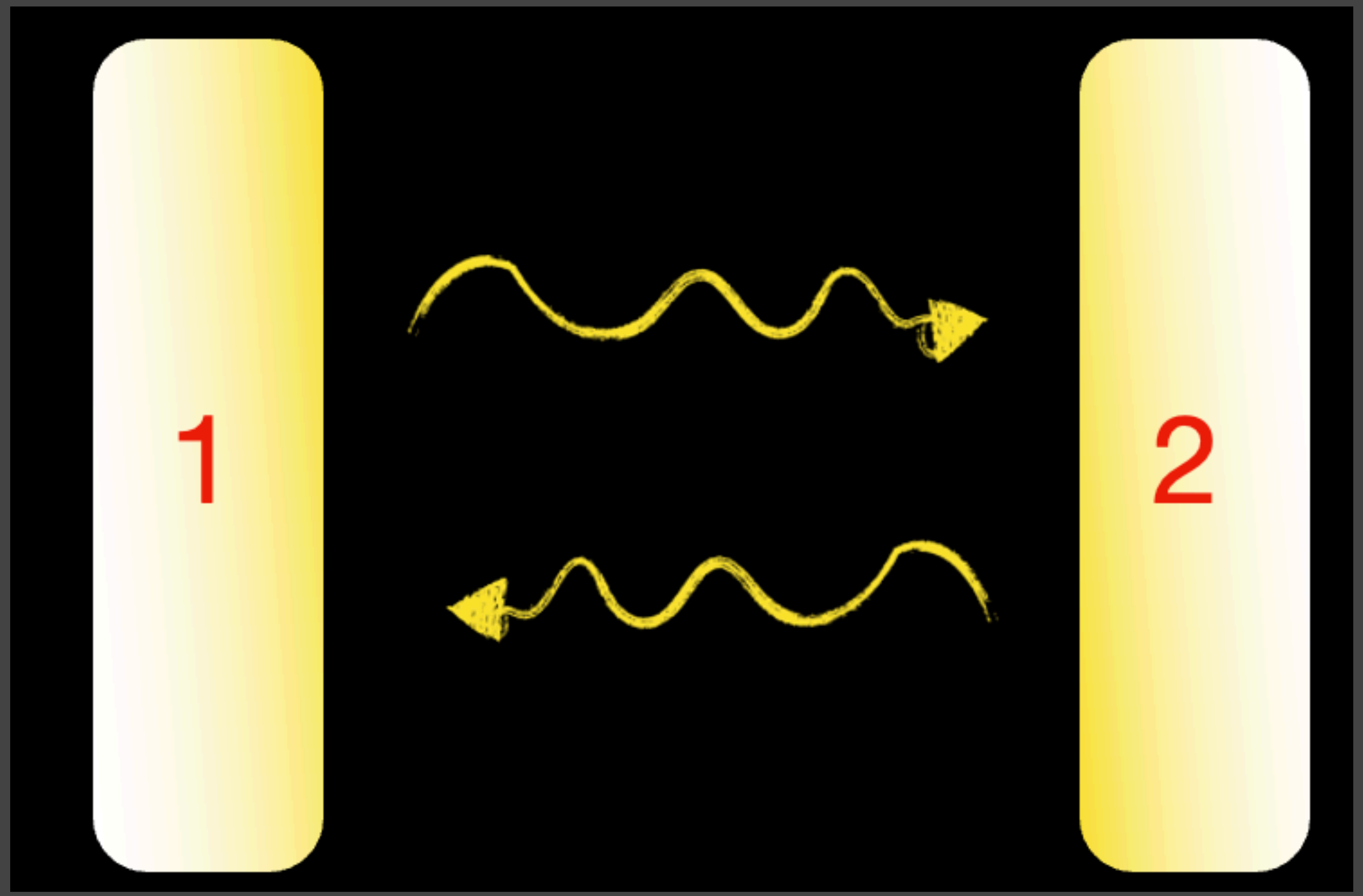
# Forces mediated by bosons

## First order


Yukawa potential | Electromagnetic forces | Gravitational forces

## Second order

Phonons, Magnons, Gravitons…



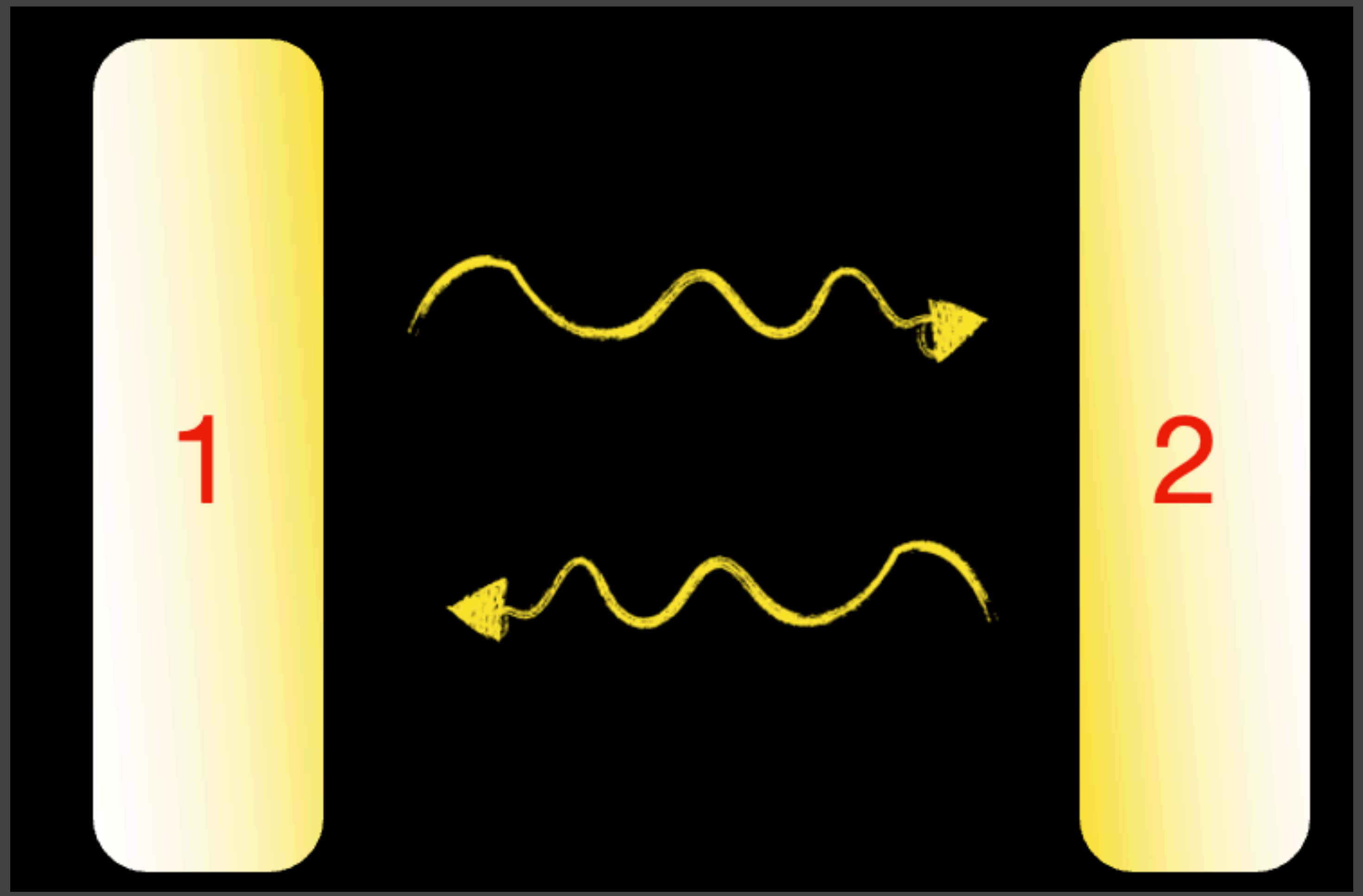
# Forces mediated by bosons

## First order


**Yukawa potential** | **Electromagnetic forces** | **Gravitational forces**

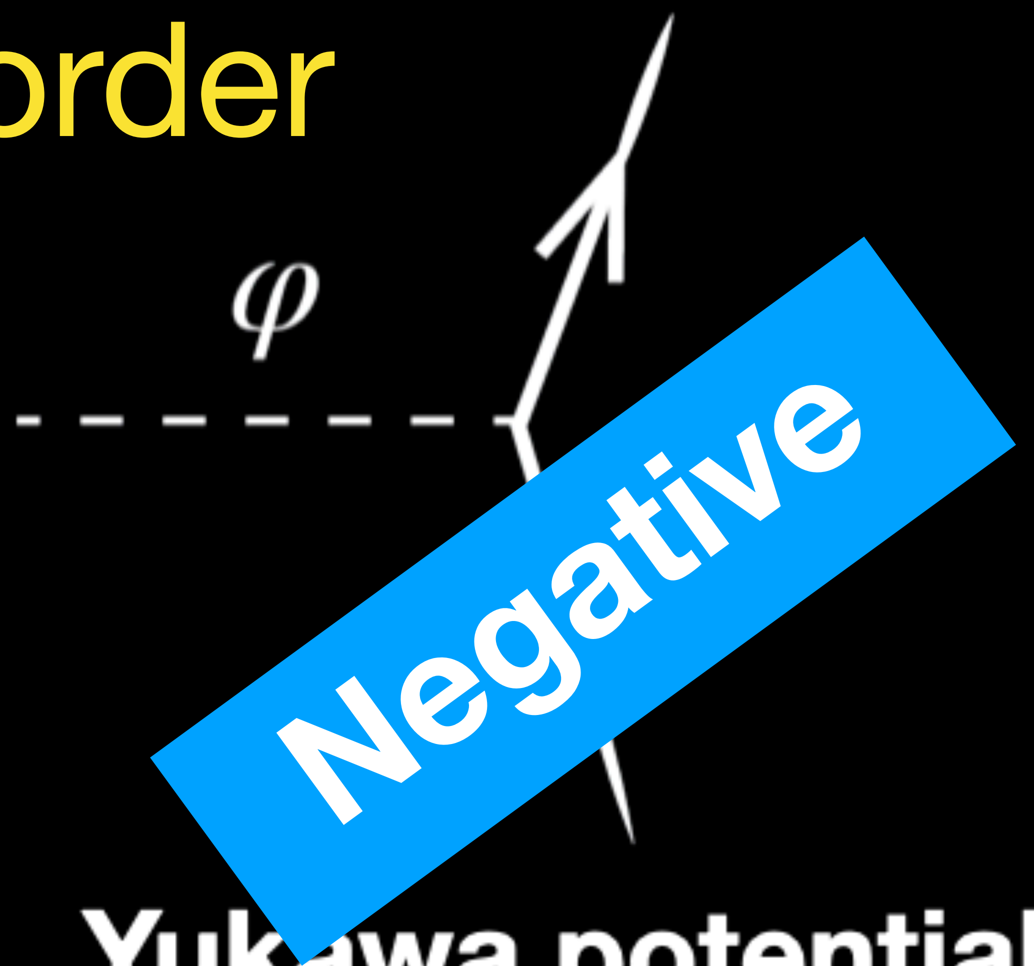
## Second order

Phonons, Magnons,
Gravitons…

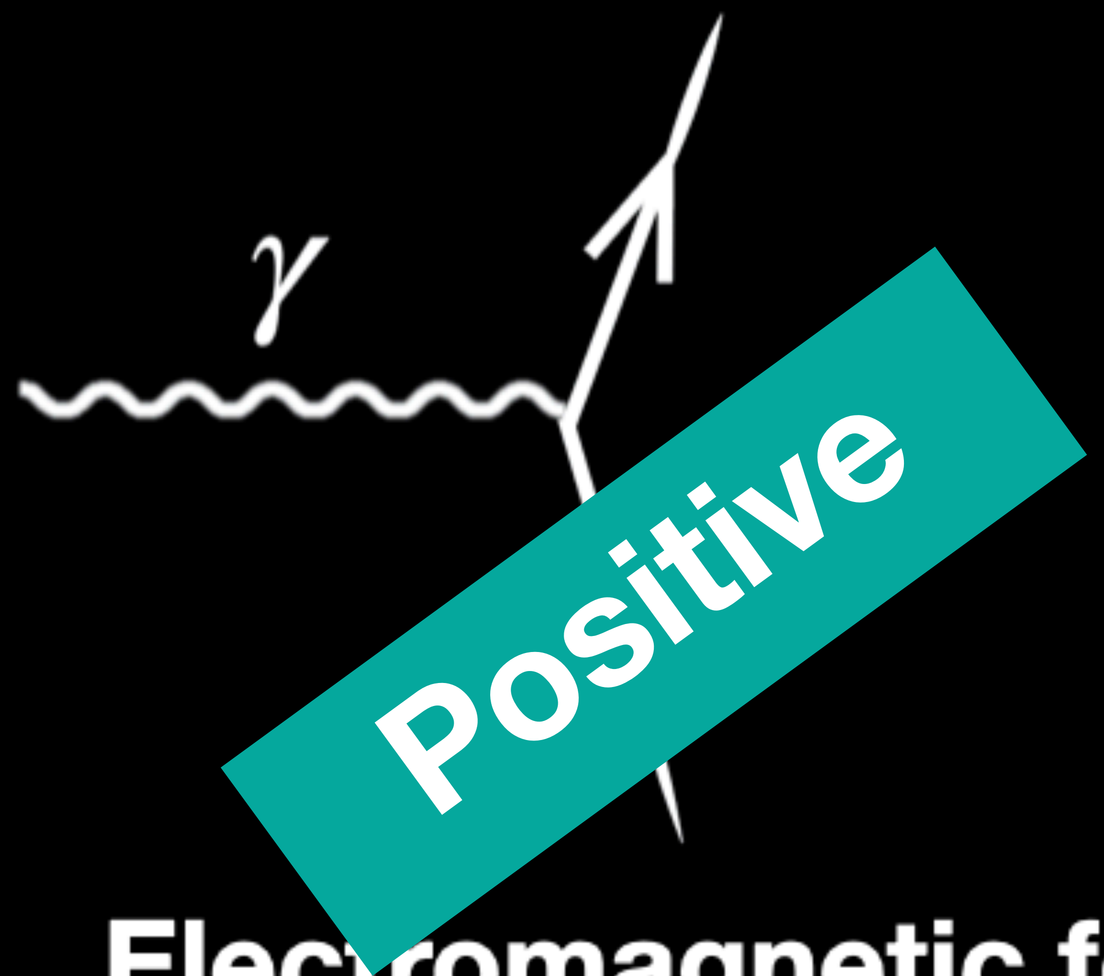


# Forces mediated by bosons

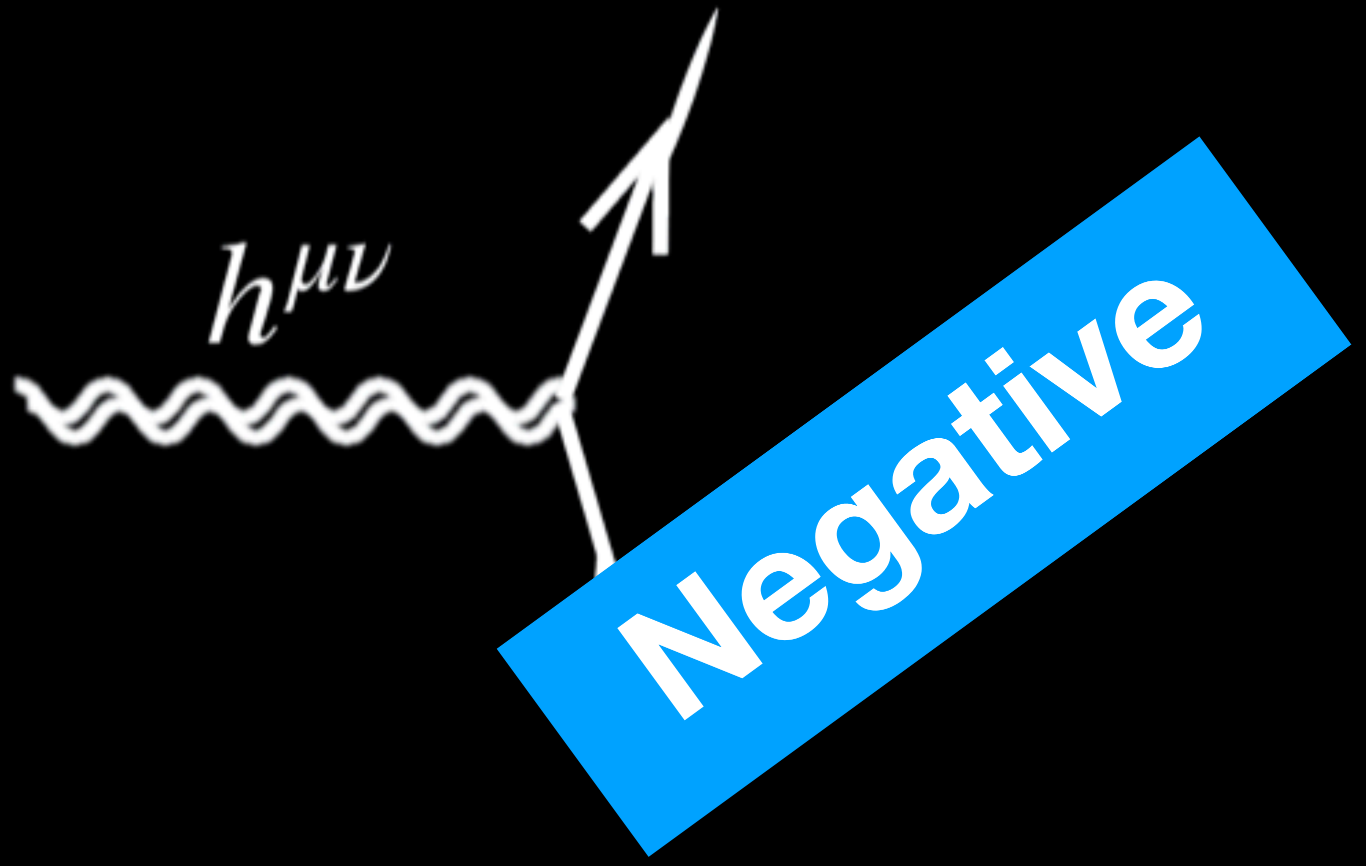
## First order




Yukawa potential




Electromagnetic forces

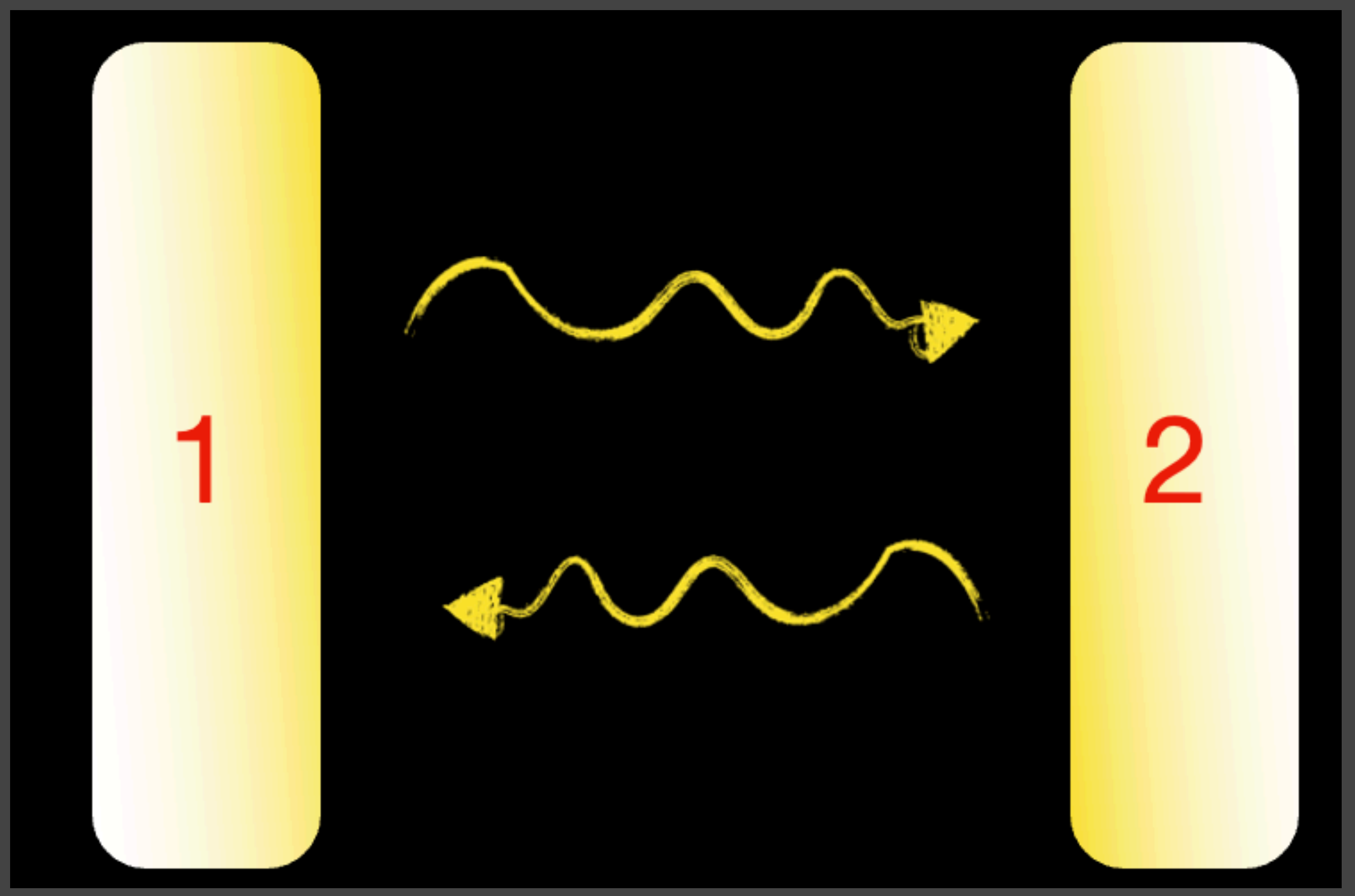



Gravitational forces

## Second order

Phonons, Magnons, Gravitons…

Negative



# What about fermions?

1. Can fermions mediate interactions?

2. If they can, what is the sign structure?

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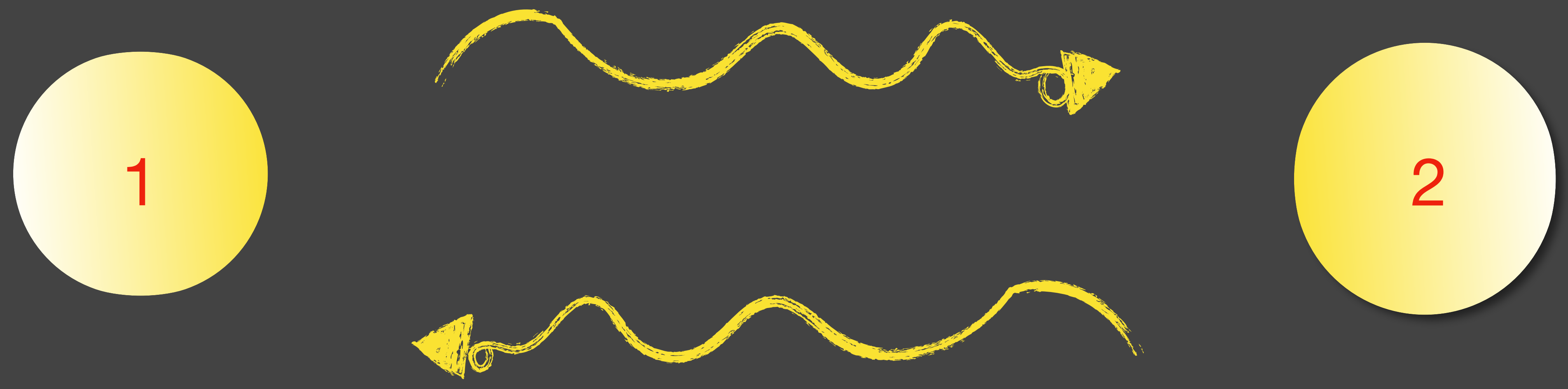
1. Can fermions mediate interactions?

2. If they can, what is the sign structure?

The lowest order is Second order

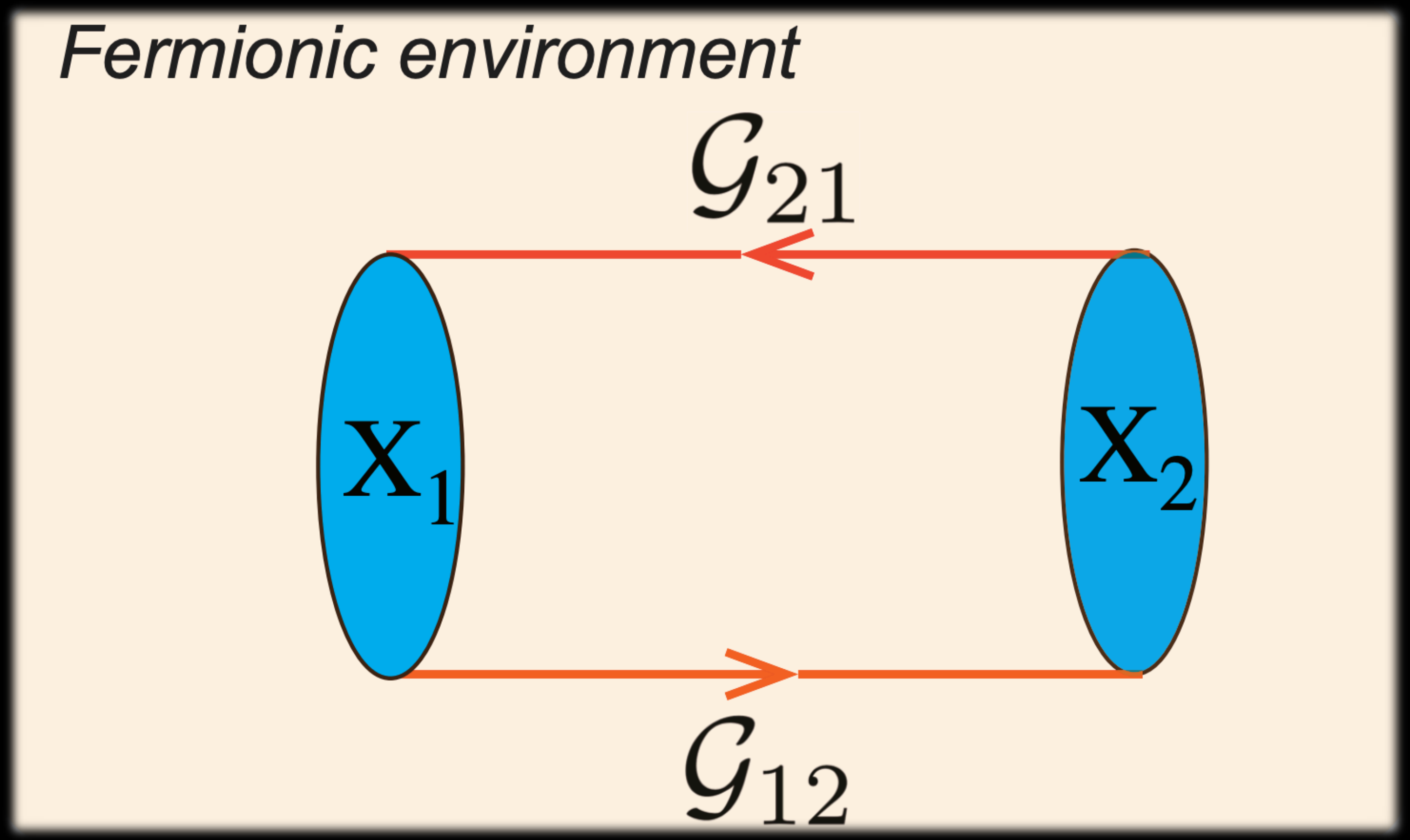



**Boson-mediated** $(-1)^{S+1}$ Vs **Fermion-mediated** ?

# Fermionic Casimir effect

$$S = \int dt d^3x \; \bar{\psi} \left( G^{-1} + V_1 + V_2 \right) \psi$$

$$Z = \int D[\bar{\psi}, \psi] e^{-S} = \mathrm{Det}_{\mathrm{t}, \vec{\mathrm{x}}} \left( \mathrm{G}^{-1} + \mathrm{V}_1 + \mathrm{V}_2 \right)$$



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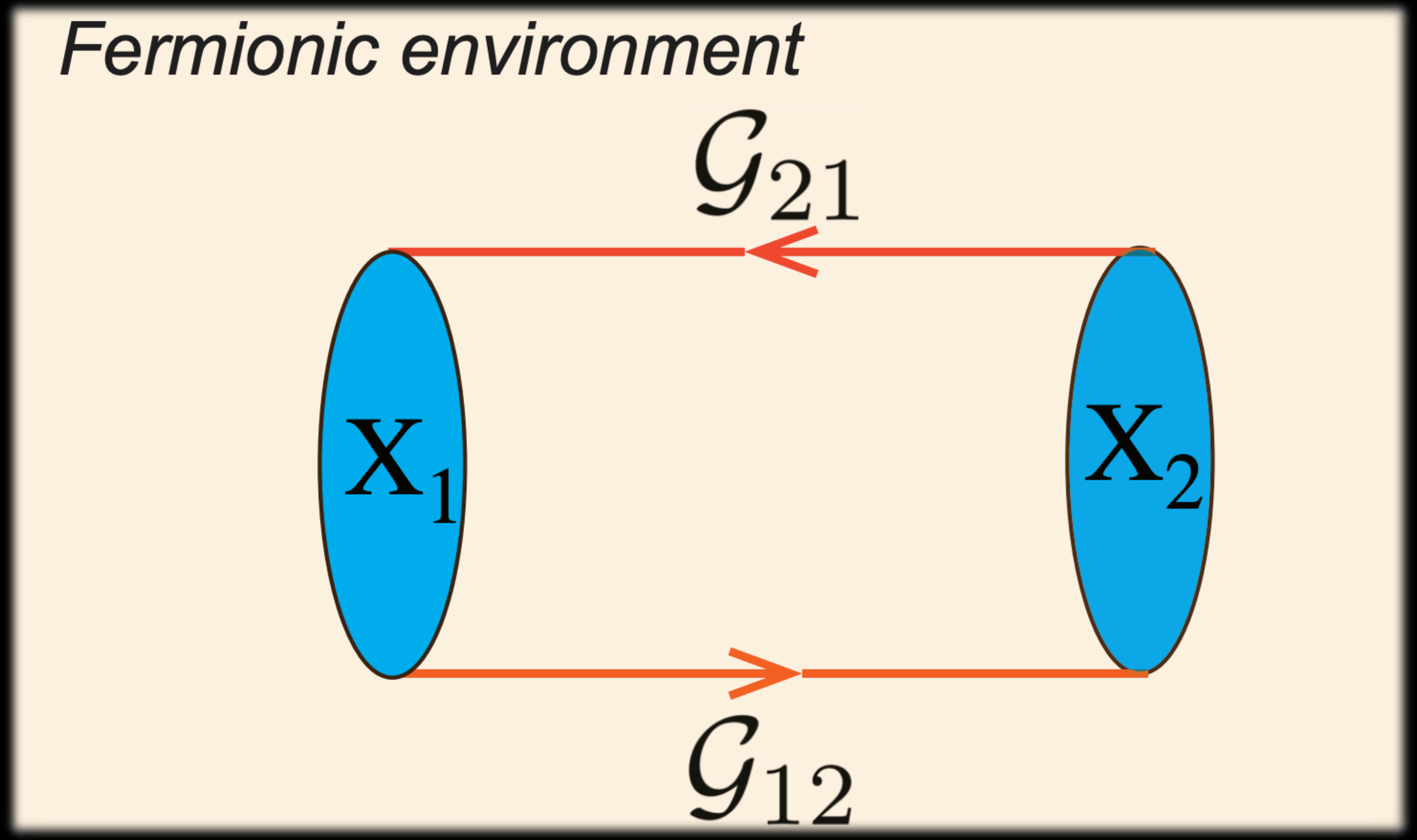



$$V_{12} = -k_B T \; \ln \frac{Z}{Z_0}$$

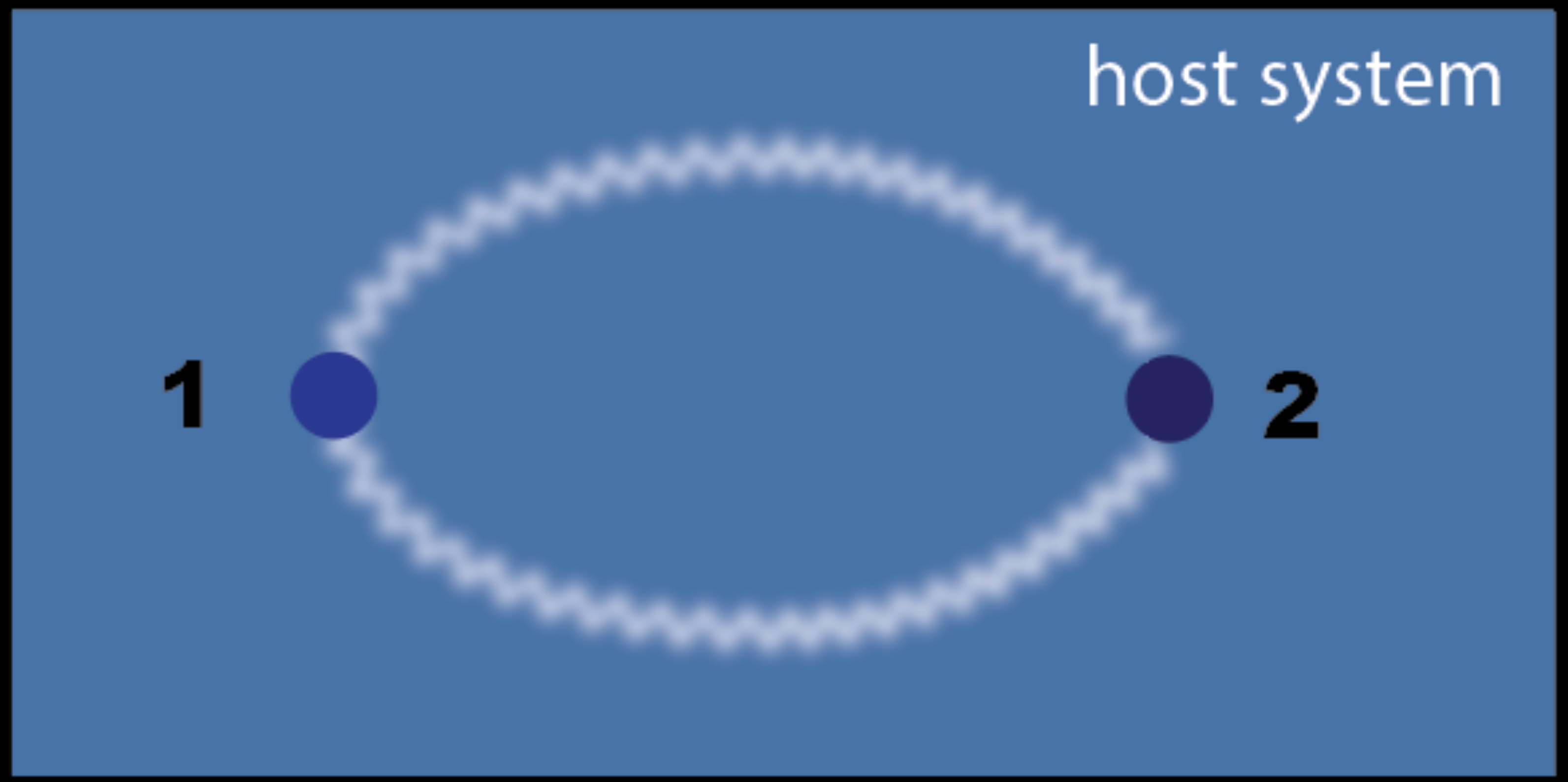
$$= -k_B T \sum_n \mathrm{Tr} \ln \left( 1 - \mathrm{G}_{12} \mathrm{T}_2 \mathrm{G}_{21} \mathrm{T}_1 \right)$$

$$\mathrm{T}_i = \langle \mathrm{r}_i | \hat{\mathrm{T}} | \mathrm{r}_i \rangle = \langle \mathrm{r}_i | \frac{\mathrm{V}}{1 + \hat{\mathrm{G}} \mathrm{V}} | \mathrm{r}_i \rangle$$

To the lowest order: $V_{12} \approx k_B T \sum_n \mathrm{Tr}\; \mathrm{G}_{12} \mathrm{T}_2 \mathrm{G}_{21} \mathrm{T}_1$

# Fermionic Casimir effect

## 1. Impurity-impurity interaction




**Surface growth/properties**

**Collective behavior of adatoms**

$$V_{12} \propto \frac{\cos(2k_F r)}{r^n}$$

**Near fermi surface, dispersion is linear; electron is massless**

*Proc. Phys. Soc. 90, 751 (1967); PRB 7, 3629 (1973); Rep. Prog. Phys. 51, 759 (1988); PRL 85, 2981 (2000);*

# Fermionic Casimir effect

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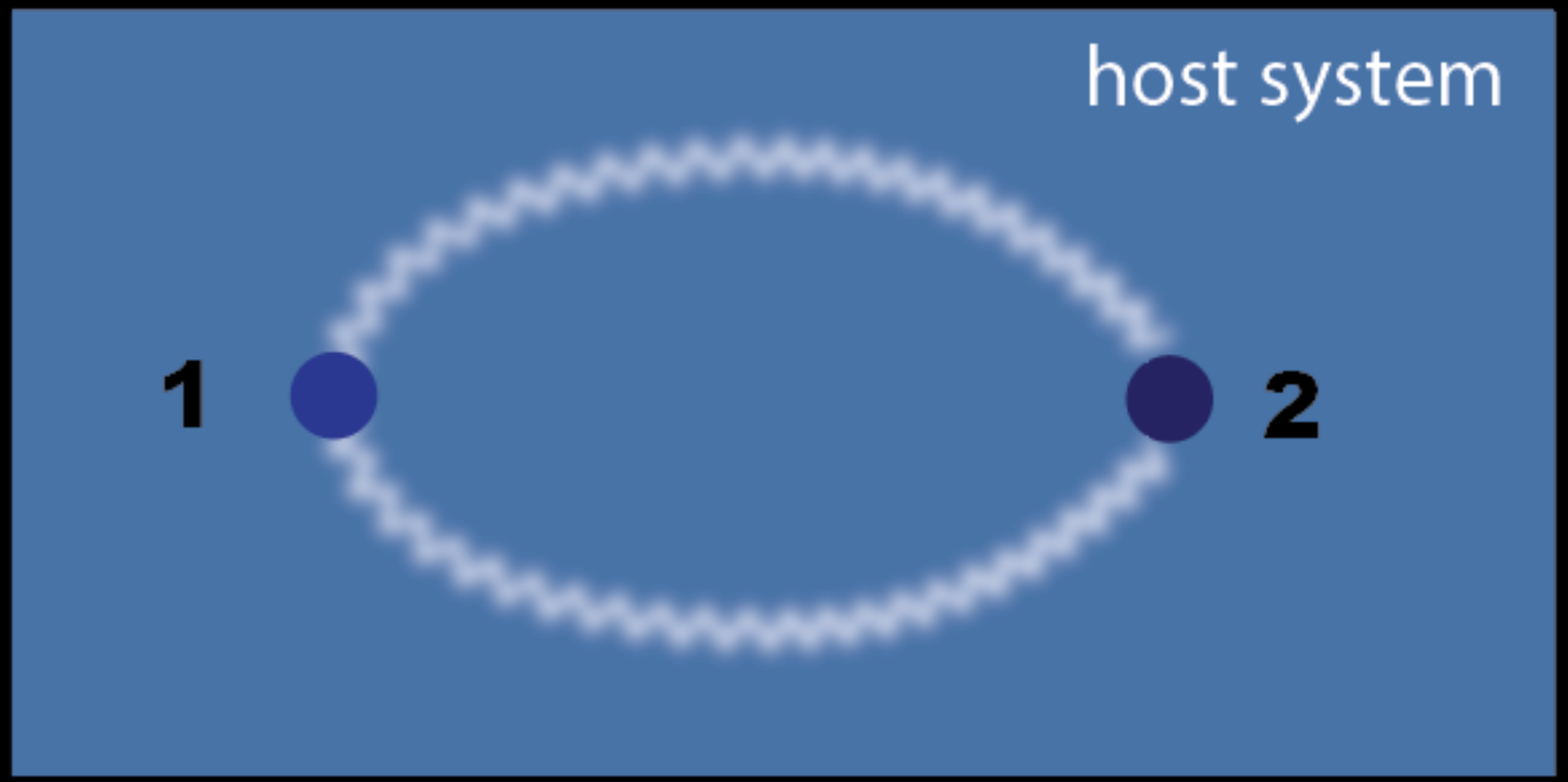


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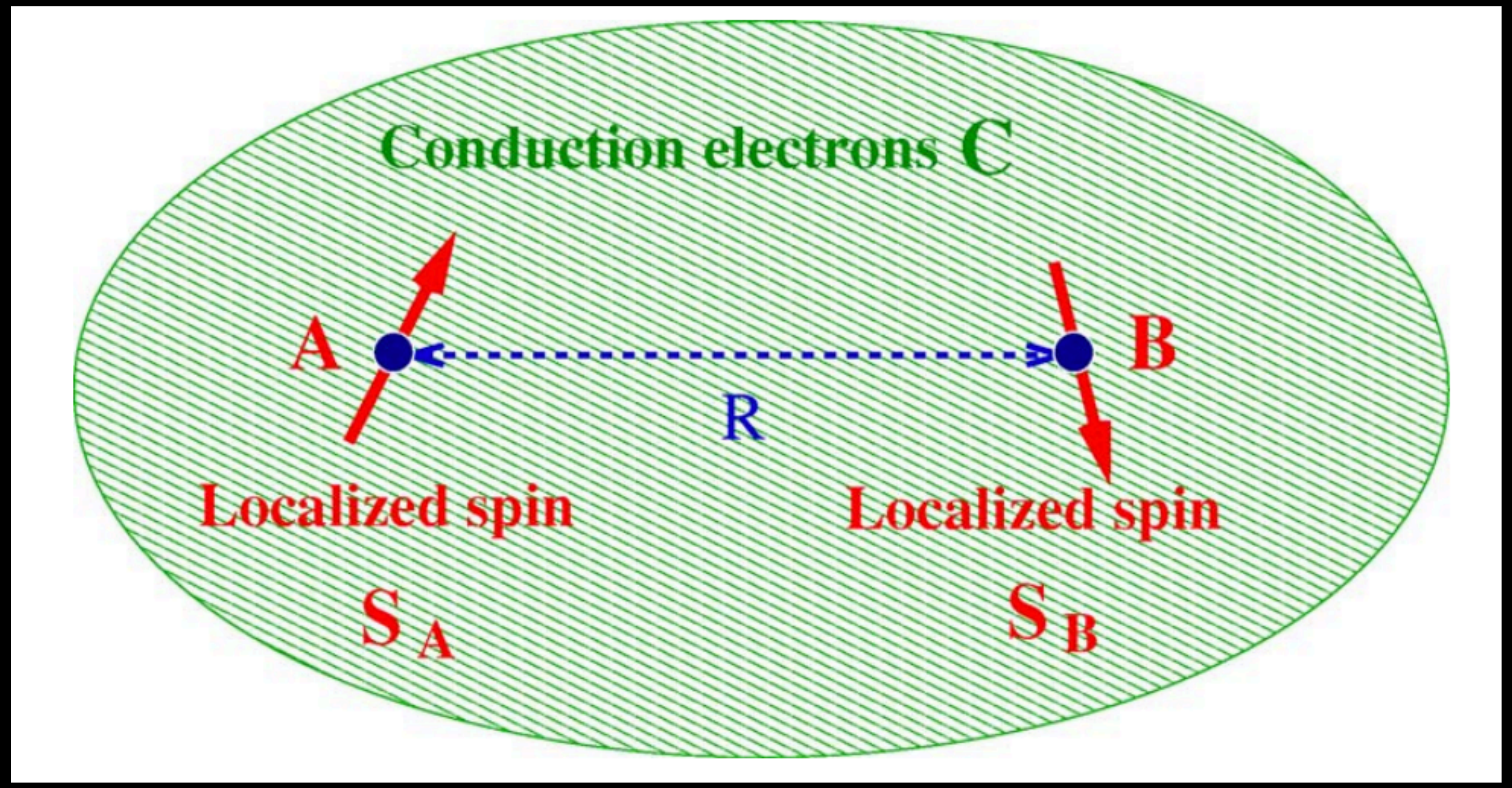
*Proc. Phys. Soc. 90, 751 (1967); PRB 7, 3629 (1973); Rep. Prog. Phys. 51, 759 (1988); PRL 85, 2981 (2000);*

## 2. Spin-spin interaction




**RKKY interaction (colinear)**

eg. GMR, FM, AFM...

**DM interaction (noncolinear)**

eg. Skyrmions...

$$V_{12} \propto \frac{\cos(2k_F r)}{r^3}\vec{s}_A \cdot \vec{s}_B$$

# Fermionic Casimir effect: The sign?

**Chiral Symmetry (CT)**

| Time-reversal Symmetry (T) | Charge-conjugate Symmetry (C) | Parity Symmetry (P) |
|---|---|---|

| class\ $\delta$ | T | C | S | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| A | 0 | 0 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 |
| AIII | 0 | 0 | 1 | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ | 0 | $\mathbb{Z}$ |
| AI | + | 0 | 0 | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |
| BDI | + | + | 1 | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ |
| D | 0 | + | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 |
| DIII | − | + | 1 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 | $2\mathbb{Z}$ |
| AII | − | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 | 0 |
| CII | − | − | 1 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 | 0 |
| C | 0 | − | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | 0 |
| CI | + | − | 1 | 0 | 0 | 0 | $2\mathbb{Z}$ | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ |

IQHE (A, $\delta=2$)
SSH (BDI, $\delta=1$)
Chiral SC (D, $\delta=2$)
QSHE (AII, $\delta=2$)
He3 B (DIII, $\delta=3$)

*Ryu, Schnyder, Furusaki, Ludwig, NJP 2010; Chiu, et. al. RMP 2016*

# Fermionic Casimir effect: The sign!

$$U_T^\dagger \mathcal{H}^*(\mathbf{k}) U_T = \mathcal{H}(-\mathbf{k})$$
$$U_C^\dagger \mathcal{H}^*(\mathbf{k}) U_C = -\mathcal{H}(-\mathbf{k})$$
$$U_S^\dagger \mathcal{H}(\mathbf{k}) U_S = -\mathcal{H}(\mathbf{k})$$

→

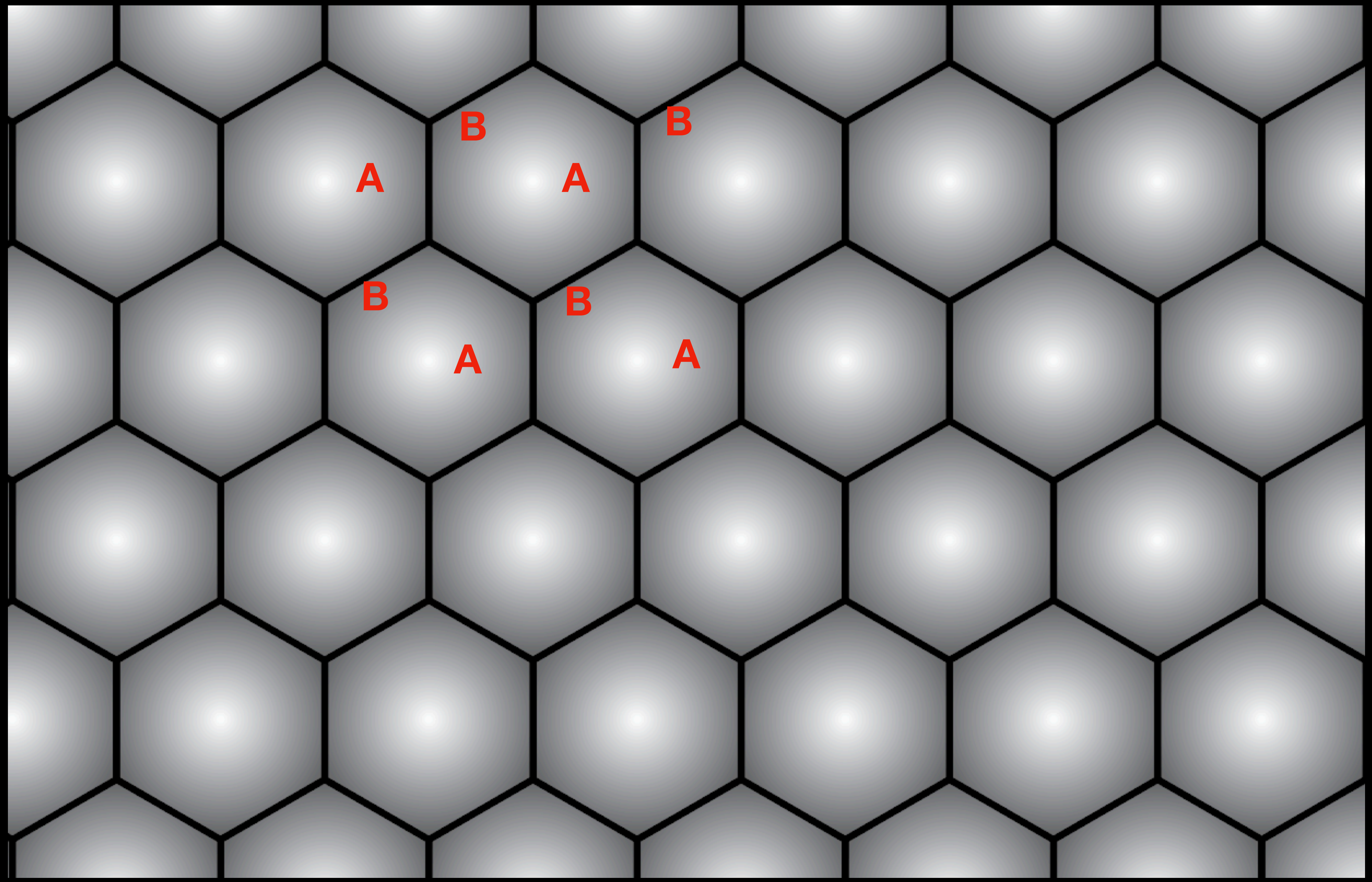
$$U_T \mathcal{G}(i\omega_n, \mathbf{k}) U_T^\dagger = \frac{1}{i\omega_n - \mathcal{H}^*(-\mathbf{k})} = \mathcal{G}^*(-i\omega_n, -\mathbf{k})$$
$$U_C \mathcal{G}(i\omega_n, \mathbf{k}) U_C^\dagger = \frac{1}{i\omega_n + \mathcal{H}^*(-\mathbf{k})} = -\mathcal{G}^*(i\omega_n, -\mathbf{k})$$
$$U_S \mathcal{G}(i\omega_n, \mathbf{k}) U_S^\dagger = -\mathcal{G}(-i\omega_n, \mathbf{k})$$
$$\mathcal{G}(i\omega_n, \mathbf{k}) = \mathcal{G}^\dagger(-i\omega_n, \mathbf{k})$$

$$U_S \mathcal{G}(i\omega_n, \mathbf{r}) U_S^\dagger = -\mathcal{G}^\dagger(i\omega_n, -\mathbf{r})$$

$$V_{12} \sim \mathrm{Tr}(G_{12}G_{21}) \propto \pm \mathrm{Tr}\left(AA^\dagger\right)$$

$$\mathrm{sgn}(U_{12}) = (-1)^\eta \chi_1 \chi_2.$$

# Fermionic Casimir effect: The sign!




*Shytov, Abanin, Levitov PRL 2009*

# The sign rule of fermionic Casimir effect

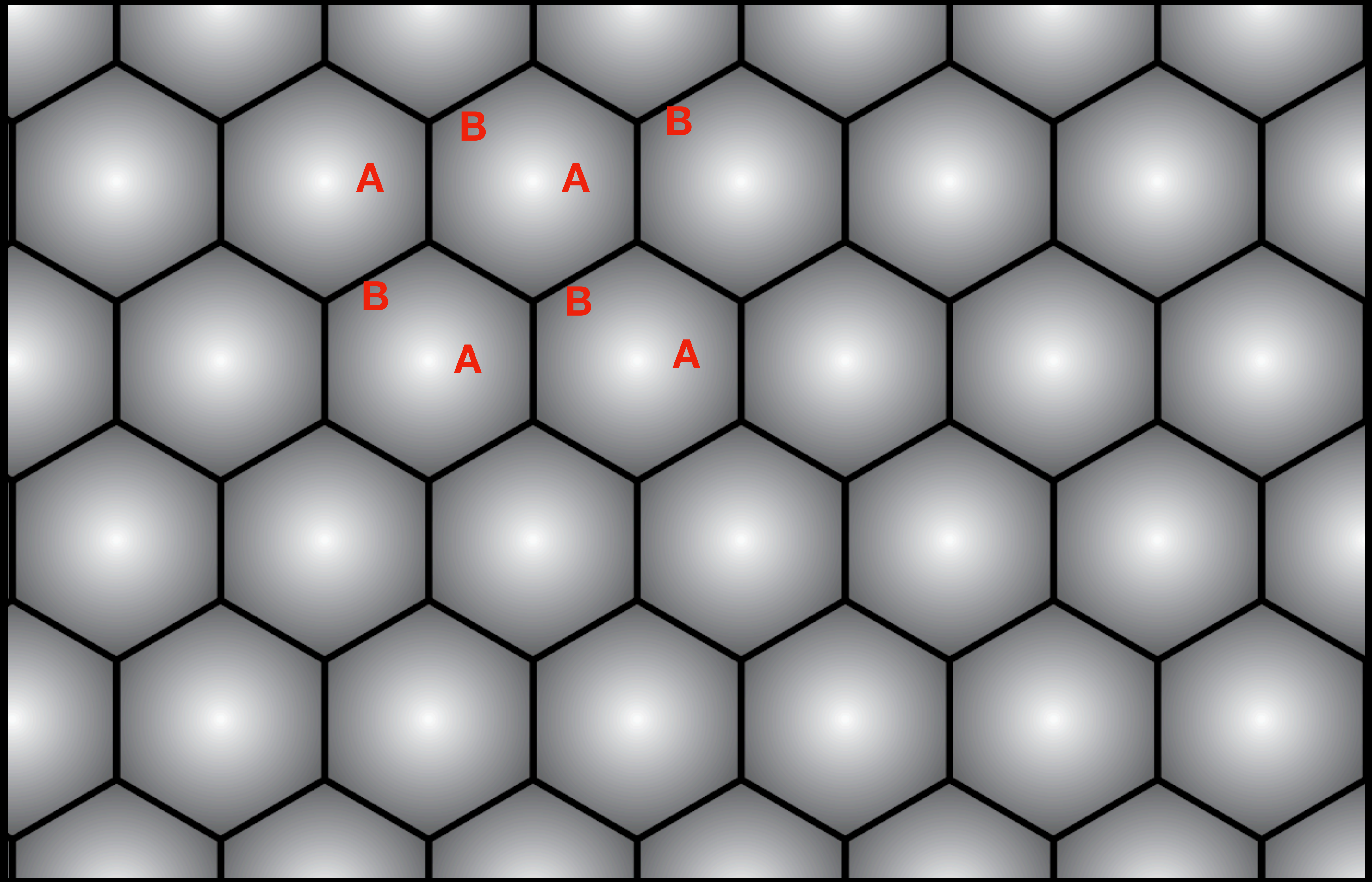
## BLG

$(\psi_{A1}, \psi_{A2}, \psi_{B1}, \psi_{B2})$

$$\mathcal{H}_{\rm BLG} = \begin{pmatrix} 0 & \mathcal{D}^*(-\mathbf{r}) \\ \mathcal{D}(\mathbf{r}) & 0 \end{pmatrix}, \mathcal{D}(\mathbf{r}) = \begin{pmatrix} -i\partial_+ & \gamma \\ 0 & -i\partial_+ \end{pmatrix}$$

## TBLG

$$\mathcal{D}_{\rm tBLG}(\mathbf{r}) = \begin{pmatrix} -v_F i\partial_+ & \mathcal{F}(\theta, \mathbf{r}) \\ \mathcal{F}(\theta, -\mathbf{r}) & -v_F i\partial_+ \end{pmatrix}$$




*Shytov, Abanin, Levitov PRL 2009*

## Hubbard model on a bipartite lattice

$$\hat{H} = \sum_{\langle i,j\rangle,\sigma} t_{ij}\hat{c}^\dagger_{i,\sigma}\hat{c}_{j,\sigma} - \mu\sum_i \hat{n}_{i\sigma} + U\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}, \quad (19)$$

*QDJ, Phys. Rev. B 103, L121107 (2021)*

- the end of lecture 1

***Thank You!***

***Take home message?***

***May the force be with you!***

# Appendices

$$A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$$

$$A = 1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$= 1 - A$$

$$\frac{1}{2} = A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$A = 1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$= 1 - A$$

$$\frac{1}{2} = A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 1$$

$$\frac{1}{2} = A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 1$$

$$B = \quad 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 1$$

$$\frac{1}{2} = A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 1$$

$$+$$

$$B = \quad 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 1$$

$$\parallel$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$\frac{1}{2} = A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$\frac{1}{4} = B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 12 +$$

$$\boxed{S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 1}$$

$$4S = 4 + 8 + 12 + 16 + 20 + 24 + 28 + 32 + 36 + 40 + 44 +$$

$$\frac{1}{2} = A = 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 -$$

$$\frac{1}{4} = B = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 - 12 +$$

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 +$$

$$-$$

$$4S = 4 + 8 + 12 + 16 + 20 +$$

$$=$$

$$-3S = 1 - 2 + 3 - 4 + 5 - 6 + 7 - 8 + 9 - 10 + 11 -$$

$$S = -\frac{1}{12}$$