Less is more:

The power of vacuum quantum fluctuations

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TSUNG-DAO LEE INSTITUTE

Quantum Connections in Sweden-16
Summer School

Lecture 1: Renormalization and Casimir Physics

Lecture 2: Casimir Torque, Friction and Spectra

Lecture 3: Quantum atmosphere

Lecture 4: Cavity Quantum Materials

What quantum fluctuations do to Symmetry?

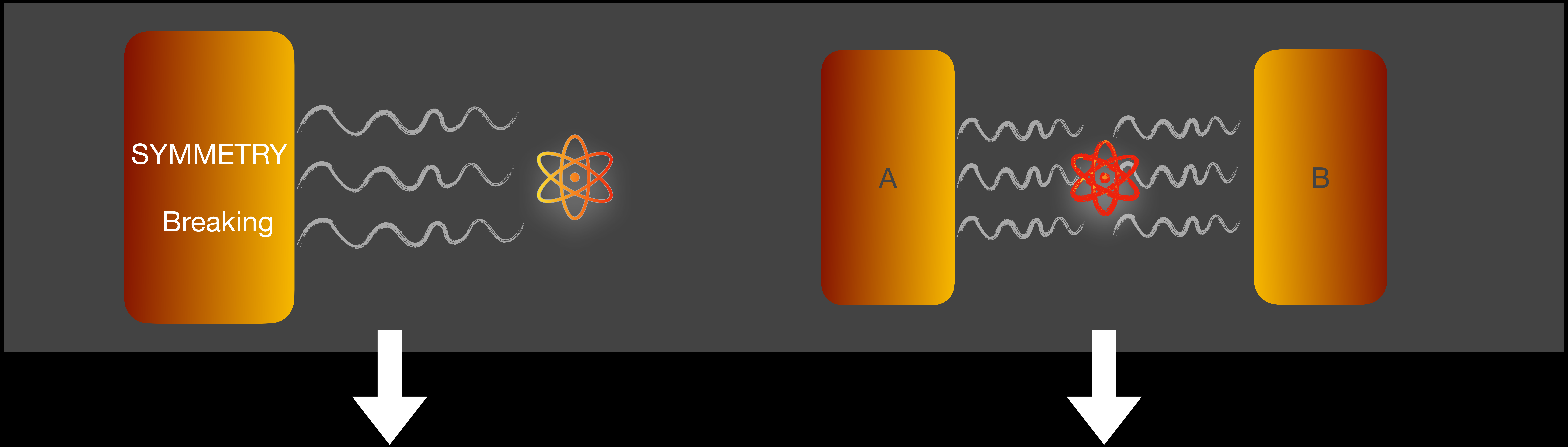
1) Quantum fluctuations can **induce** symmetry breaking

Quantum Anomaly

2) Quantum fluctuations can **transmit** symmetry breaking

Quantum Atmosphere (Less known)

Qualitative Change and Quantative Enhancement



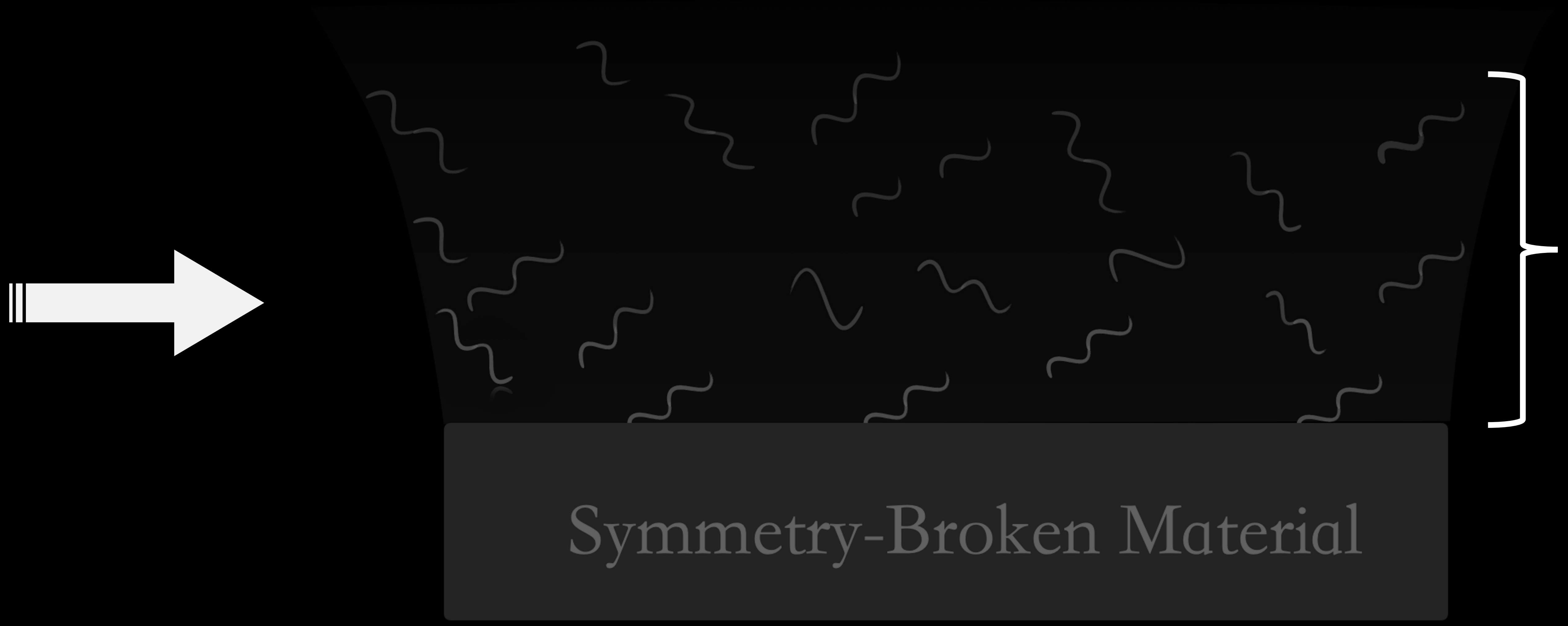
Quantum Atmosphere

Chiral Lamb Shift

Quantum Atmosphere

Classical

Quantum Atmosphere



QDJ and Frank Wilczek, PRB 99, 201104 (R) (2019)

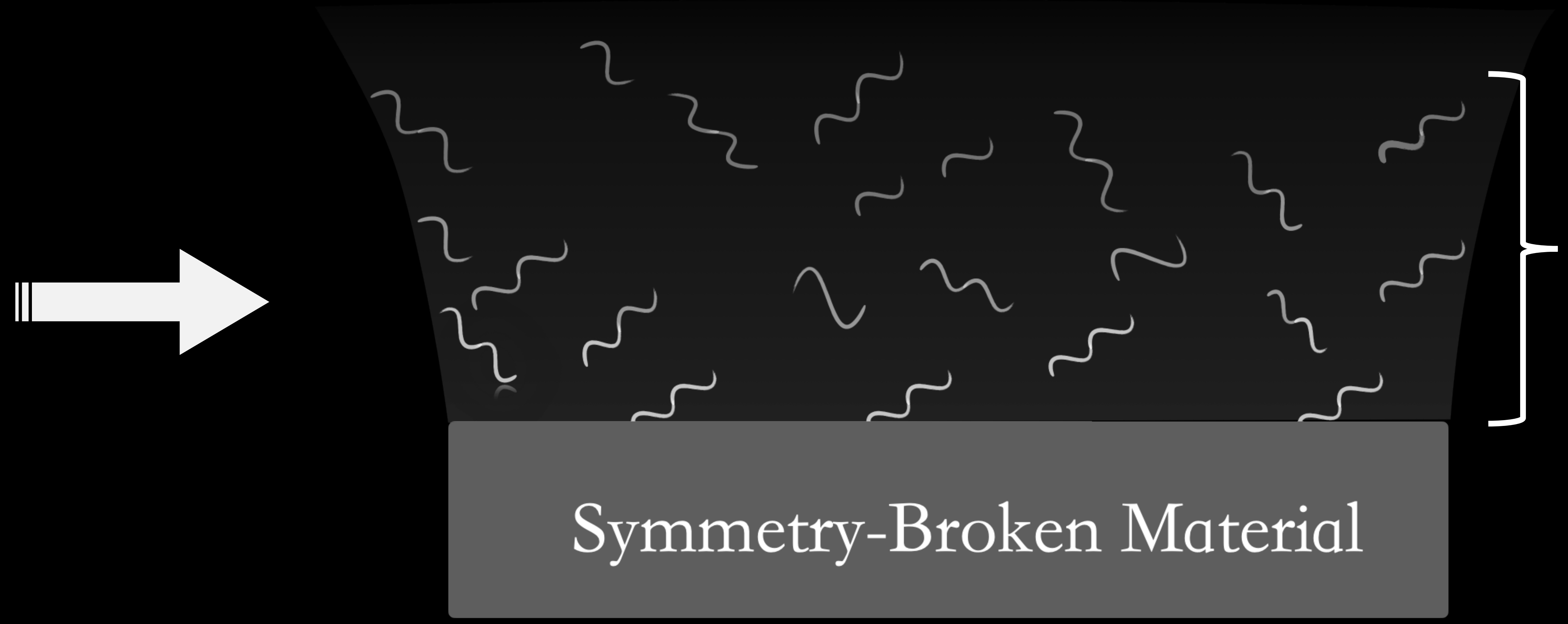
Reading the atmosphere, Nature Materials, **17**, 951 (News and Views)

Quanta Magazine: Quantum Atmosphere May Reveal Secret of Matter

Quantum Atmosphere

Classical

Quantum Atmosphere



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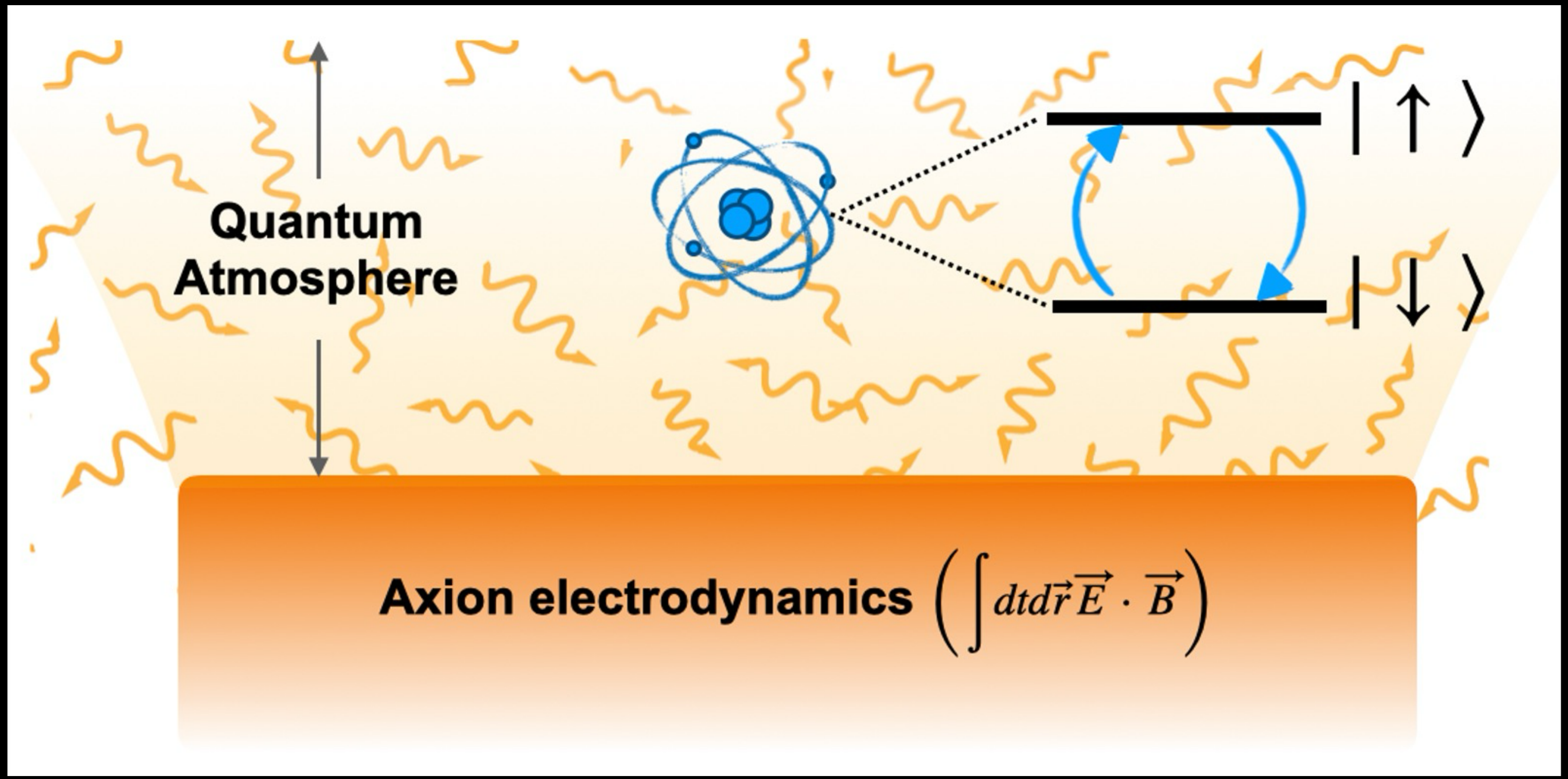
Quantum atmosphere in AI's eyes

How to observe Quantum Atmosphere?

T-broken quantum atmosphere

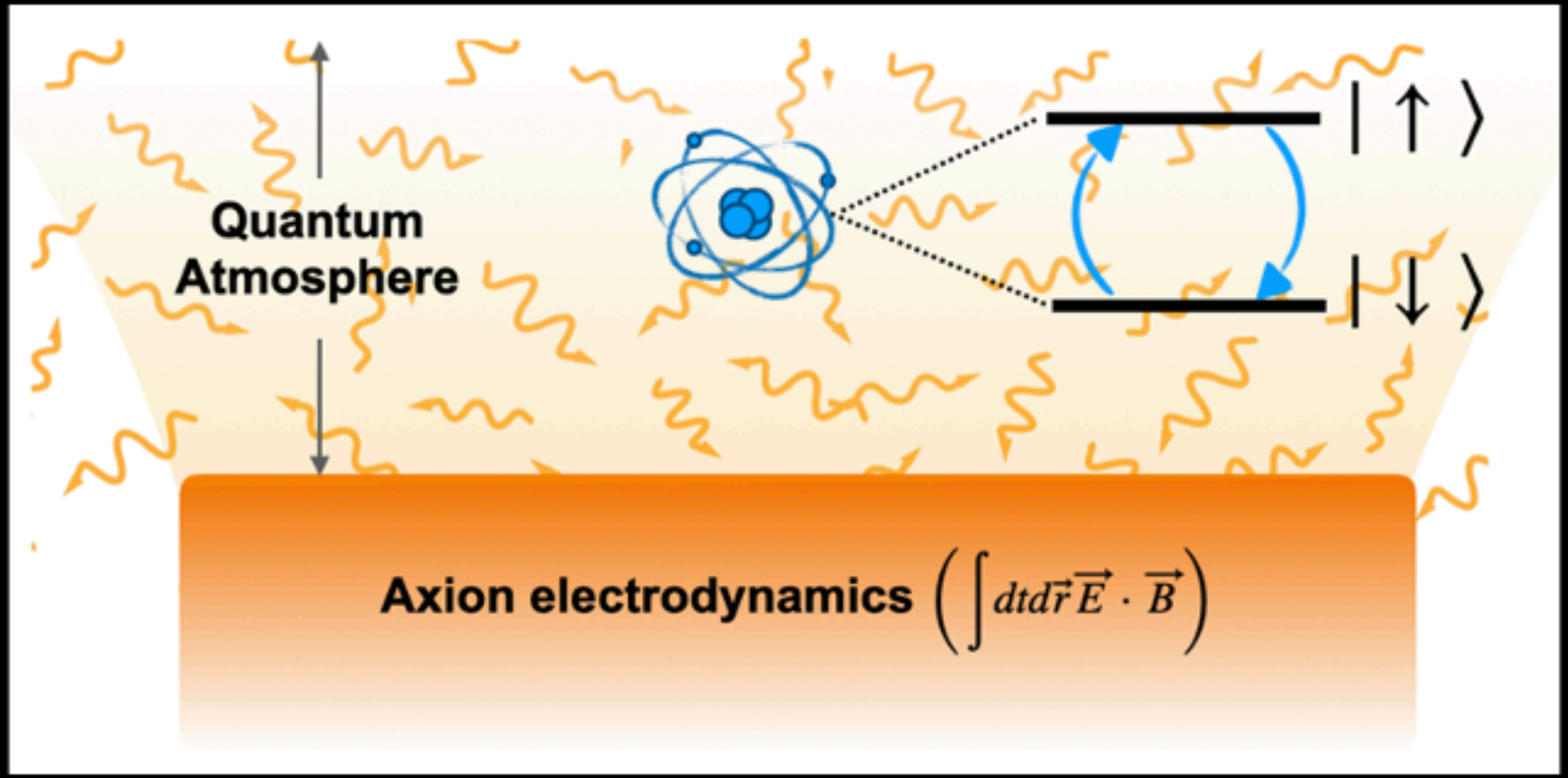


No real magnetic field
but Zeeman effect

QDJ and Frank Wilczek, PRB 99, 201104(R) (2019)

How to calculate/detect Quantum Atmosphere?

T-broken quantum atmosphere



No real magnetic field
but Zeeman effect

Jiang and Wilczek, PRB 99, 201104(R) (2019)

$$\theta\, E \cdot B$$

TRS operation

$$E \rightarrow E$$

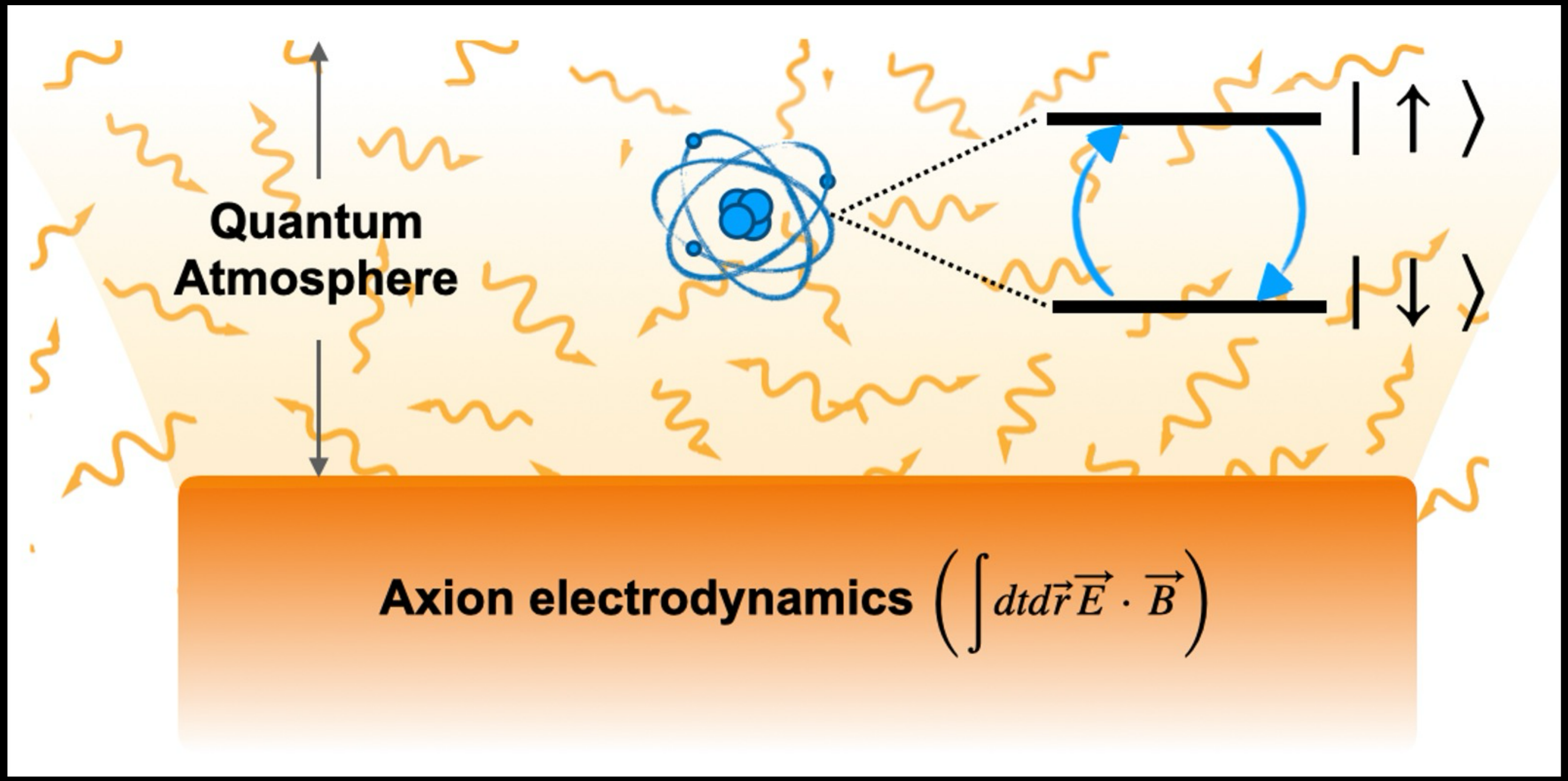
$$B \rightarrow -B$$

$$\theta\, E \cdot B$$

Breaks time-reversal symmetry

How to observe Quantum Atmosphere?

T-broken quantum atmosphere



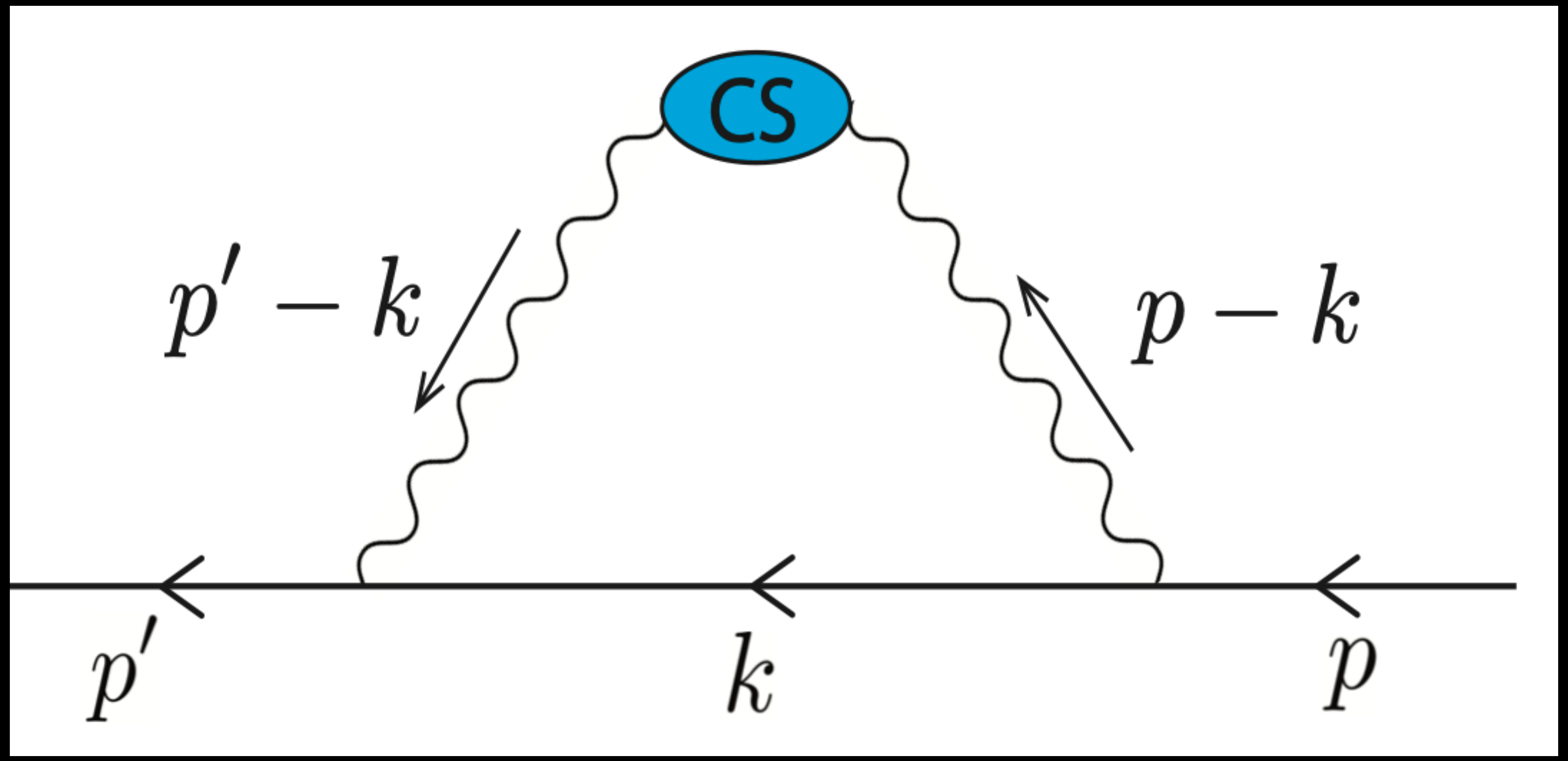
No real magnetic field
but Zeeman effect

QDJ and Frank Wilczek, PRB 99, 201104(R) (2019)

Chern-Simons Lagrangian

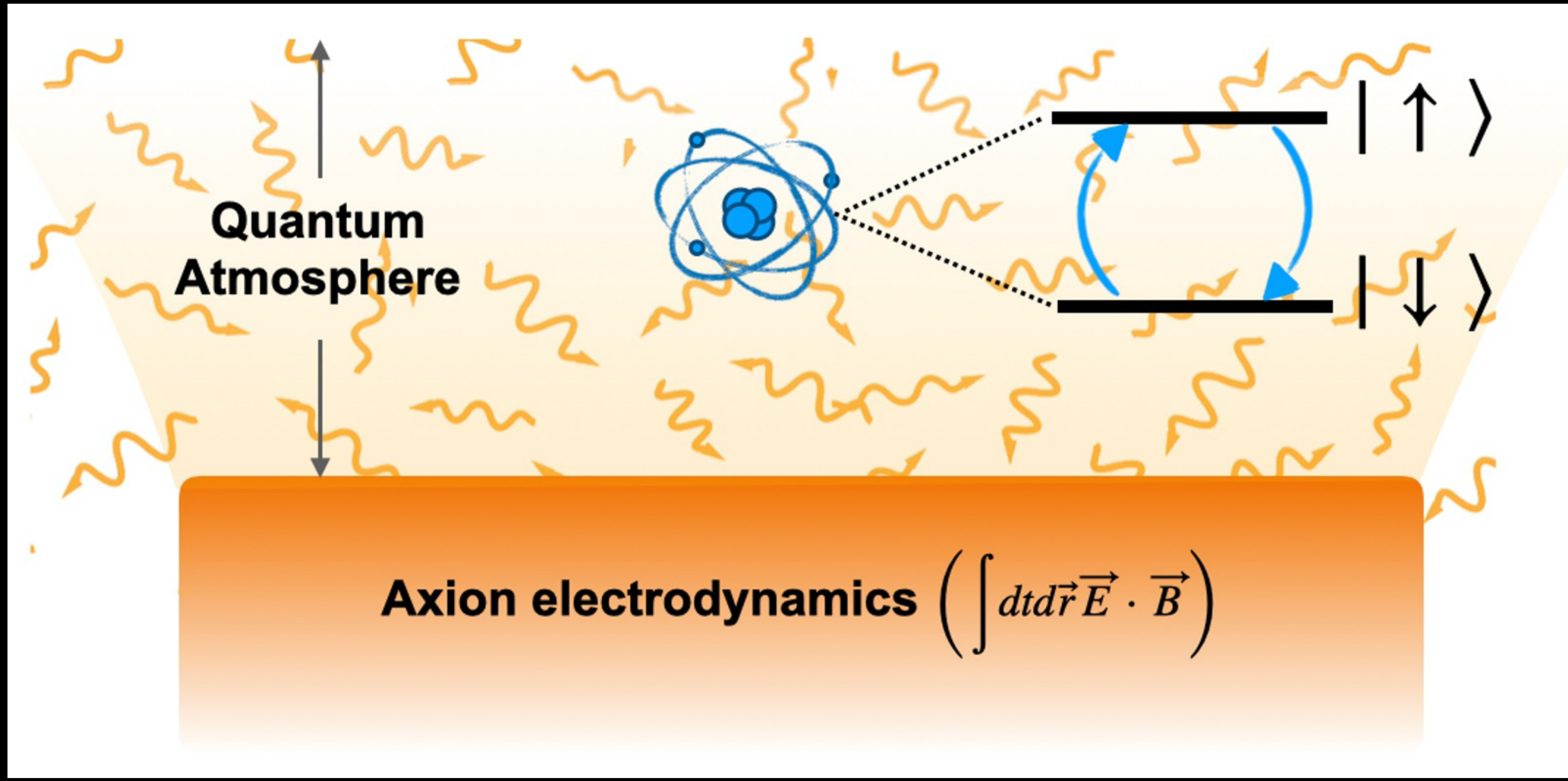
$$L_{cs} = \epsilon_{ijk} A_i \partial_j A_k$$

P breaking and T breaking



How to observe Quantum Atmosphere?

T-broken quantum atmosphere



No real magnetic field
but Zeeman effect

QDJ and Frank Wilczek, PRB 99, 201104(R) (2019)

$$L_{cs} = \epsilon_{ijk} A_i \partial_j A_k$$

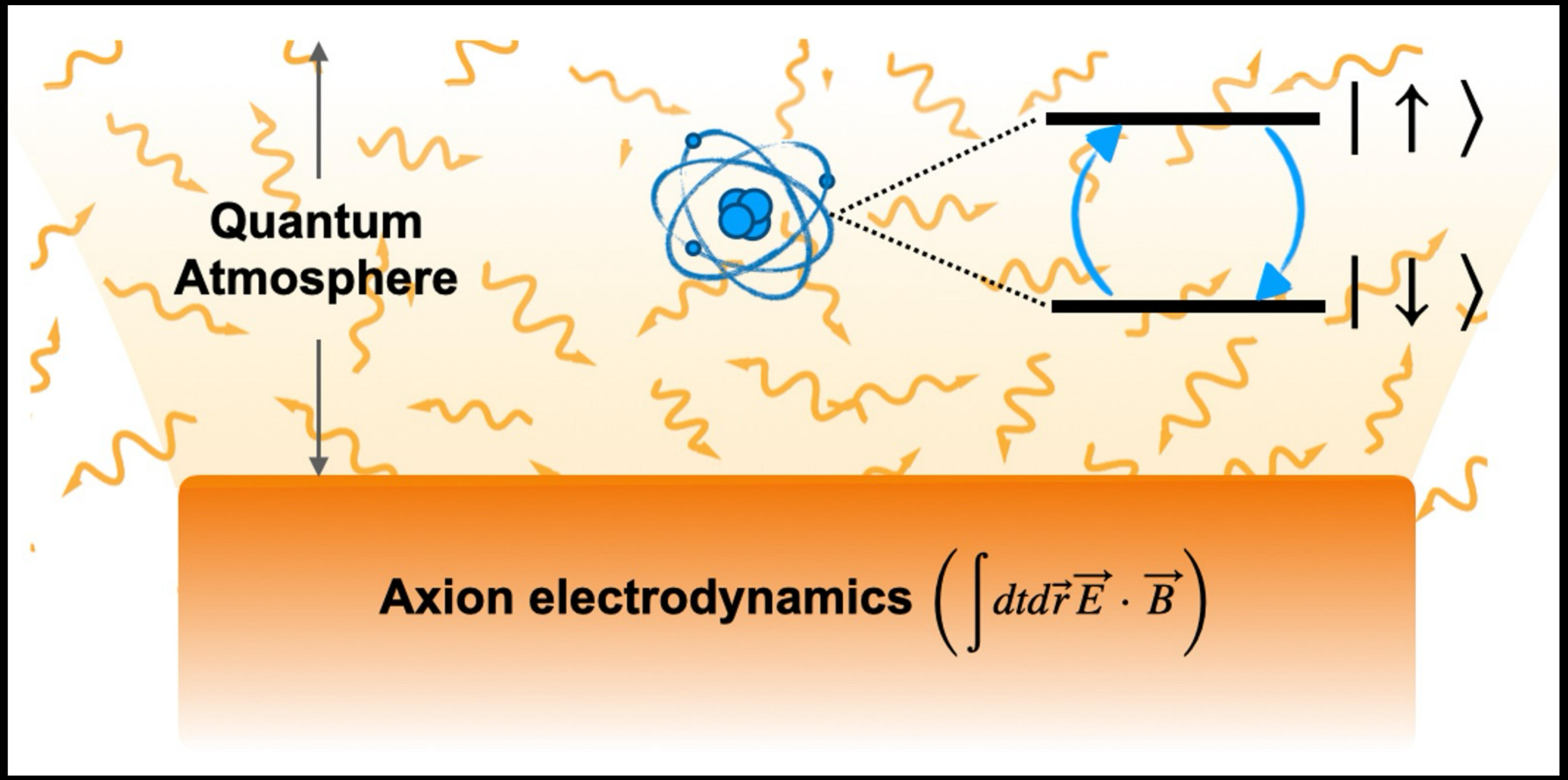
P breaking and T breaking

$$L_{int.} = \left(\frac{10\mathrm{nm}}{r}\right)^2 \frac{e\hat{n}\cdot s}{m}\, 10\ \mathrm{gauss}$$

How to measure?
NV center or STM

Classical Stray Field Vs Quantum Atmosphere?

T-broken quantum atmosphere



No real magnetic field

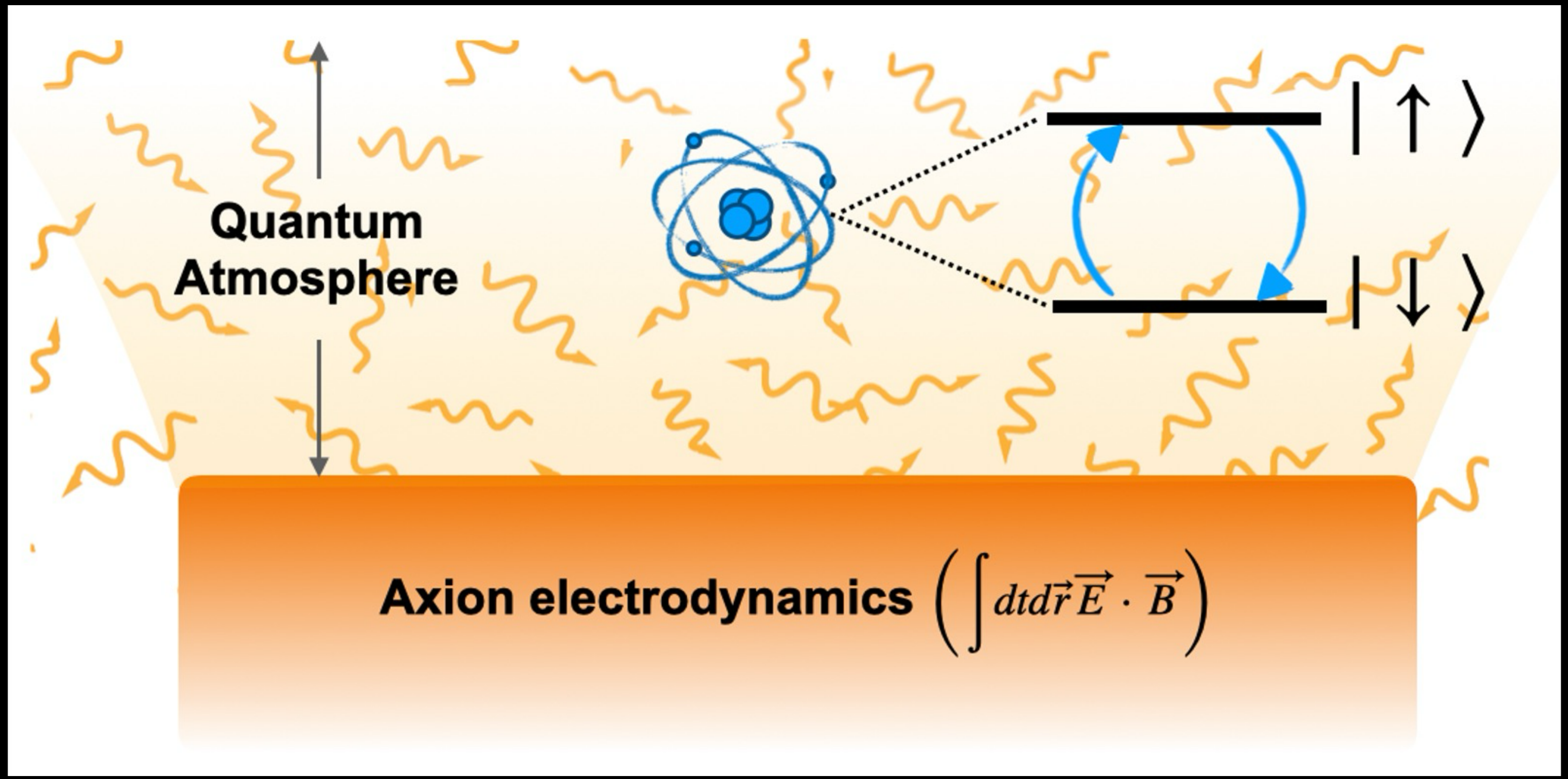
but Zeeman effect

QDJ and Frank Wilczek, PRB 99, 201104(R) (2019)

Where is QUANTUM?

Classical Stray Field Vs Quantum Atmosphere?

T-broken quantum atmosphere



No real magnetic field
but Zeeman effect

QDJ and Frank Wilczek, PRB 99, 201104(R) (2019)

Where is QUANTUM?

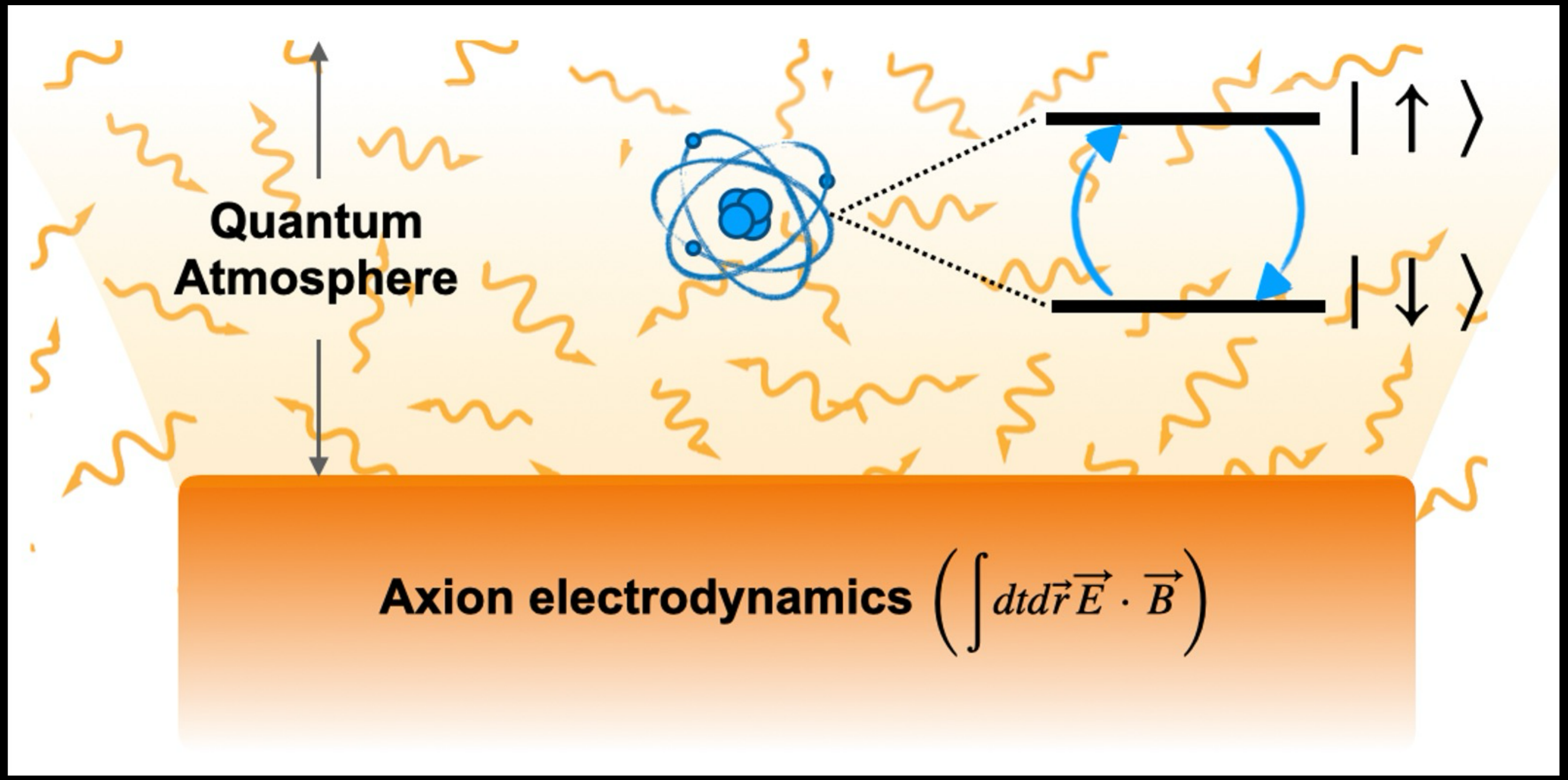
① Asymptotic behavior

$$B_{\{cl\}} \rightarrow a - b\,\mathrm{r}^2$$

$$B_{\{qa\}} \rightarrow c/r^2$$

Classical Stray Field Vs Quantum Atmosphere?

T-broken quantum atmosphere



No real magnetic field
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QDJ and Frank Wilczek, PRB 99, 201104(R) (2019)

Where is QUANTUM?

① Asymptotic behavior

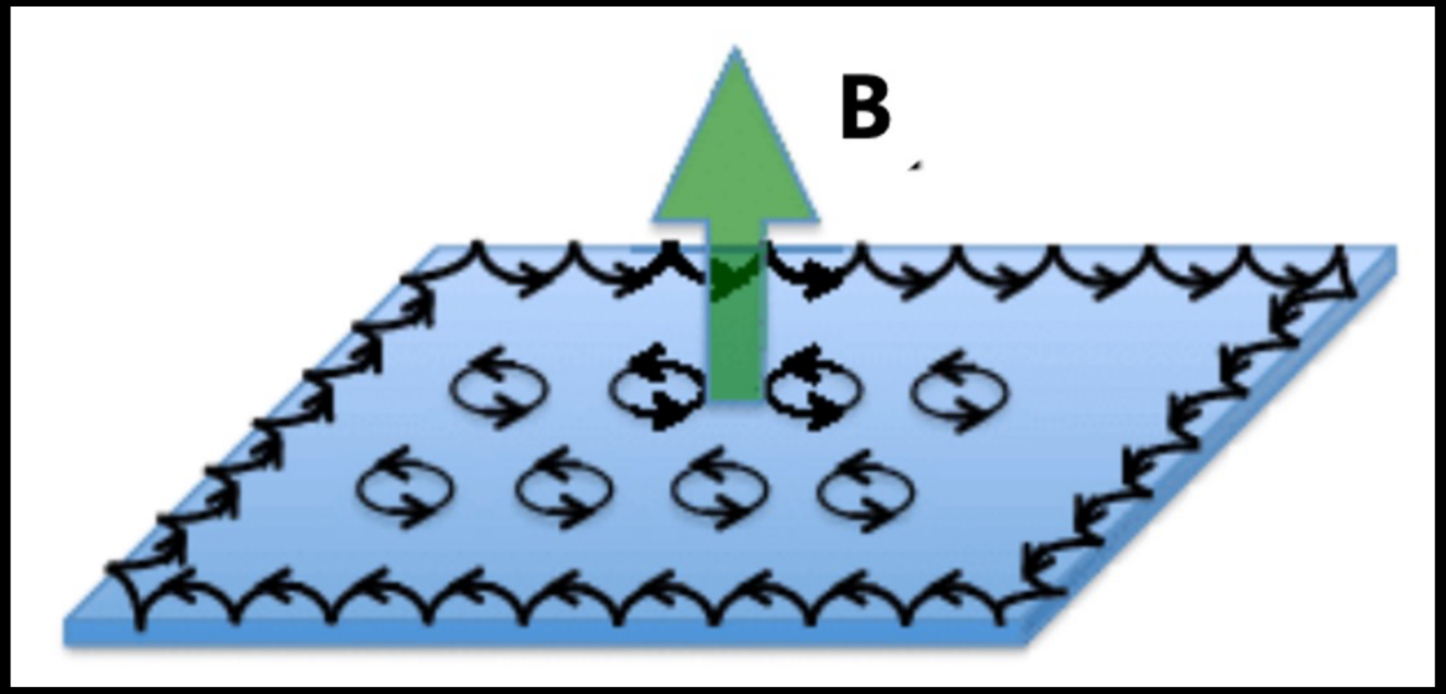
$$B_{\{cl\}} \rightarrow a - b\, \mathrm{r}^2$$

$$B_{\{qa\}} \rightarrow c/r^2$$

② c is quantized

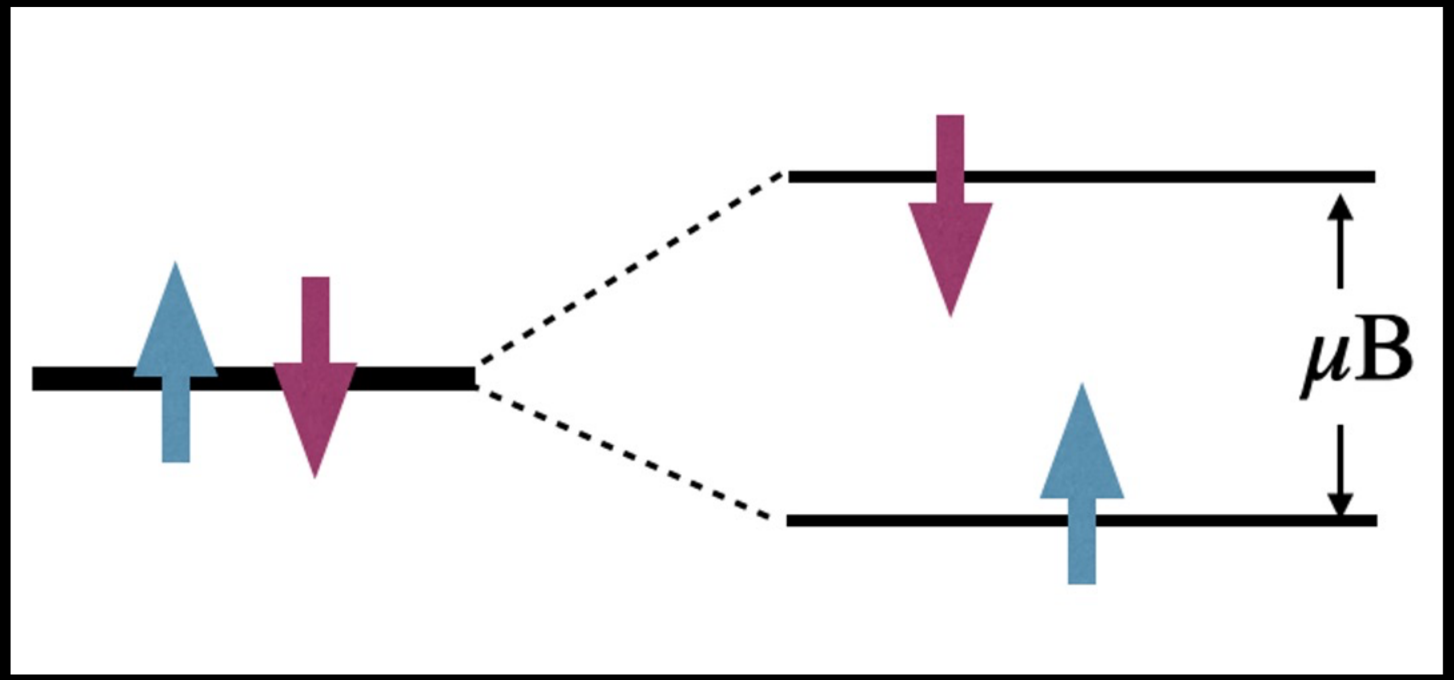
Quantum fluctuations can transmit symmetry breaking

With Magnetic Field



Orbital

Quantum Hall effect

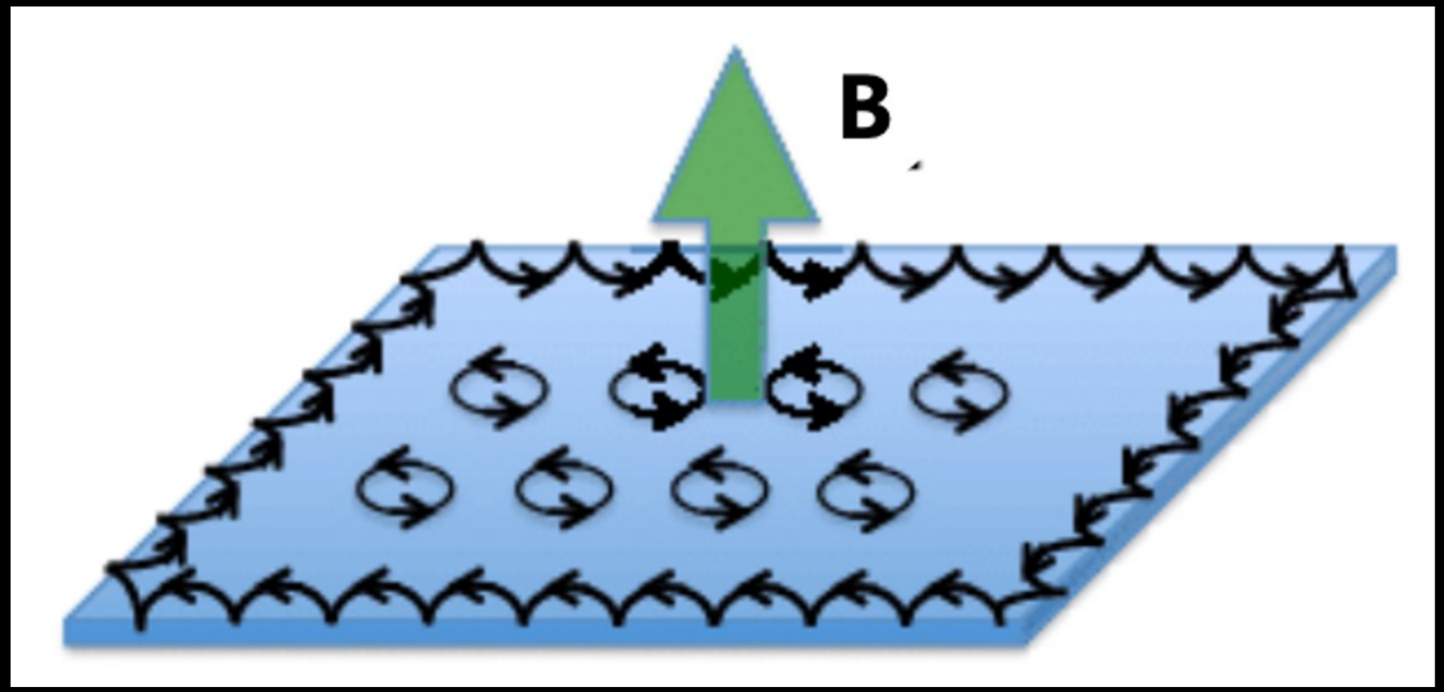


Spin

Zeeman effect

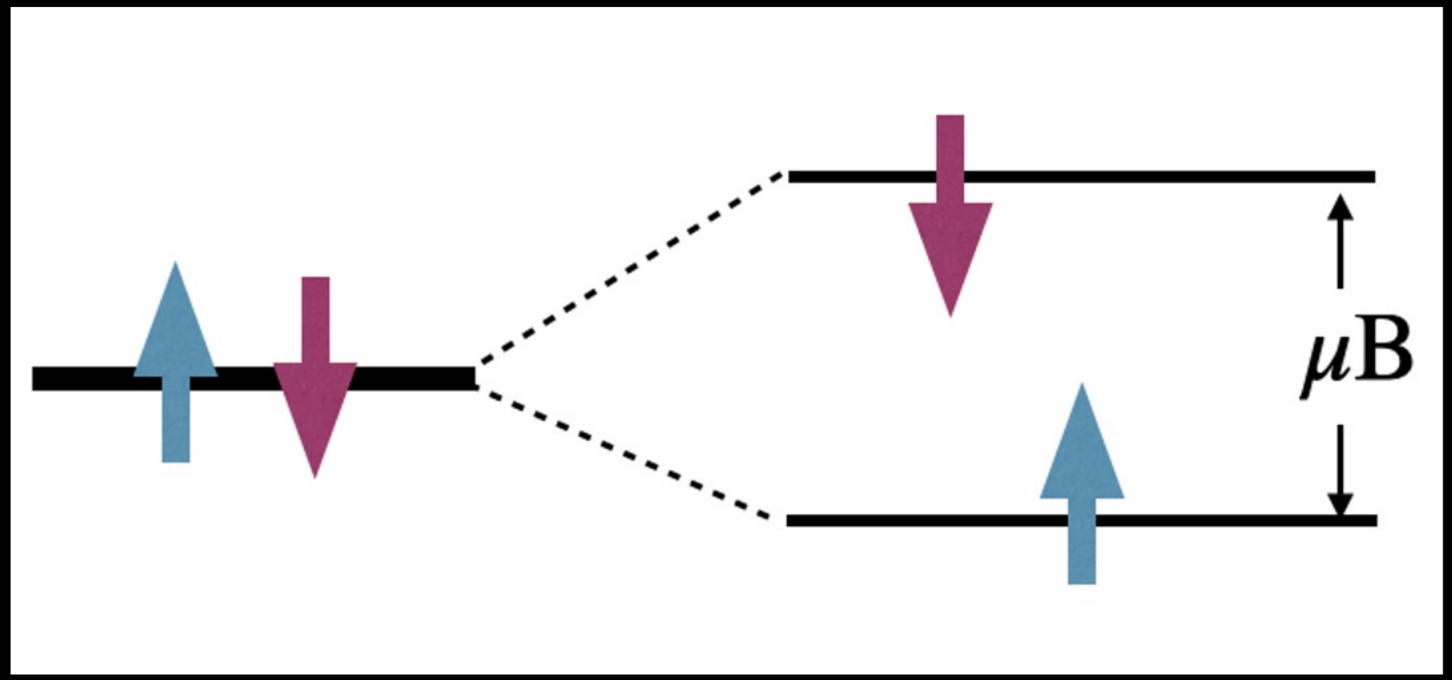
Time-reversal symmetry breaking & quantum effects

With Magnetic Field



Quantum Hall effect

Orbital



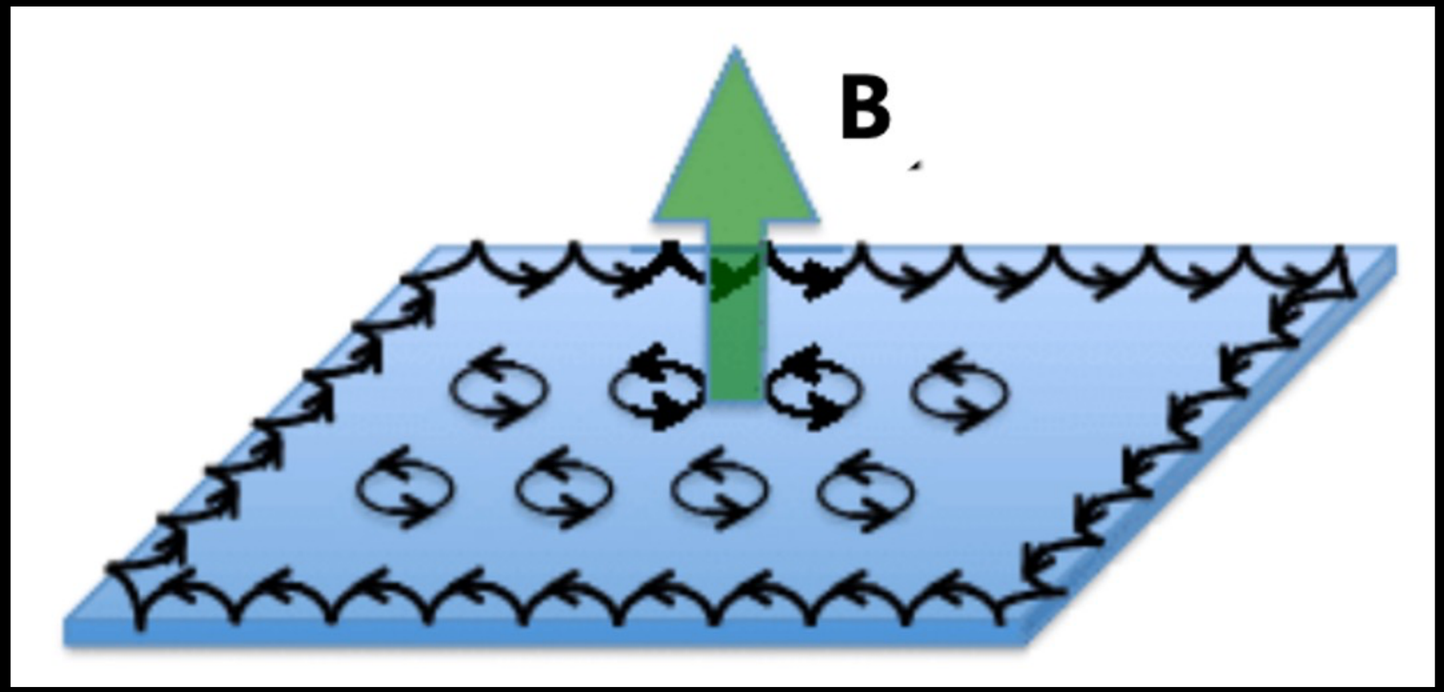
Zeeman effect

Spin

Do we need Magnetic Field?

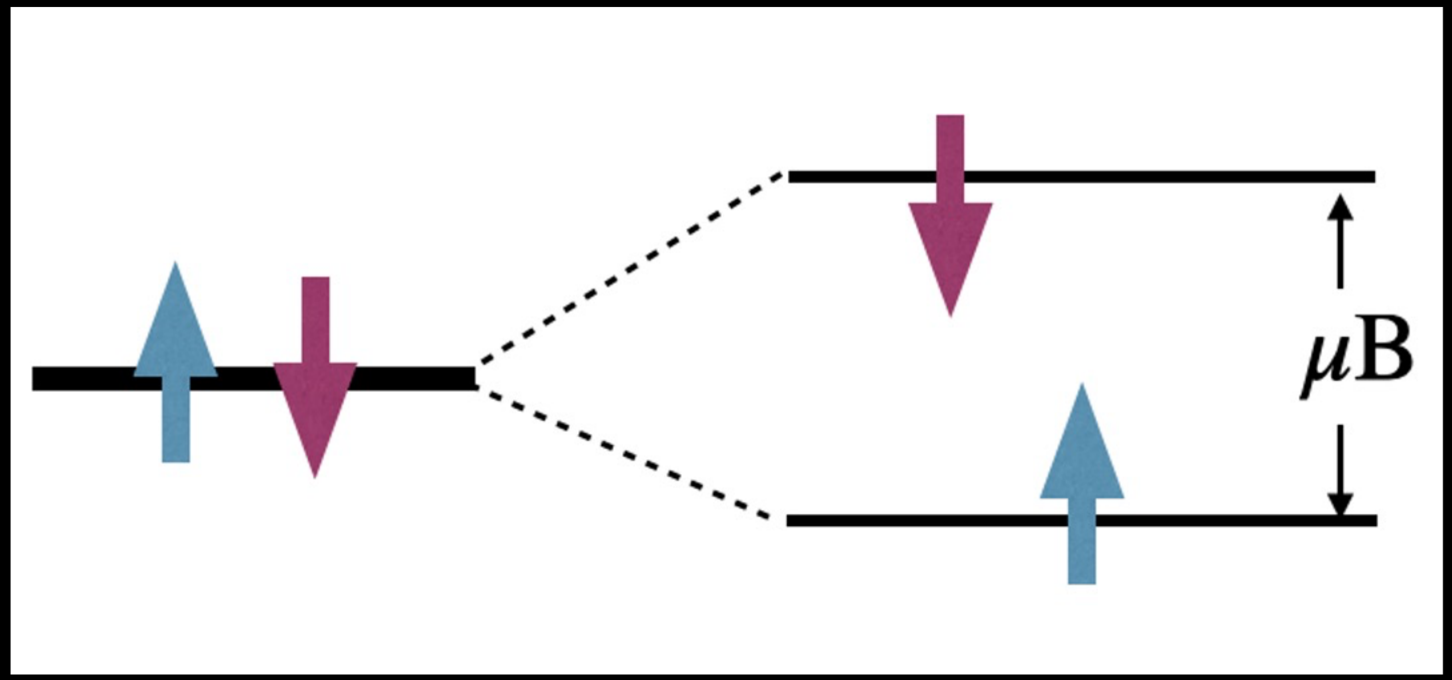
Time-reversal symmetry breaking & quantum effects

With Magnetic Field



Quantum Hall effect

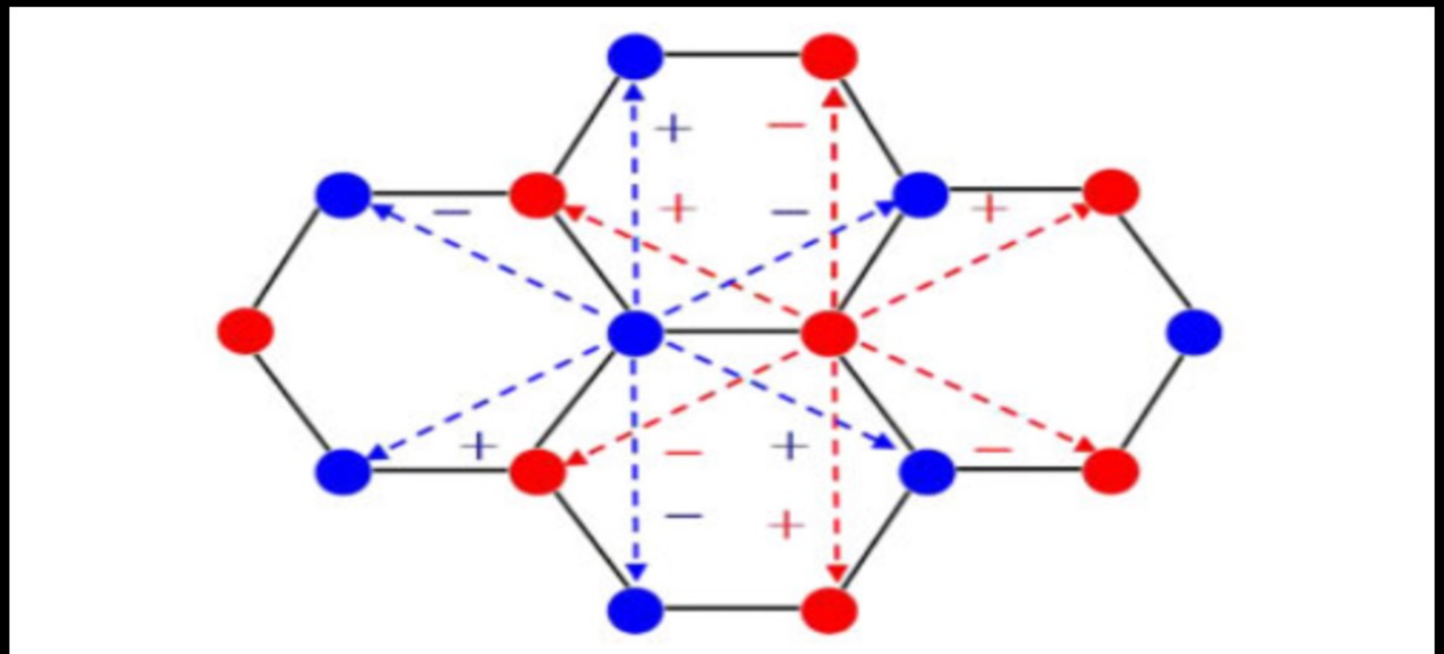
Orbital



Zeeman effect

Spin

Without Magnetic Field



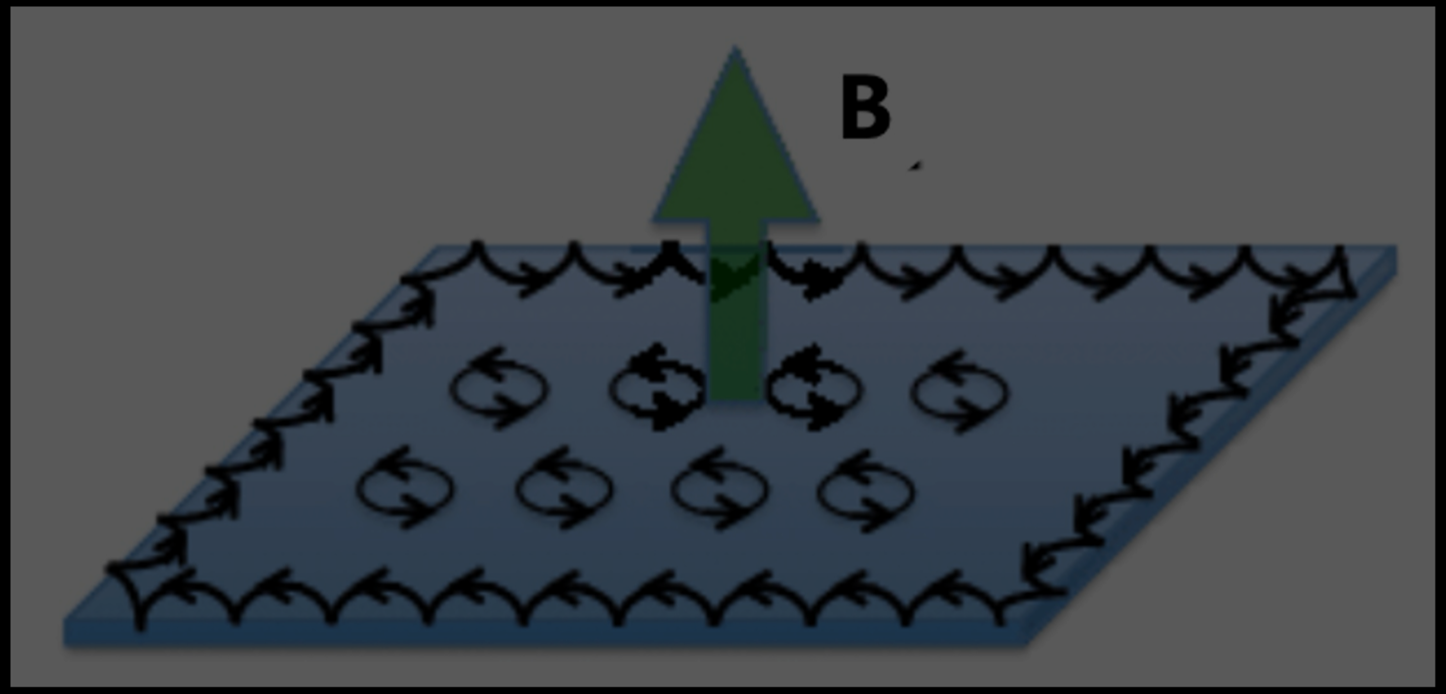
Quantum anomalous Hall

D. Haldane
1988

Time-reversal symmetry breaking & quantum effects

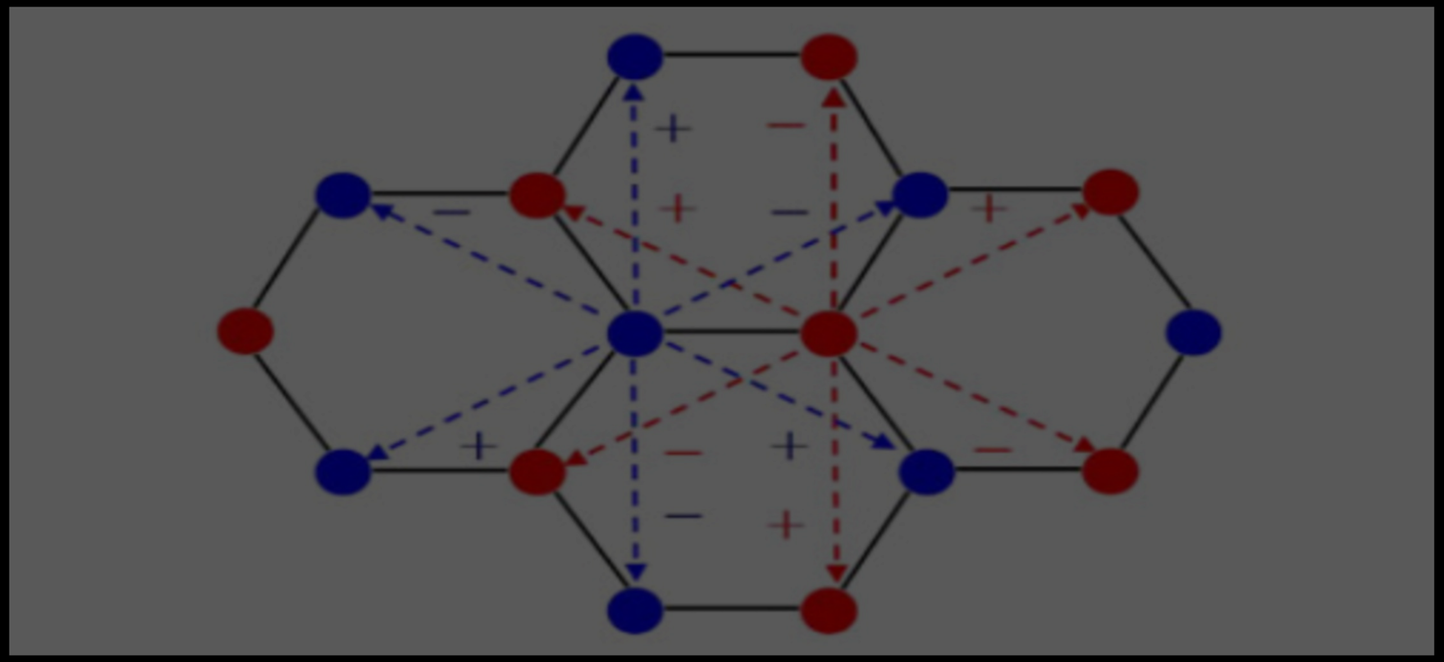
With Magnetic Field

Without Magnetic Field



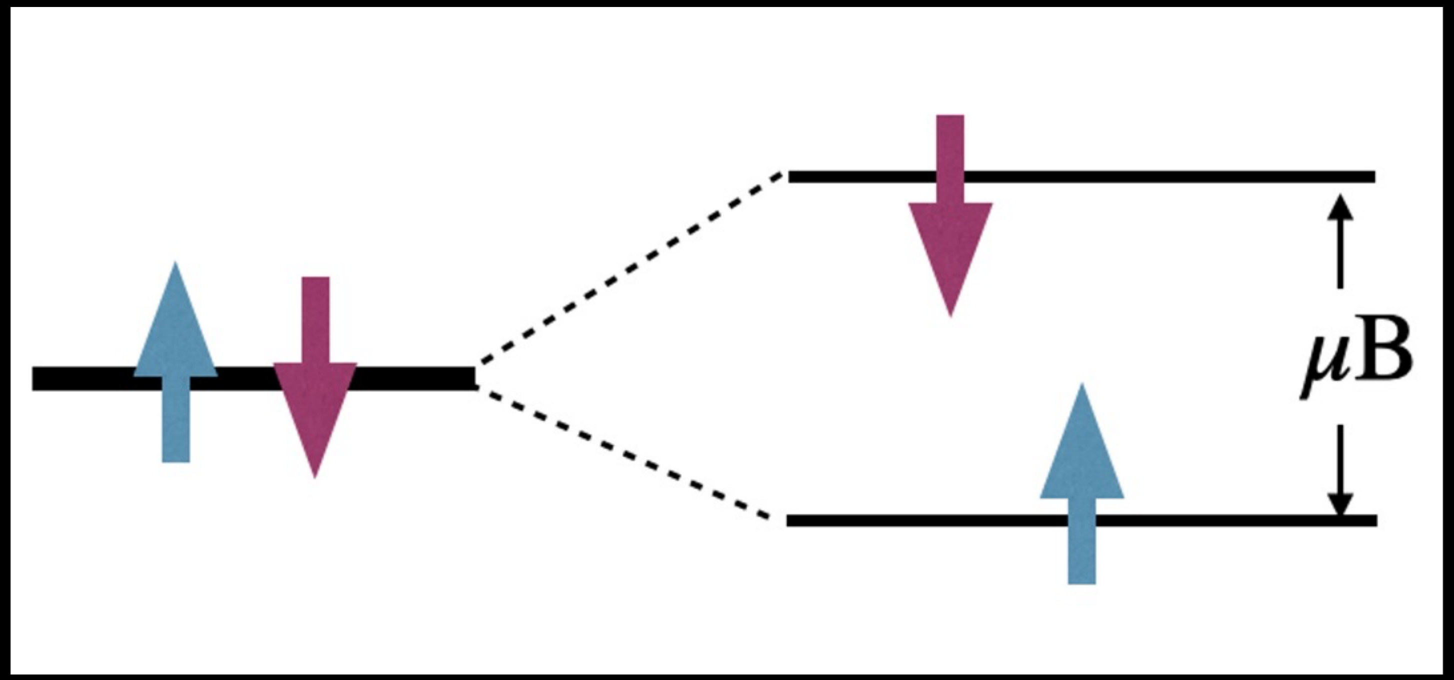
Quantum Hall effect

Orbital



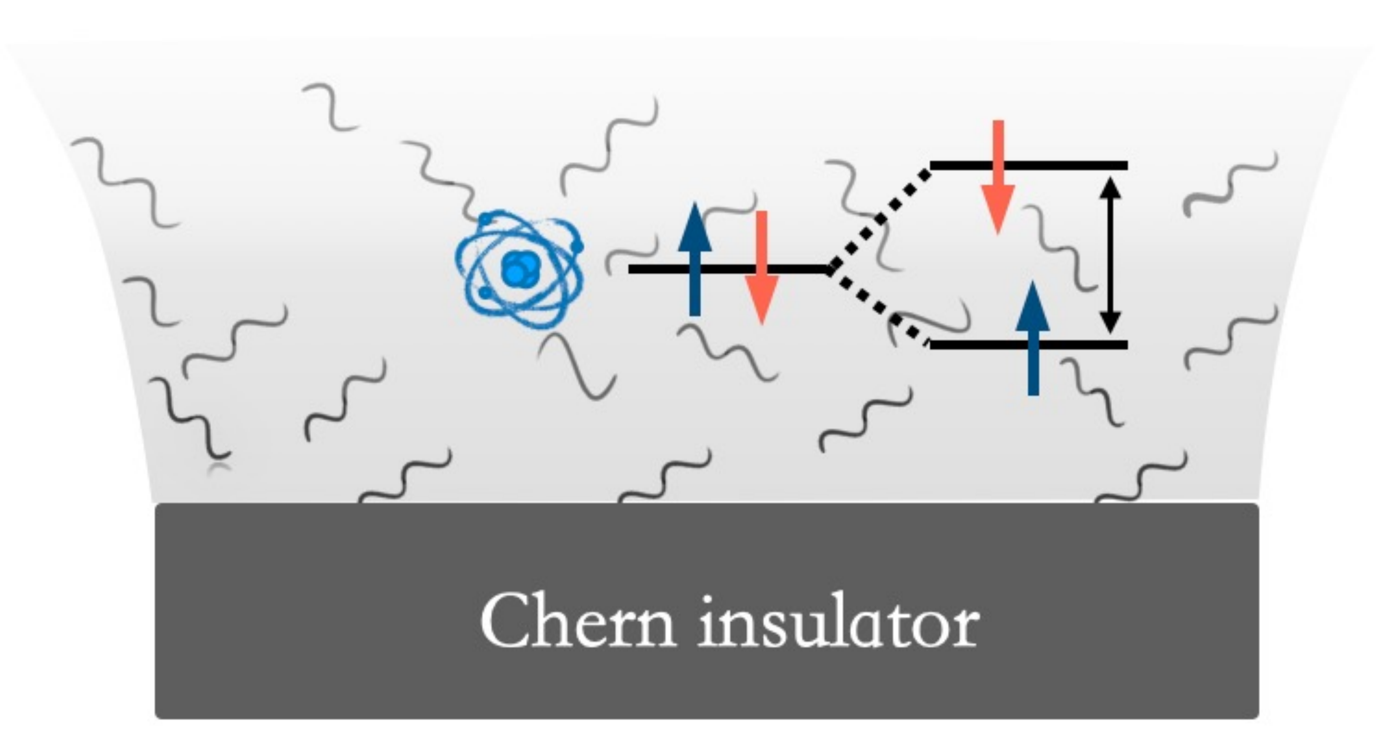
Quantum anomalous Hall

D. Haldane 1988



Zeeman effect

Spin



Quantum Atmosphere

PRB 99, 201104 (R) (2019);
PRL 131, 223601 (2023); PRL 132, 166901 (2024)

More types of Quantum Atmospheres

$$\vec{D} = c_e\vec{E} + c_{a1}\vec{B} + c_1\frac{\partial\vec{E}}{\partial t} + c_4\nabla\times\vec{B} + c_5\nabla\times\vec{E} + c_8\frac{\partial\vec{B}}{\partial t}$$

$$\vec{H} = c_b\vec{B} + c_{a2}\vec{E} + c_2\frac{\partial\vec{B}}{\partial t} + c_3\nabla\times\vec{E} + c_6\nabla\times\vec{B} + c_7\frac{\partial\vec{E}}{\partial t}$$

Preserve T & P

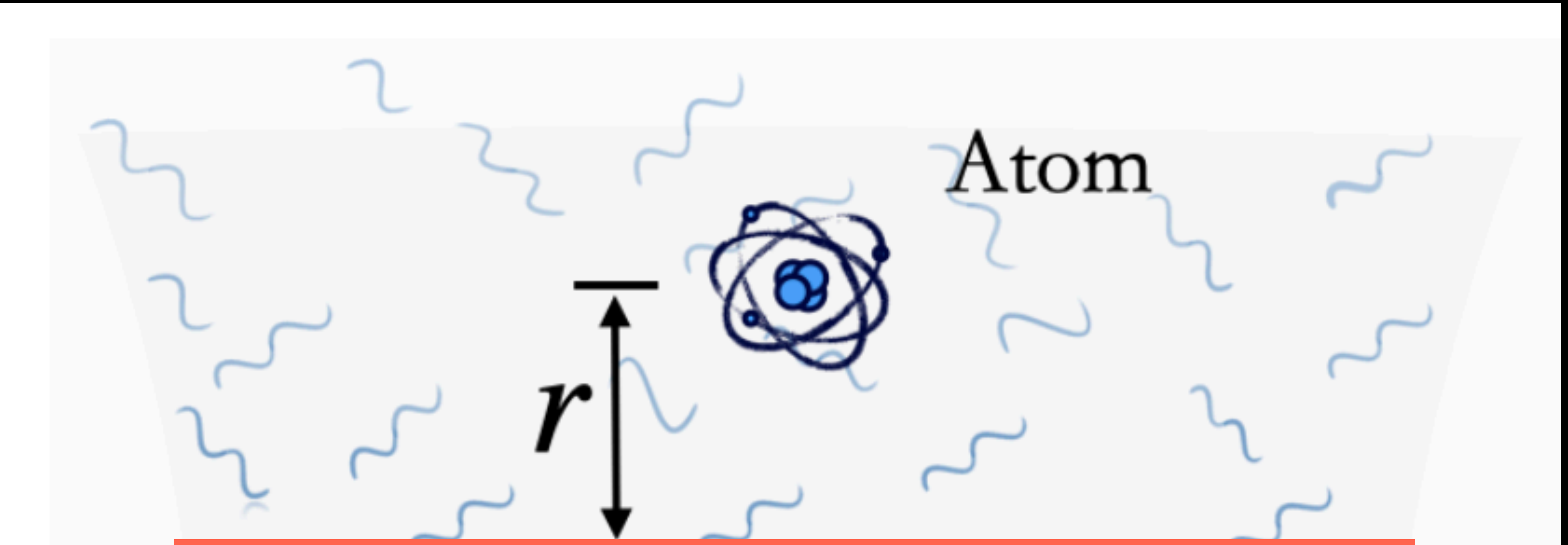
$\vec{E}\cdot\vec{B}$

Break T & P

$\vec{B}\cdot(\nabla\times\vec{B})$

Preserve T
But break P

Quantum atmosphere: more symmetries and symmetries breaking



$$\delta H = \overrightarrow{B} \cdot (\nabla \times \overrightarrow{B})$$

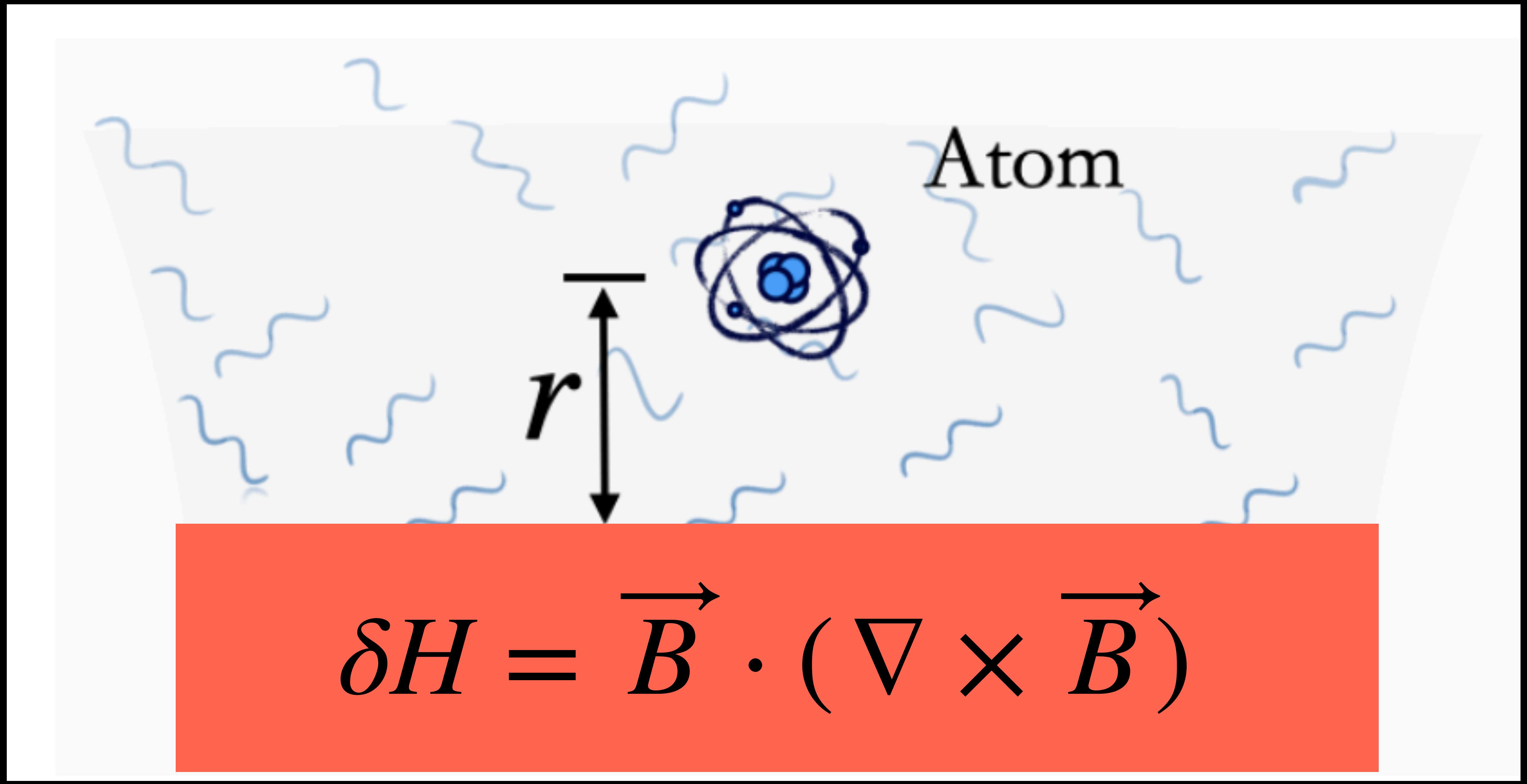
$\langle j \rangle = 0$

$\langle L \rangle = 0$

T preserved

$\langle L \cdot j \rangle \neq 0$ P broken

Quantum atmosphere: more symmetries and symmetries breaking



1: to see the symmetry

T preserved   P broken

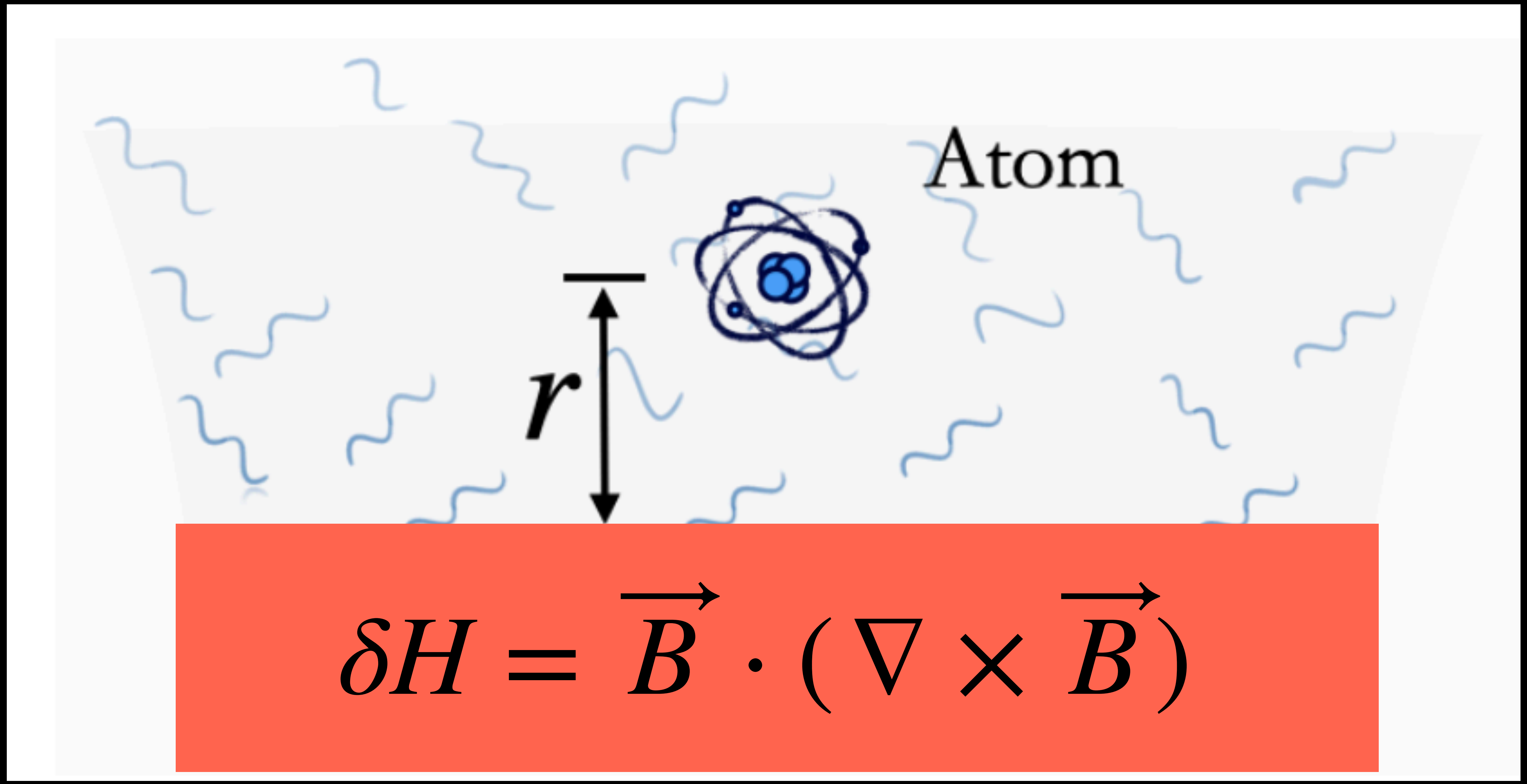
$\langle j \rangle = 0$

$\langle L \rangle = 0$

T preserved

$\langle L \cdot j \rangle \neq 0$ P broken

Quantum atmosphere: more symmetries and symmetries breaking



$\langle j \rangle = 0$

$\langle L \rangle = 0$

T preserved

$\langle L \cdot j \rangle \neq 0$ P broken

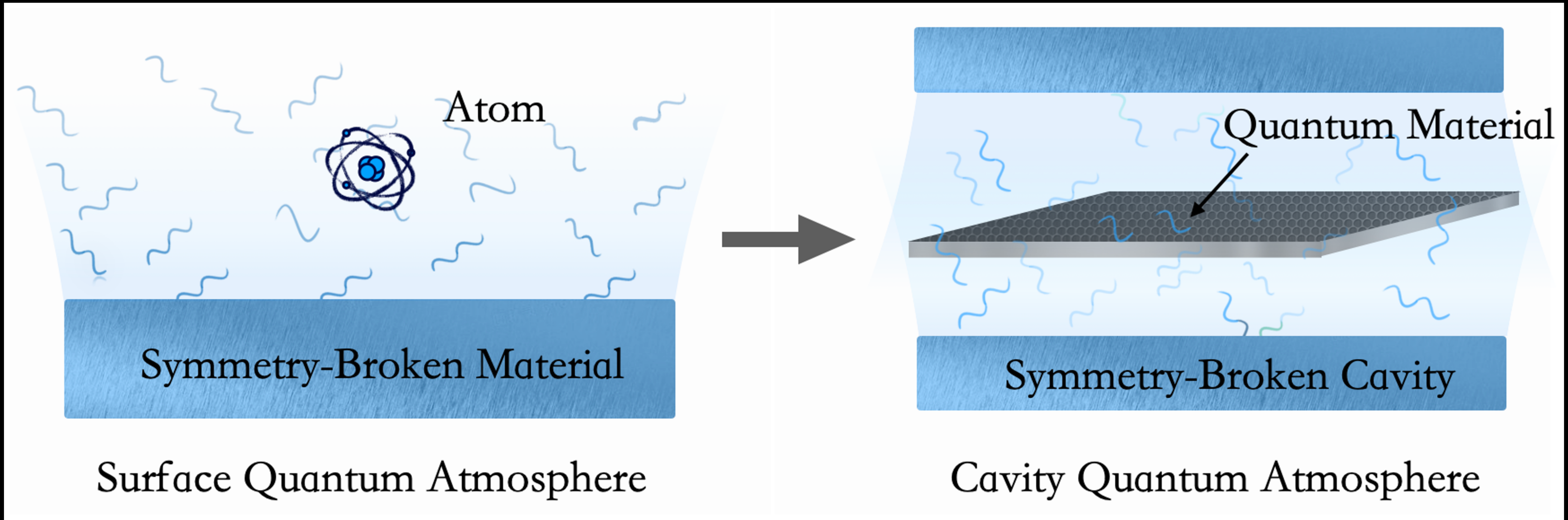
1: to see the symmetry

T preserved P broken

2: to use the atmosphere

Design symmetry-breaking required phenomena without illumination

How to enhance QA effect?



Enhanced Quantum Fluctuations in Cavity

$$H_{\mathrm{HO}} = \frac{1}{2}\left(\frac{p^2}{m} + m\Omega^2 x^2\right) = \frac{m}{2}\left(\dot{x}^2 + \Omega^2 x^2\right)$$

Ground state root-mean-square

$$\Delta x = (\hbar/2\Omega m)^{1/2}$$

$$H_{\mathrm{RF}} = \frac{\varepsilon_0 V}{2}\left(E^2 + c^2 B^2\right) = \frac{\varepsilon_0 V}{2}\left(\dot{A}^2 + c^2 |\mathbf{k} \times \mathbf{A}|^2\right)$$

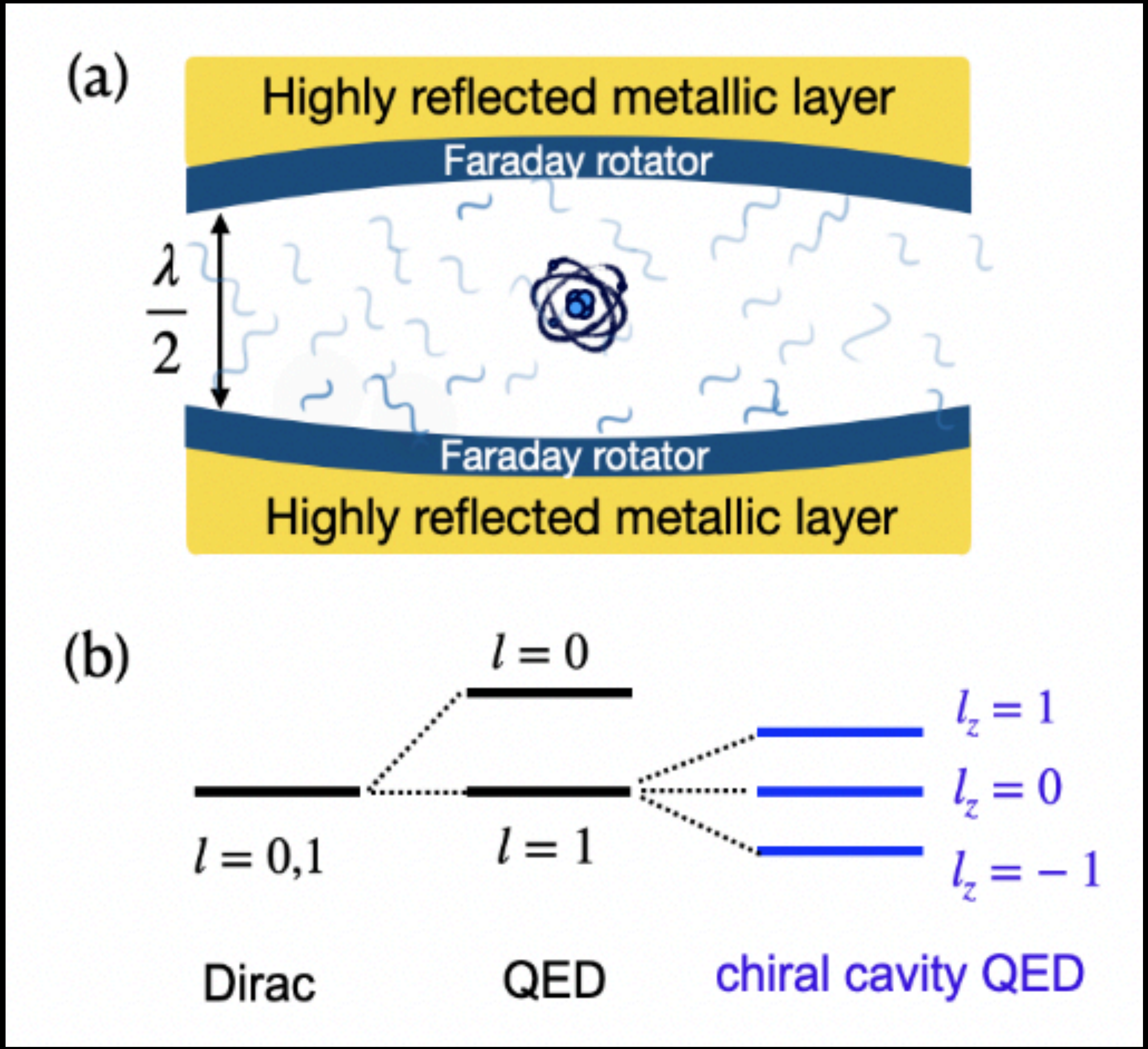
$$m \rightarrow \varepsilon_0 V \qquad \Omega \rightarrow c\mathrm{k}$$

$$\Delta A = (\hbar/2\Omega \varepsilon_0 V)^{1/2}$$

Benea-Chelmus et al., Nature, 568, 202 (2019)

Riek et al., Science, 350, 420 (2015)

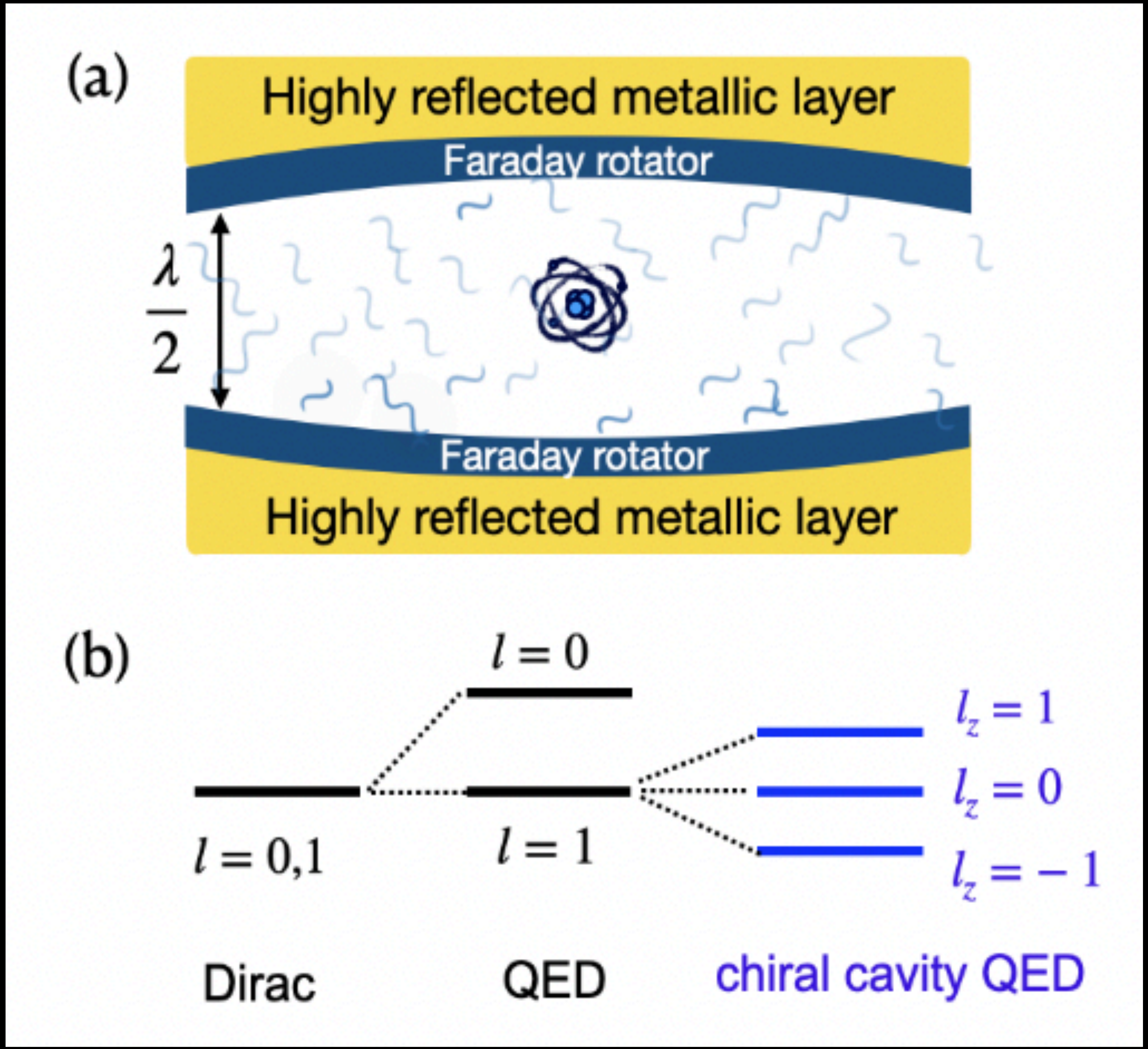
Angular-momentum dependent Lamb shift in a chiral vacuum cavity



$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - q\hat{\mathbf{A}}\right)^2 + V(\mathbf{r}) + \hbar\omega_c \hat{a}^\dagger \hat{a}$$

$$\hat{\mathbf{A}} = A_0\left(\boldsymbol{\varepsilon}^* \hat{a}^\dagger + \boldsymbol{\varepsilon}\hat{a}\right) \qquad A_0 = \sqrt{\frac{\hbar}{2\epsilon_0 V \omega_c}}$$

Angular-momentum dependent Lamb shift in a chiral vacuum cavity



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$$\hat{\mathbf{A}} = A_0\left(\boldsymbol{\varepsilon}^* \hat{a}^\dagger + \boldsymbol{\varepsilon}\hat{a}\right) \qquad A_0 = \sqrt{\frac{\hbar}{2\epsilon_0 V \omega_c}}$$

Dimensionless Parameter

$$g = \sqrt{\frac{(qA_0)^2}{m\hbar\omega_c}}$$

$$10^{-1} \leqslant g \leqslant 1$$

Strong coupling

$$1 \leqslant g$$

Deep strong coupling

Angular-momentum dependent Lamb shift in a chiral vacuum cavity

$$\hat{H} = \frac{1}{2m}\left(\hat{\mathbf{p}} - q\hat{\mathbf{A}}\right)^2 + V(\mathbf{r}) + \hbar\omega_c \hat{a}^\dagger \hat{a} \qquad \hat{\mathbf{A}} = A_0\left(\boldsymbol{\varepsilon}^* \hat{a}^\dagger + \boldsymbol{\varepsilon}\hat{a}\right)$$

$$\hat{U} = e^{-i\frac{\xi}{\hbar}\hat{p}\cdot\hat{\pi}} \qquad \xi = \frac{g}{1+g^2}\sqrt{\frac{\hbar}{m\omega_c}}$$

$$\hat{H}'(\xi) = \hat{U}^\dagger \hat{H} \hat{U} = \frac{\hat{\mathbf{p}}^2}{2m_{\text{eff}}} + V\left(\mathbf{r} + \xi\hat{\boldsymbol{\pi}} + \frac{\xi^2}{2\hbar}\hat{\mathbf{p}} \times \mathbf{e_z}\right) + \hbar\omega_{\text{eff}}\hat{a}^\dagger \hat{a}$$

Angular-momentum dependent Lamb shift in a chiral vacuum cavity

$$\hat{H}'(\xi) = \hat{U}^\dagger \hat{H} \hat{U} = \frac{\hat{\mathbf{p}}^2}{2m_{\mathrm{eff}}} + V\left(\mathbf{r} + \xi\hat{\boldsymbol{\pi}} + \frac{\xi^2}{2\hbar}\hat{\mathbf{p}} \times \mathbf{e_z}\right) + \hbar\omega_{\mathrm{eff}}\hat{a}^\dagger\hat{a} \qquad \xi = \frac{g}{1+g^2}\sqrt{\frac{\hbar}{m\omega_c}}$$

$$g \to 0 \qquad \xi \to 0$$

$$g \to \infty \qquad \xi \to 0$$

Expansion $$V\left(\mathbf{r} + \xi\hat{\boldsymbol{\pi}} + \frac{\xi^2}{2\hbar}\hat{\mathbf{p}} \times \mathbf{e_z}\right) \sim V(r) + \nabla V \cdot \left(\xi\hat{\boldsymbol{\pi}} + \frac{\xi^2}{2\hbar}\hat{\mathbf{p}} \times \mathbf{e_z}\right)$$

What do you do to this expansion?

AM-dependent spectral shift and cavity Lamb shift

$$\delta E_n = \langle \Psi_n | \delta \hat{H} | \Psi_n \rangle = \delta E_n^{\mathrm{AM}} + \delta E_n^{\mathrm{CL}}$$

$$\delta E_n^{\mathrm{AM}} = \frac{\xi^2}{2\hbar} \langle \psi_n | \frac{1}{r} \frac{dV(r)}{dr} \hat{L}_z | \psi_n \rangle$$

$$\delta E_n^{\mathrm{CL}} = \frac{\xi^2}{4} \langle \psi_n | \nabla^2 V(r) | \psi_n \rangle$$

- **2D Harmonic oscillator**

$$\Delta E_{n,l_z}^{\mathrm{CL}} = \frac{m\xi^2}{4} \omega^2$$

$$\Delta E^{\mathrm{AM}} = \frac{\xi^2}{2\hbar} m\omega^2 L_z$$

QDJ, Phys. Rev. B 111, 205405 (2025)

AM-dependent spectral shift and cavity Lamb shift

$$\delta E_n = \langle \Psi_n | \delta \hat{H} | \Psi_n \rangle = \delta E_n^{\mathrm{AM}} + \delta E_n^{\mathrm{CL}}$$

$$\delta E_n^{\mathrm{AM}} = \frac{\xi^2}{2\hbar} \langle \psi_n | \frac{1}{r} \frac{dV(r)}{dr} \hat{L}_z | \psi_n \rangle$$

$$\delta E_n^{\mathrm{CL}} = \frac{\xi^2}{4} \langle \psi_n | \nabla^2 V(r) | \psi_n \rangle$$

QDJ, Phys. Rev. B 111, 205405 (2025)

- **2D Harmonic oscillator**

$$\Delta E_{n,l_z}^{\mathrm{CL}} = \frac{m\xi^2}{4} \omega^2$$

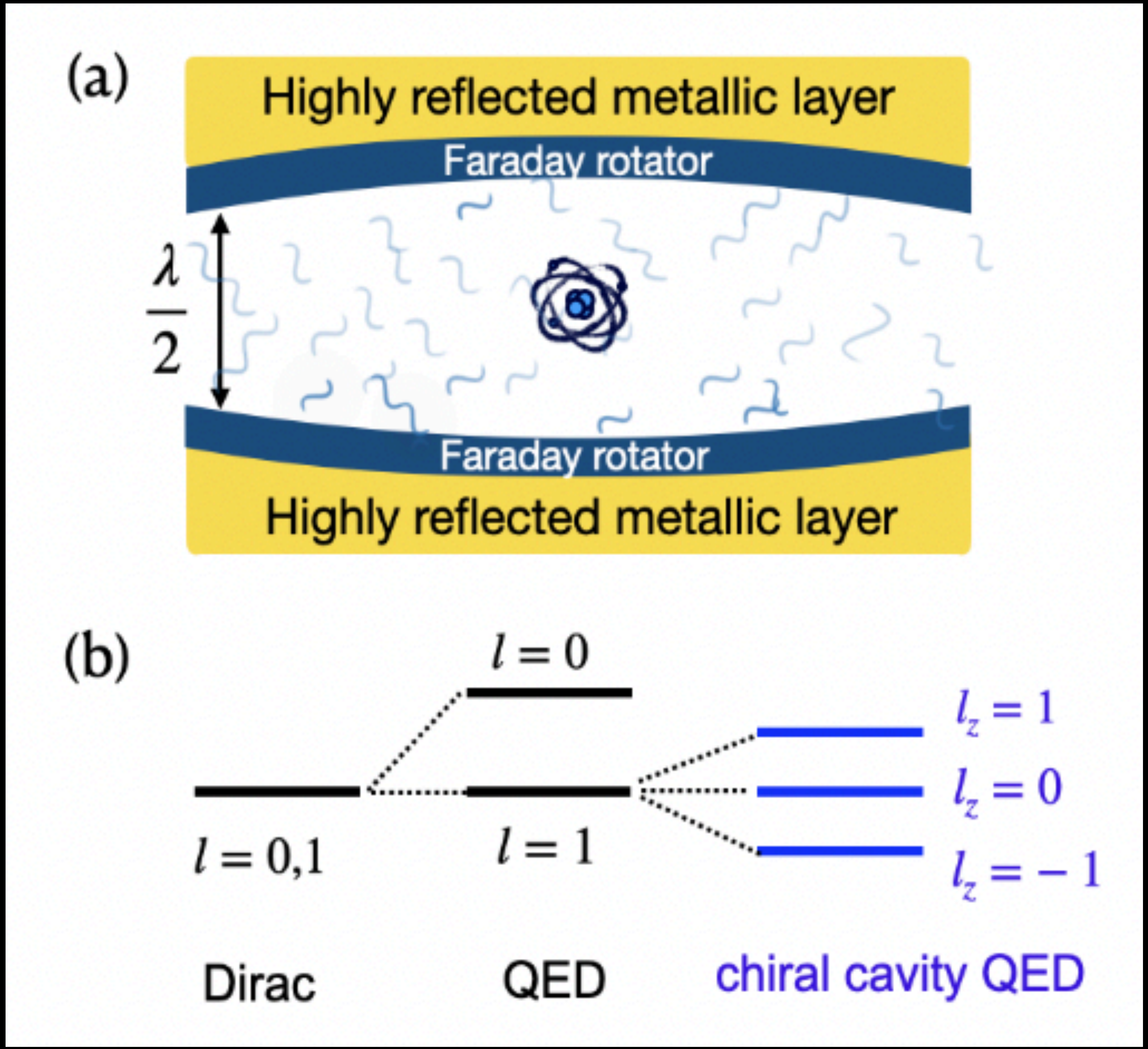
$$\Delta E^{\mathrm{AM}} = \frac{\xi^2}{2\hbar} m\omega^2 L_z$$

- **Hydrogen atom**

$$\Delta E_{n,l,l_z}^{\mathrm{CL}} = \frac{\pi \xi^2 k}{n^3 a_{\mathrm{eff}}^3} \; \delta_{l,0} \; \delta_{l_z,0}$$

$$\Delta E_{n,l,l_z}^{\mathrm{AM}} = \frac{l_z \xi^2 k}{2 a_{\mathrm{eff}}^3 n^3 l (l + \frac{1}{2})(l + 1)}$$

AM-dependent spectral shift and cavity Lamb shift



How Big?

$$g \sim 0.01 \quad \delta E \sim 0.3\ meV$$

Using vacuum to Selecting chirality

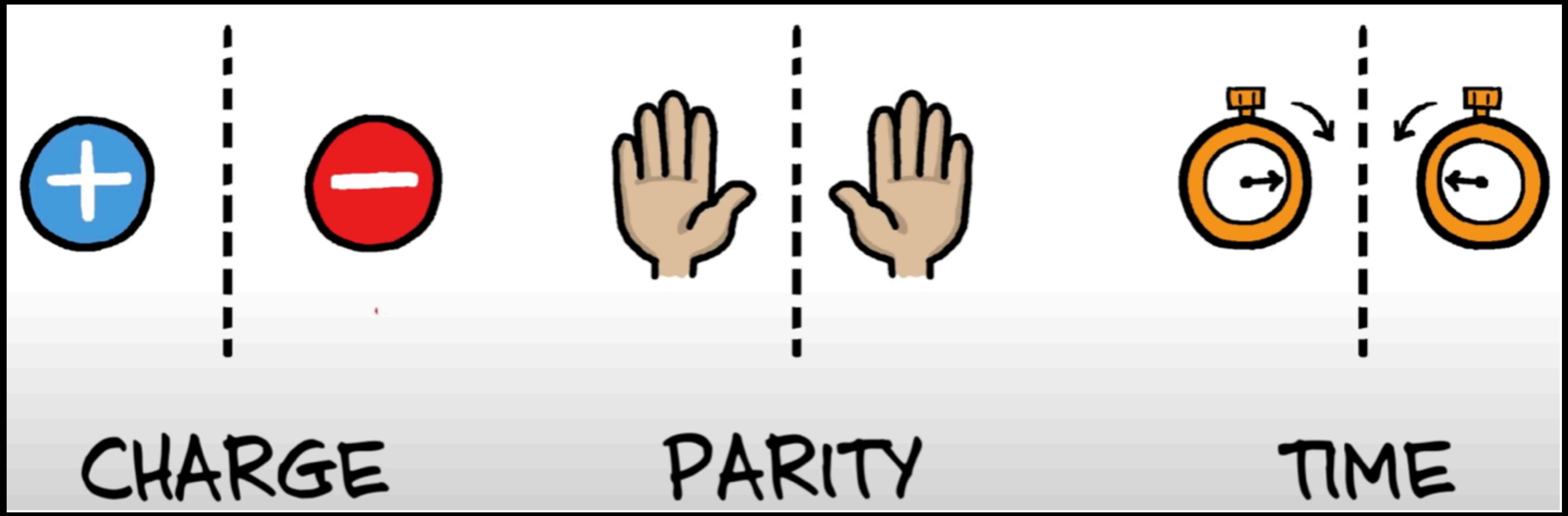
Question 1: What is Chirality?

2019.12.29

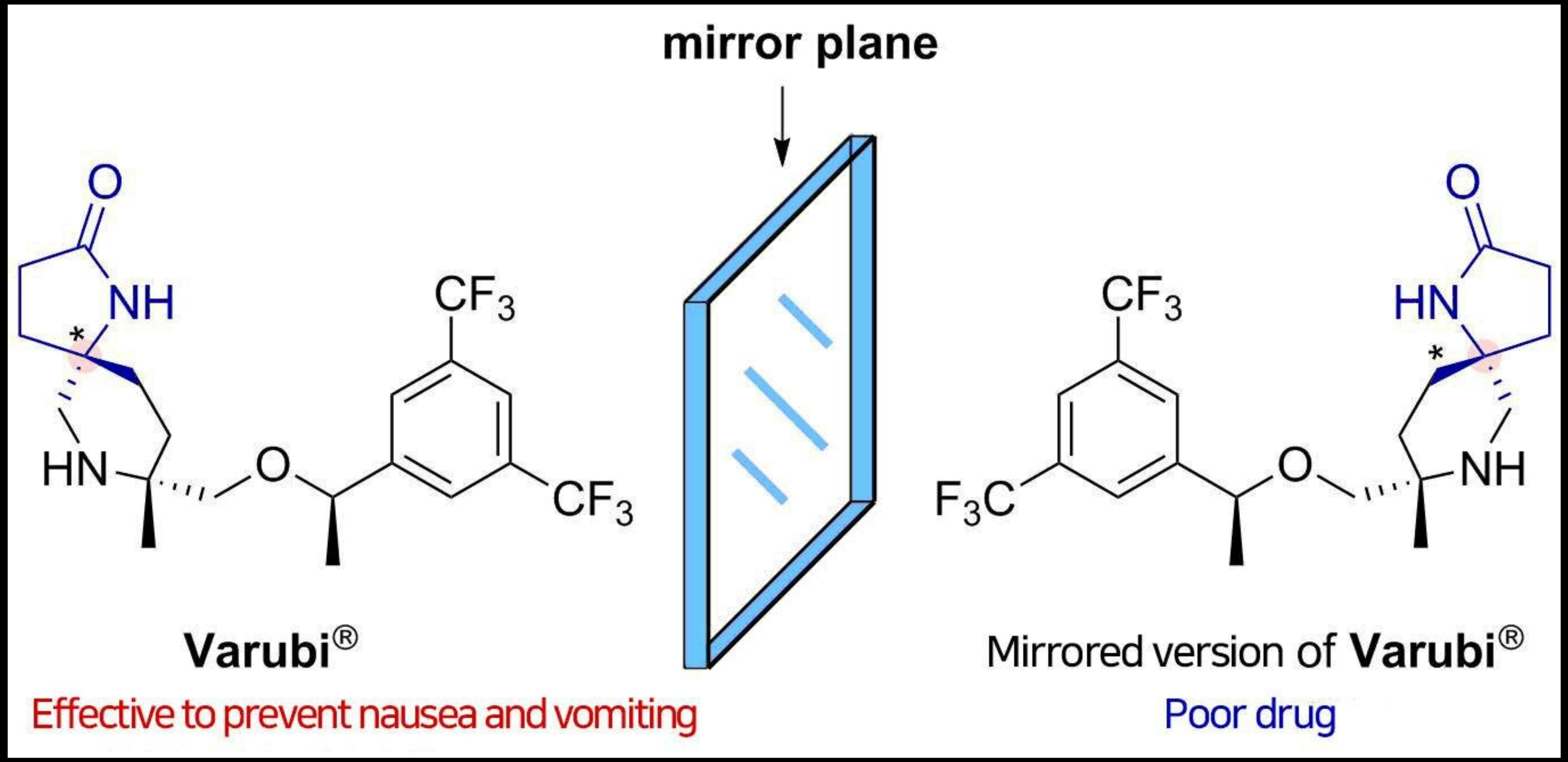
This playground is **handy!**

Linnégatan 81, Stockholm

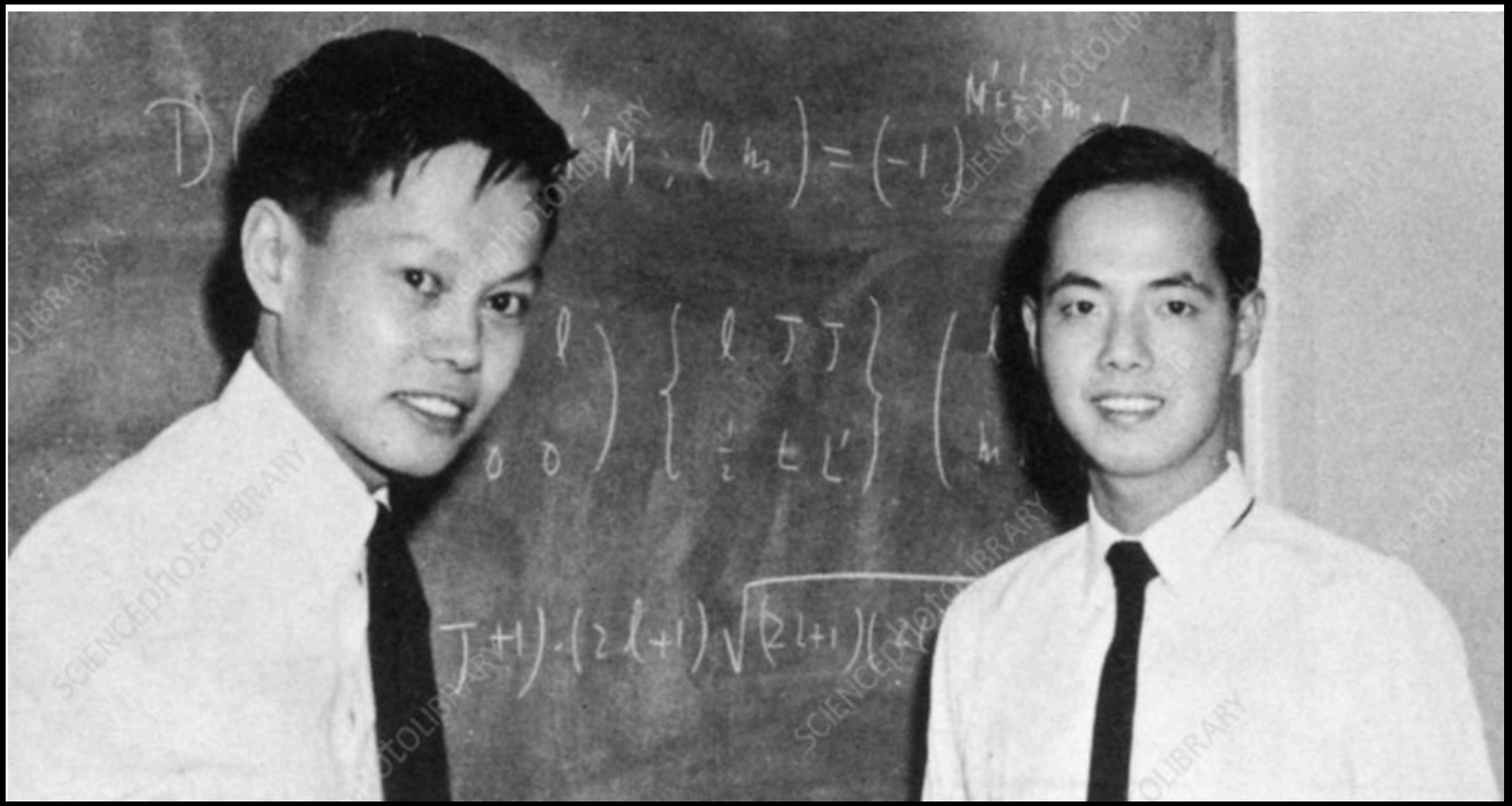
What is chirality?



Chirality: Lack of parity symmetry



Chirality of fundamental physics



PHYSICAL REVIEW VOLUME 104, NUMBER 1 OCTOBER 1, 1956

Question of Parity Conservation in Weak Interactions*

T. D. LEE, *Columbia University, New York, New York*

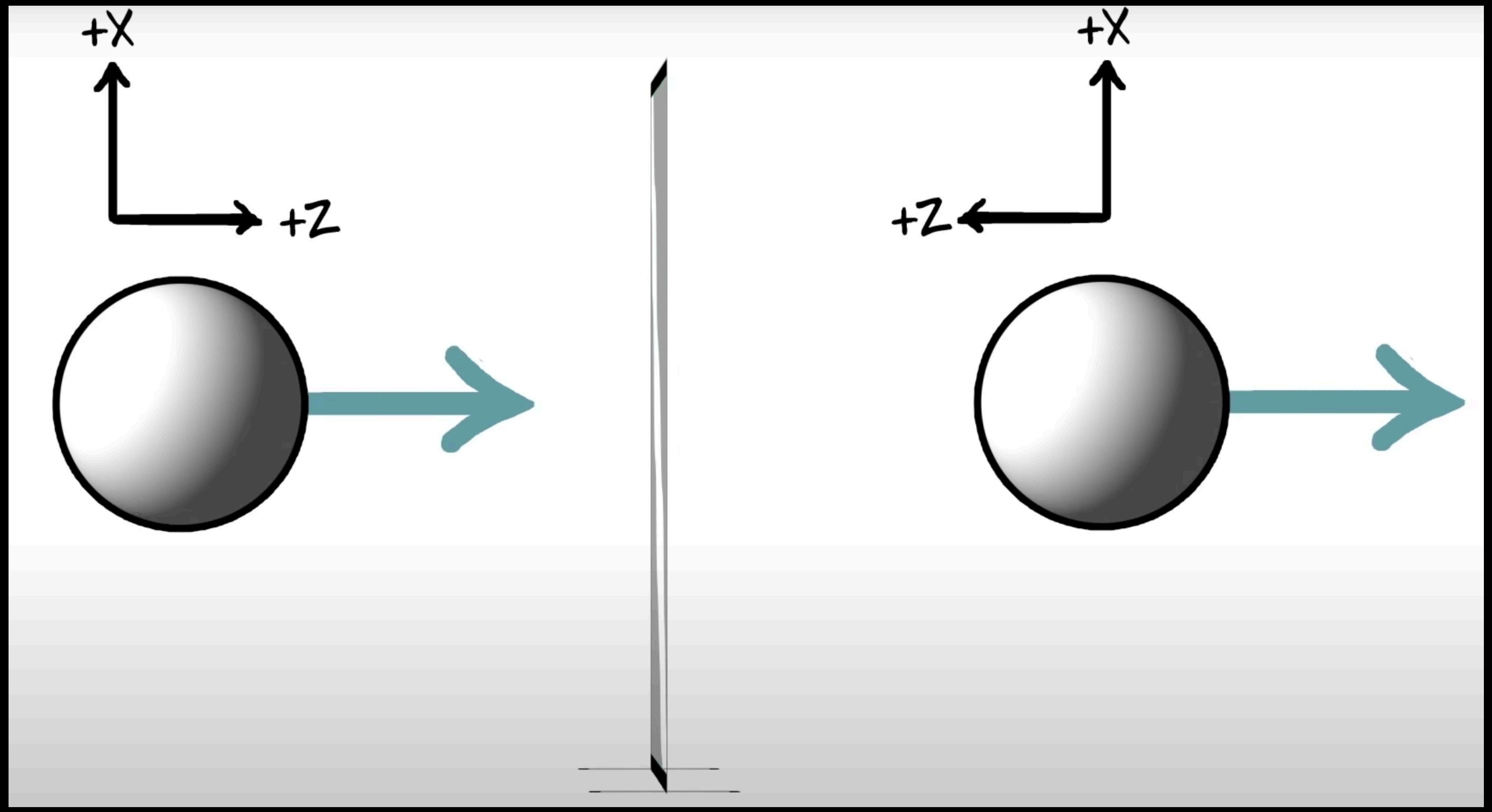
AND

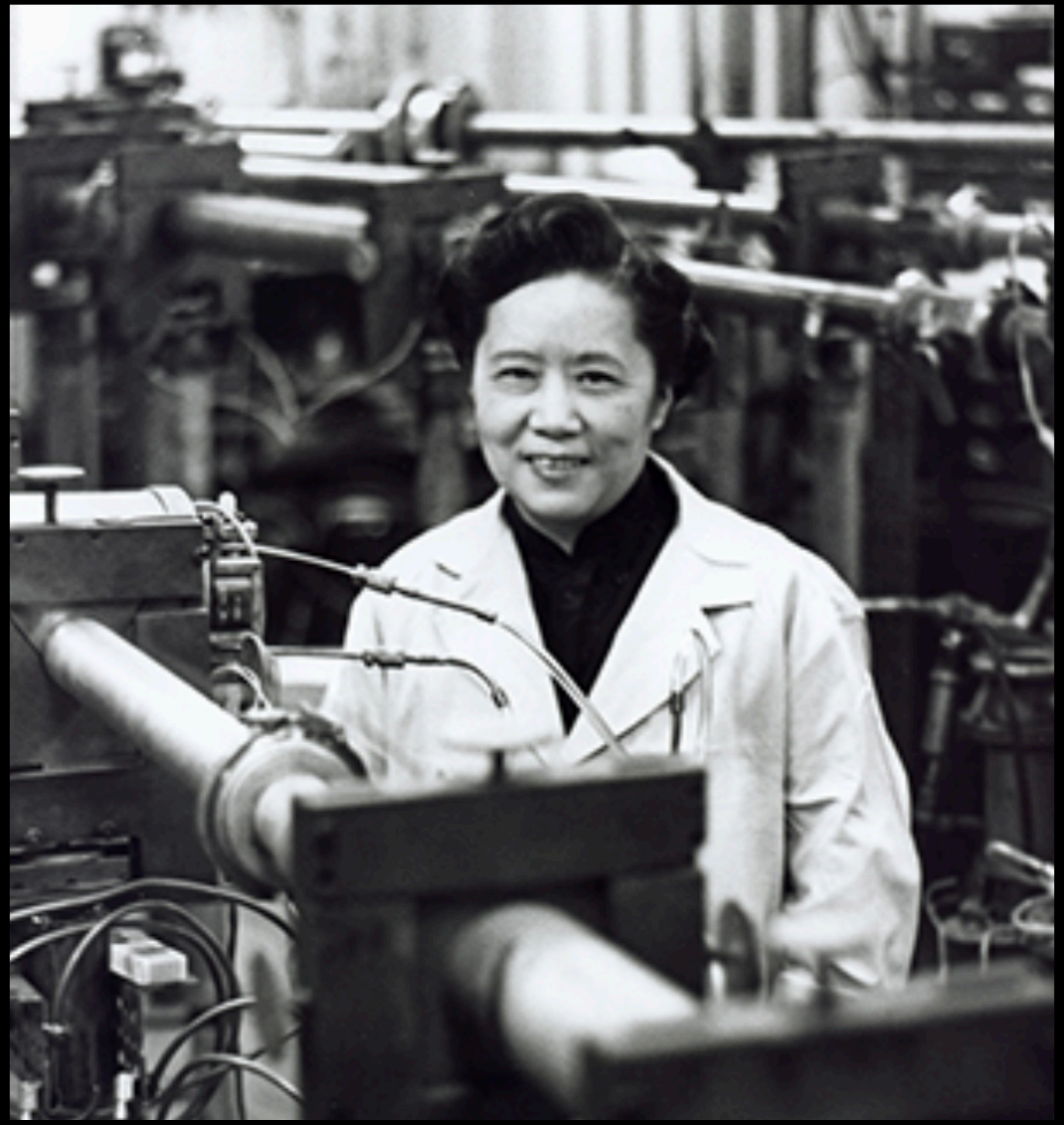
C. N. YANG,† *Brookhaven National Laboratory, Upton, New York*

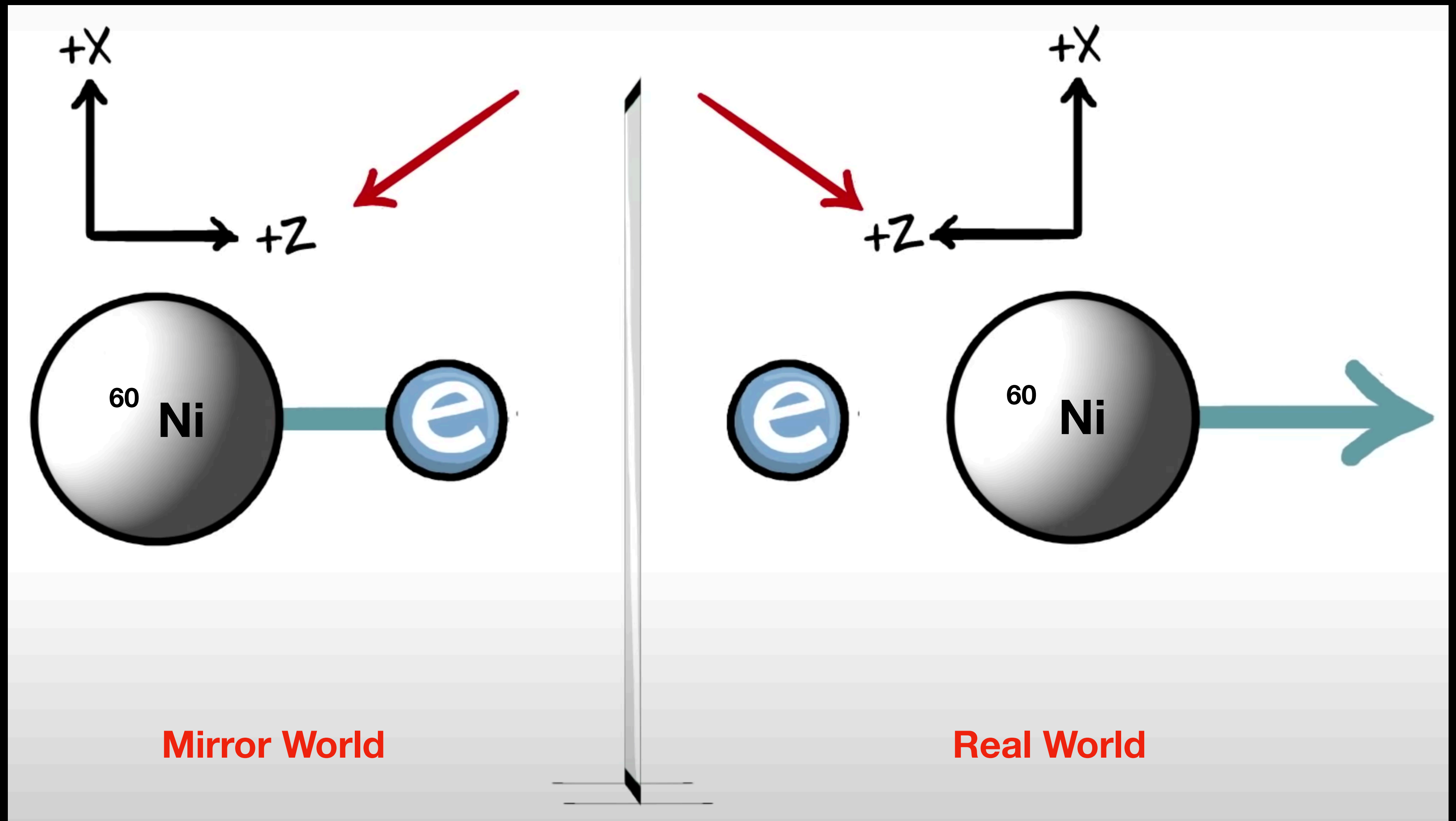
(Received June 22, 1956)

The question of parity conservation in β decays and in hyperon and meson decays is examined. Possible experiments are suggested which might test parity conservation in these interactions.

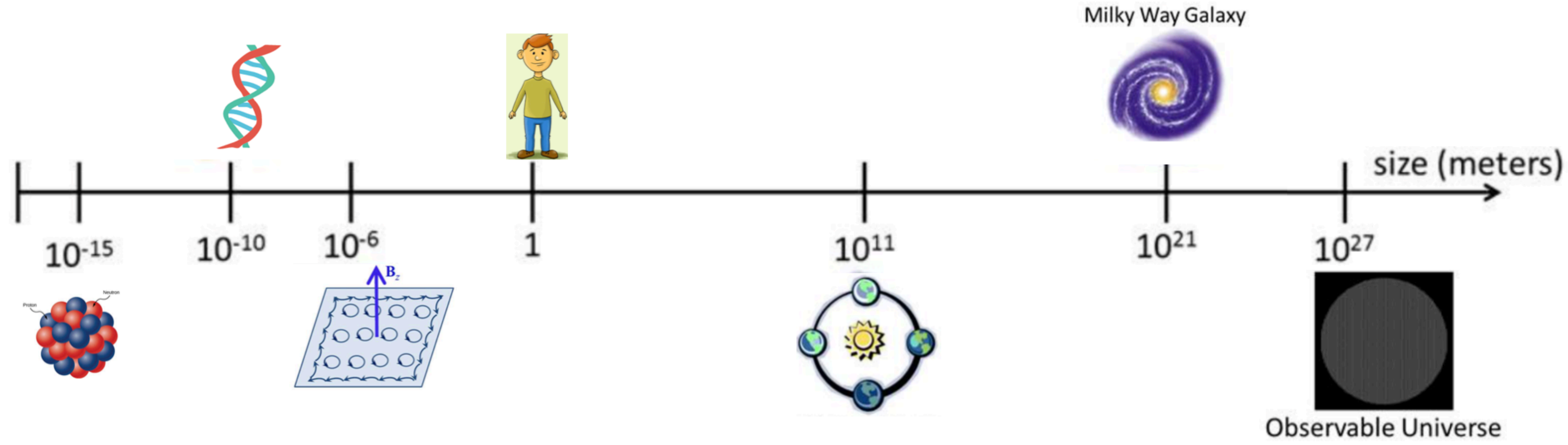
Chirality of fundamental physics





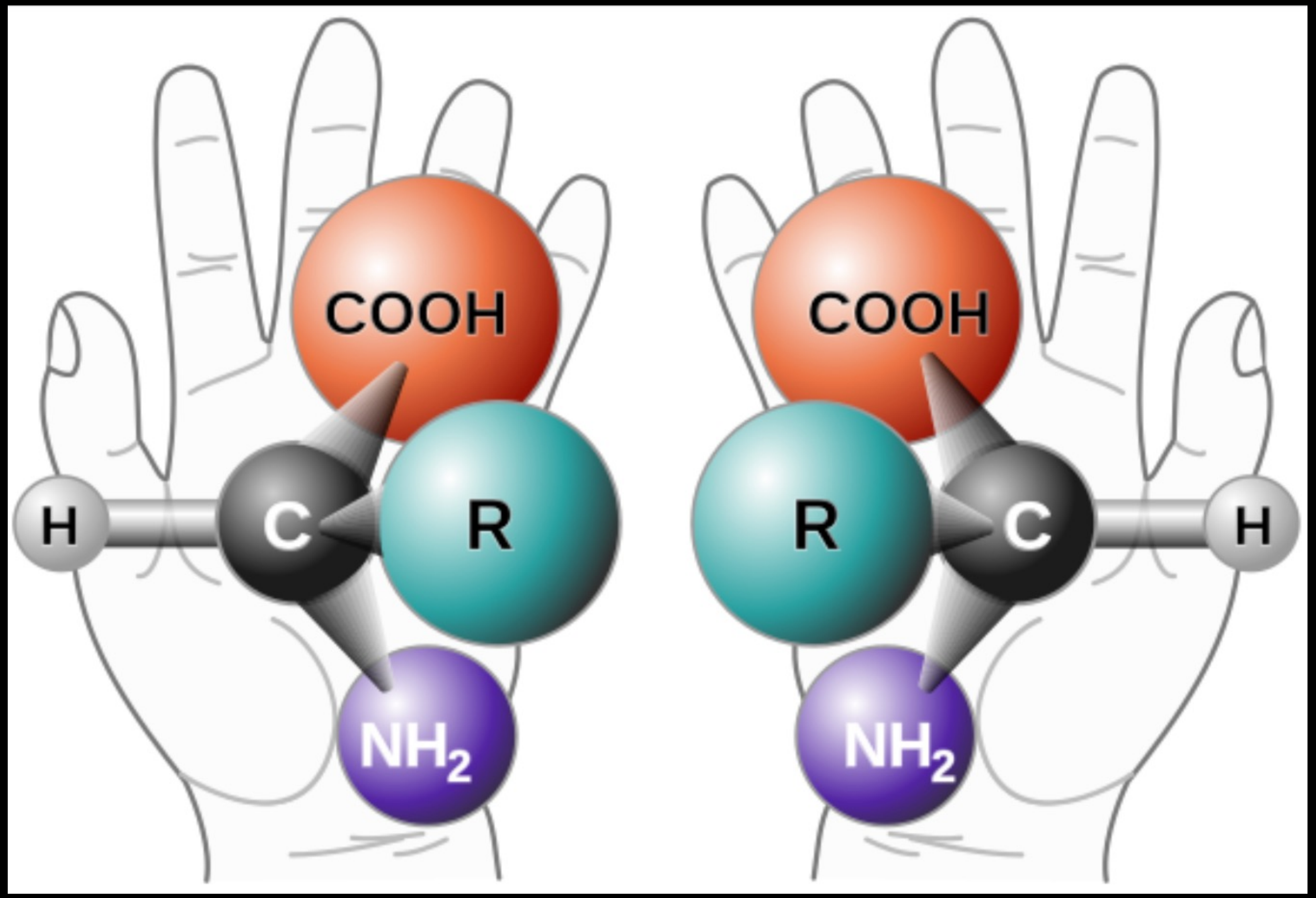


Chirality at different length scales



Chirality + Quantum fluctuations

Cavity vacuum select Chirality



- Enantiomer ~ pair of opposite chiralities
- When chiral molecules are synthesized in the laboratory from achiral building blocks, equal amounts of the **L- and D-enantiomers** are produced.

Key Question:

Can we find a better way to select chirality in chemical reactions?

Answer:

Vacuum!

Chiral molecules in a cavity

Model of Chiral Molecules

$$\widehat{H}_{mol} = \widehat{T}_n + \widehat{V}_n(\hat{\mathbf{R}}) + \widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}}).$$

Chiral molecules in a cavity

Model of Chiral Molecules

$$\widehat{H}_{mol} = \widehat{T}_n + \widehat{V}_n(\hat{\mathbf{R}}) + \widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}}).$$

Model of chiral molecules: BO approximation

$$\widehat{H}_{mol} = \widehat{T}_n + \widehat{V}_n(\hat{\mathbf{R}}) + \widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}}).$$

$$\underbrace{\widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}})}_{H_{el}(\hat{R})}$$

STEP 1: Solve electron part

$E_{el,i}(R)$ $\quad |\phi_i(R)\rangle$

Model of chiral molecules: BO approximation

$$\widehat{H}_{mol} = \widehat{T}_n + \widehat{V}_n(\hat{\mathbf{R}}) + \widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}}).$$

$$\underbrace{\widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}})}_{H_{el}(\hat{R})}$$

STEP 1: Solve electron part

$E_{el,i}(R) \qquad |\phi_i(R)\rangle$

Chiral molecule

$$\widehat{H}_{el}(\mathcal{J}_R\hat{\mathbf{R}}) \neq \widehat{H}_{el}(\mathcal{J}_P\hat{\mathbf{R}})$$

Model of chiral molecules: BO approximation

$$\widehat{H}_{mol} = \widehat{T}_n + \widehat{V}_n(\hat{\mathbf{R}}) + \underbrace{\widehat{T}_{el} + \widehat{V}_{el}(\hat{\mathbf{r}}) + \widehat{V}_{n-el}(\hat{\mathbf{R}}, \hat{\mathbf{r}})}_{H_{el}(\hat{R})}.$$

STEP 1: Solve electron part

$$E_{el,i}(R) \qquad |\phi_i(R)\rangle$$

Chiral molecule

$$\widehat{H}_{el}(\mathcal{J}_R\hat{\mathbf{R}}) \neq \widehat{H}_{el}(\mathcal{J}_P\hat{\mathbf{R}})$$

STEP 2: Promote R to be an operator

$$\widehat{V}_i(\hat{\mathbf{R}}) \equiv \widehat{V}_n(\hat{\mathbf{R}}) + E_{el,i}(\hat{\mathbf{R}})|\phi_i(\hat{\mathbf{R}})\rangle\langle\phi_i(\hat{\mathbf{R}})|.$$

$$\widehat{H}^{(i)}_{mol} = \widehat{T}_n(\hat{\mathbf{R}}) + \widehat{V}_i(\hat{\mathbf{R}}).$$

Energy shift of chiral molecules: 2nd order perturbation

$$\Delta E_0 = -\sum_{i,I,F} p(I) \frac{\left|\left\langle \phi_i, F \left| \widehat{\mathbf{d}} \cdot \widehat{\mathbf{E}} + \widehat{\mathbf{m}} \cdot \widehat{\mathbf{B}} \right| \phi_0, I \right\rangle\right|^2}{E_{i0} + \Omega_{FI}}.$$

↓

$$\Delta E_0^{a\chi} = -\sum_{i,I,F} p(I) \frac{\left|\mathbf{d}_{0i} \cdot \mathbf{E}_{IF}\right|^2 + \left|\mathbf{m}_{0i} \cdot \mathbf{B}_{IF}\right|^2}{E_{i0} + \Omega_{FI}}$$

Energy shift of chiral molecules: 2nd order perturbation

$$\Delta E_0 = -\sum_{i,I,F} p(I) \frac{\left|\left\langle \phi_i, F \left| \widehat{\mathbf{d}} \cdot \widehat{\mathbf{E}} + \widehat{\mathbf{m}} \cdot \widehat{\mathbf{B}} \right| \phi_0, I \right\rangle\right|^2}{E_{i0} + \Omega_{FI}}.$$

↓

$$\Delta E_0^{a\chi} = -\sum_{i,I,F} p(I) \frac{|\mathbf{d}_{0i} \cdot \mathbf{E}_{IF}|^2 + |\mathbf{m}_{0i} \cdot \mathbf{B}_{IF}|^2}{E_{i0} + \Omega_{FI}}$$

$$\Delta E_0^{\chi} = -\sum_{i,I,F} p(I) \frac{2\,\mathrm{Re}\left[(\mathbf{d}_{0i} \cdot \mathbf{E}_{IF})(\mathbf{m}_{i0} \cdot \mathbf{B}_{FI})\right]}{E_{i0} + \Omega_{FI}}$$

$$\langle \Delta E_0^{\chi} \rangle = -\sum_{i,I,F} \frac{2\,p(I)\,\mathrm{Re}\left[\mathcal{R}_{i0}\,(\mathbf{E}_{IF} \cdot \mathbf{B}_{FI})\right]}{3\,(E_{i0} + \Omega_{FI})},$$

Energy shift of chiral molecules: 2nd order perturbation

$$\Delta E_0 = -\sum_{i,I,F} p(I) \frac{\left|\left\langle \phi_i, F \left| \hat{\mathbf{d}} \cdot \hat{\mathbf{E}} + \hat{\mathbf{m}} \cdot \hat{\mathbf{B}} \right| \phi_0, I \right\rangle\right|^2}{E_{i0} + \Omega_{FI}}.$$

↓

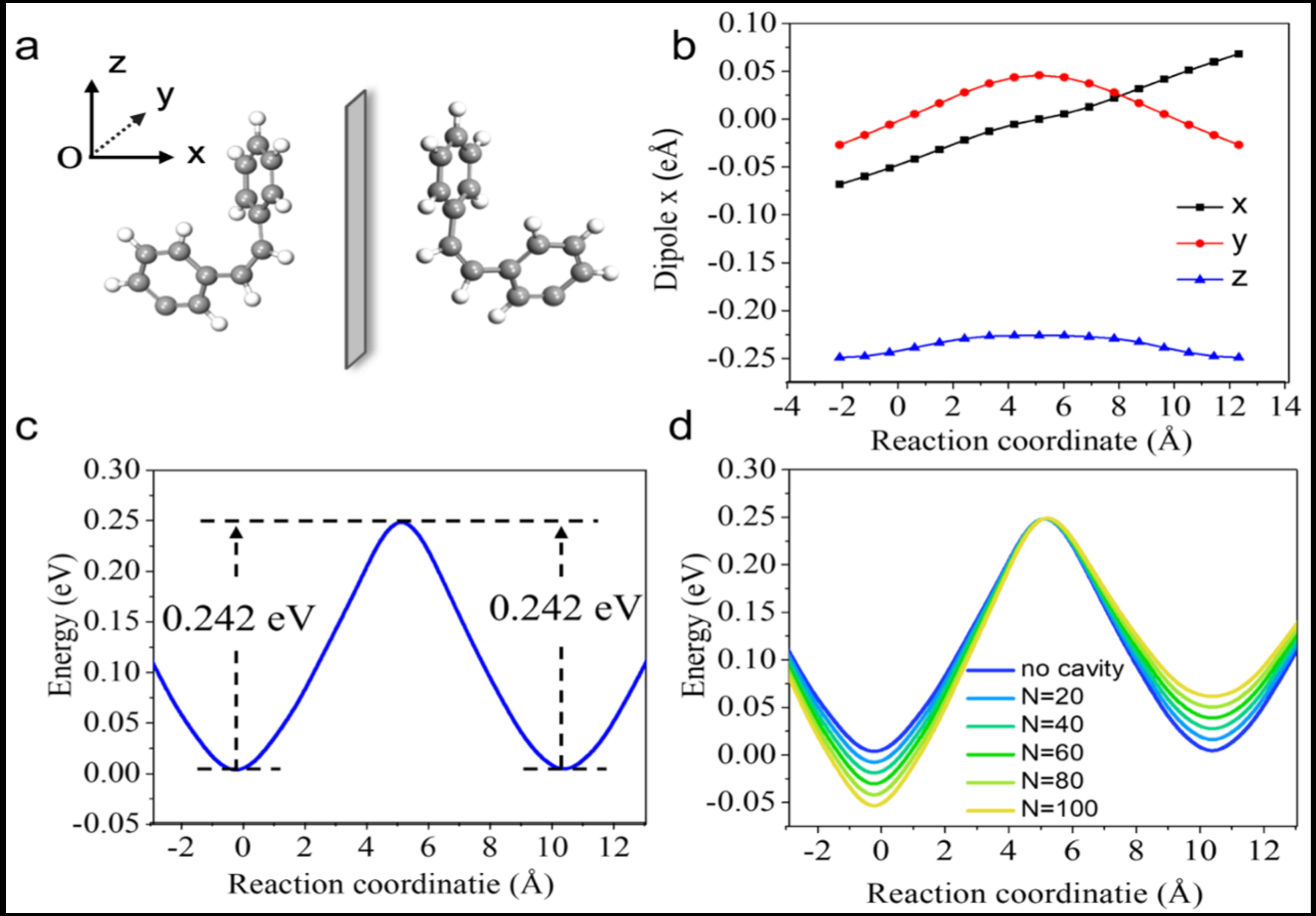
$$\Delta E_0^{a\chi} = -\sum_{i,I,F} p(I) \frac{\left|\mathbf{d}_{0i} \cdot \mathbf{E}_{IF}\right|^2 + \left|\mathbf{m}_{0i} \cdot \mathbf{B}_{IF}\right|^2}{E_{i0} + \Omega_{FI}}$$

$$\Delta E_0^{\chi} = -\sum_{i,I,F} p(I) \frac{2\,\mathrm{Re}\left[\left(\mathbf{d}_{0i} \cdot \mathbf{E}_{IF}\right)\left(\mathbf{m}_{i0} \cdot \mathbf{B}_{FI}\right)\right]}{E_{i0} + \Omega_{FI}}$$

Think:

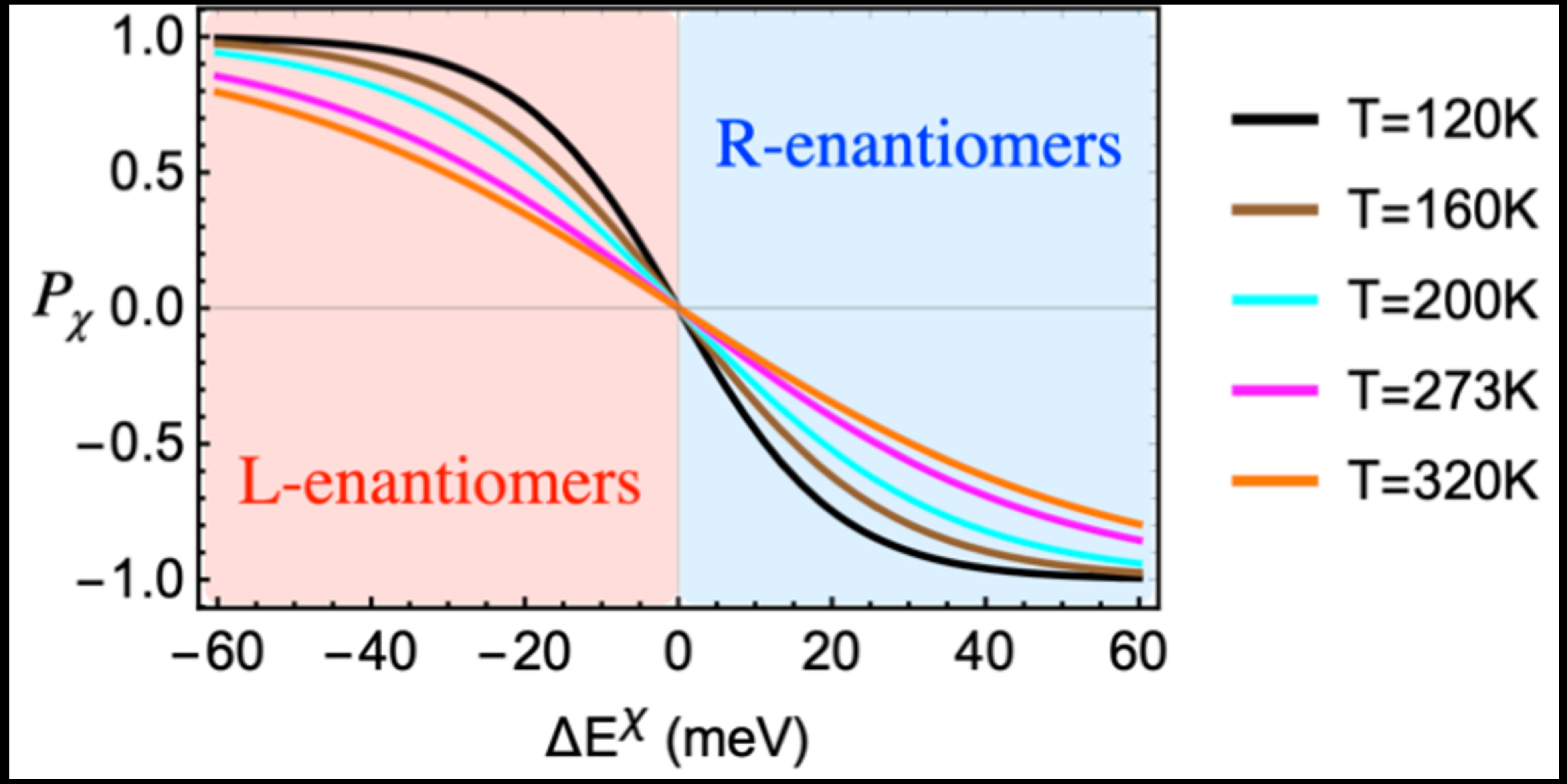
Why is it chiral?

Energy shift of chiral molecules



Chirality-Selective Rate

$$P_\chi = \frac{k_L - k_R}{k_L + k_R} = \frac{1 - \exp\left(-\beta \Delta E_0^\chi\right)}{1 + \exp\left(-\beta \Delta E_0^\chi\right)}$$



Y. Ke, Z. Song, QD Jiang, PRL, **131**, 223601 (2023)

Power–Zienau–Woolley (PZW) transformation

$$H = \sum_i \frac{\left[\mathbf{p}_i - q_i \mathbf{A}(\mathbf{r}_i)\right]^2}{2m_i} + V_{\text{Coul}} + H_{\text{EM}}.$$

↓

$$H = H_{\text{matter}} + H_{\text{field}} + H_{\text{int}}.$$

Power–Zienau–Woolley (PZW) transformation

$$H = \sum_i \frac{\left[\mathbf{p}_i - q_i \mathbf{A}(\mathbf{r}_i)\right]^2}{2m_i} + V_{\text{Coul}} + H_{\text{EM}}.$$

↓

$$H = H_{\text{matter}} + H_{\text{field}} + H_{\text{int}}.$$

↓

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}.$$

Power–Zienau–Woolley (PZW) transformation

$$H = \sum_i \frac{\left[\mathbf{p}_i - q_i \mathbf{A}(\mathbf{r}_i)\right]^2}{2m_i} + V_{\mathrm{Coul}} + H_{\mathrm{EM}}.$$

↓

$$U = \exp\left[-\frac{i}{\hbar}\int d^3r\, \mathbf{P}(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r})\right]$$

$$\mathbf{P}(\mathbf{r}) = \sum_i q_i \int_0^1 d\lambda\, \mathbf{r}_i \delta(\mathbf{r} - \lambda \mathbf{r}_i).$$

↓

$$H_{\mathrm{int}} = -\int d^3r\, \mathbf{P}(\mathbf{r}) \cdot \mathbf{E}_\perp(\mathbf{r}) - \int d^3r\, \mathbf{M}(\mathbf{r}) \cdot \mathbf{B}(\mathbf{r}) + \cdots$$

Power–Zienau–Woolley (PZW) transformation

$$H = \sum_i \frac{[\mathbf{p}_i - q_i \mathbf{A}(\mathbf{r}_i)]^2}{2m_i} + V_{\text{Coul}} + H_{\text{EM}}.$$

$$U = \exp\left[-\frac{i}{\hbar}\int d^3r\,\mathbf{P}(\mathbf{r})\cdot\mathbf{A}(\mathbf{r})\right]$$

$$\mathbf{P}(\mathbf{r}) = \sum_i q_i \int_0^1 d\lambda\,\mathbf{r}_i \delta(\mathbf{r} - \lambda\mathbf{r}_i).$$

$$H_{\text{int}} = -\int d^3r\,\mathbf{P}(\mathbf{r})\cdot\mathbf{E}_\perp(\mathbf{r}) - \int d^3r\,\mathbf{M}(\mathbf{r})\cdot\mathbf{B}(\mathbf{r}) + \cdots$$

$$\langle E\rangle = \langle B\rangle = 0$$

- the end of lecture 3

Thank You!

Take home message?

May the atmosphere be with you!

Appendices