Quantum Connections in Sweden-16
Summer School

Symmetry theory of magnetism

Qihang Liu (刘奇航)

Department of Physics
Southern University of Science and Technology (SUSTech)

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

2026. 06. 08 & 06.10 Högberga Gård, Sweden

Outline

- Lecture I: Spin group theory

  - From magnetic group to spin group
  - Spin point groups and spin space groups.
  - Enumeration and representation of spin groups.
  - Classification of magnetic order using spin space groups.

- Lecture II: Unconventional magnetism

  - Introduction: conventional magnets and spintronics.
  - Unconventional magnetism: spin-split AFM and anomalous Hall AFM.
  - Other facets: topological magnons, quantum geometry, multiferroicity, etc.
  - Prediction of new magnetic structures

Acknowledgements

Collaborators

Theory: Xiangang Wan (NJU), Caiheng Li (SUSTech), J. Manuel Perez-Mato (Bilbao, Spain)

Xi Dai (HKUST), Stefan Blügel (Jülich, Germany), Alessandro Stroppa (CNR-SPIN, Italy),

Shixuan Du (CAS-IOP)

ARPES: Prof. Chang Liu (SUSTech), Shan Qiao (SIMIT)

Group Members

Xiaobing Chen, Pengfei Liu, Yuntian Liu, Yutong Yu, Mingqiang Gu, Weizhao Chen, Haiyuan Zhu, Jiayu Li, Yongqian Zhu, Jun Ren, etc.

国家自然科学基金委员会 1986 NATIONAL NATURAL SCIENCE FOUNDATION OF CHINA

中华人民共和国科学技术部
Ministry of Science and Technology of the People's Republic of China

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

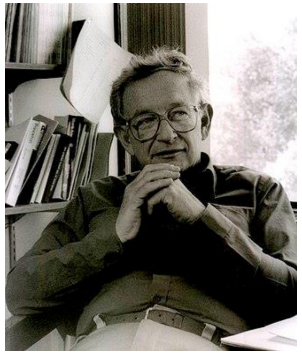
Lecture I: Spin group theory

- Introduction
  - Magnetic groups and spin groups
- Theories of spin groups
  - Group extension
  - Enumeration of spin groups
  - Representation of spin groups
- Classification of magnetic orders
  - Dichotomy of ferromagnets/antiferromagnets
  - Unifying spin group and magnetic group
  - FINDSPINGROUP

Symmetry is the kernel of physics

"I believe what characterizes 20th-century physics, so as to distinguish it from the flavor of physics in past centuries, are three concepts: Quantization, phase factor, and symmetry."
- C. N. YANG, in Hong Kong University (2000)

"It is only slightly overstating the case to say that physics is the study of symmetry."
- Philip Anderson, Nobel Laureate (1977)



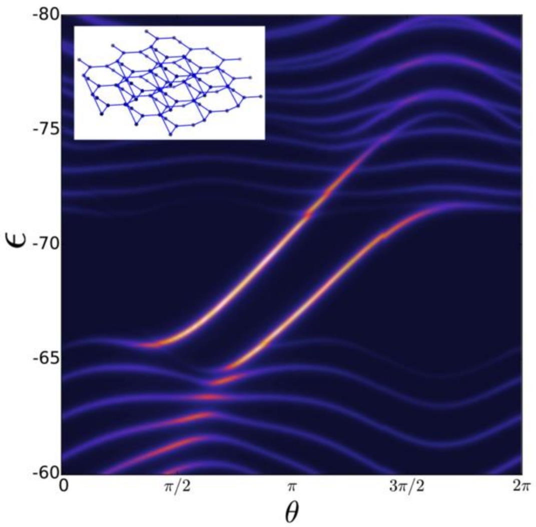
The connection is Group Theory !

Group theory in condensed matter physics

Group representations dictate

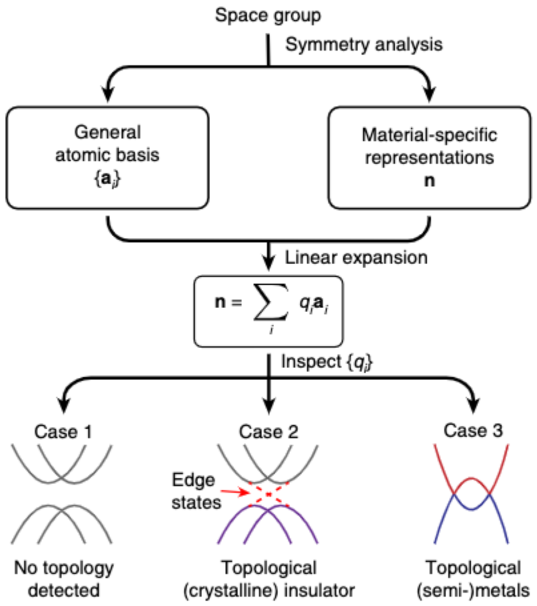
- Energy degeneracy
- Selection rule
- Physical responses
- Topological classification
- …

New quasiparticles



Science **353**, 558 (2016)

Symmetry indicator



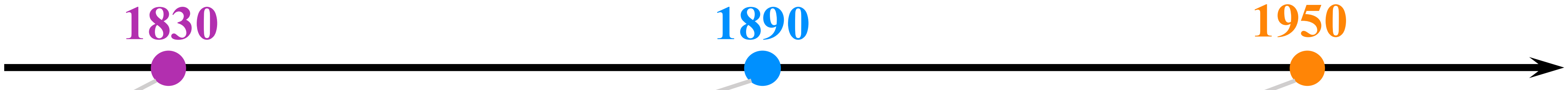
Nature **547**, 298 (2017)
Nat. Phys. 15, 470 (2019)

Linear response

Crystal system	Point group	$\chi^{(0)}$
triclinic	1	$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix}$
monoclinic	2	$\begin{pmatrix} x_{11} & 0 & x_{13} \\ 0 & x_{22} & 0 \\ x_{31} & 0 & x_{33} \end{pmatrix}$
	m	$\begin{pmatrix} 0 & x_{12} & 0 \\ x_{21} & 0 & x_{23} \\ 0 & x_{32} & 0 \end{pmatrix}$
orthorhombic	222	$\begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & 0 \\ 0 & 0 & x_{33} \end{pmatrix}$
	$mm2$	$\begin{pmatrix} 0 & x_{12} & 0 \\ x_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
tetragonal	4	$\begin{pmatrix} x_{11} & -x_{21} & 0 \\ x_{21} & x_{11} & 0 \\ 0 & 0 & x_{33} \end{pmatrix}$

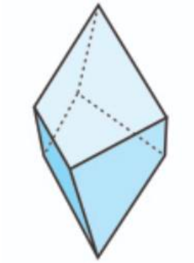
PRB **95**, 014403 (2017)

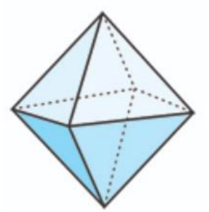
Wisdom of symmetry description for crystals



Point groups

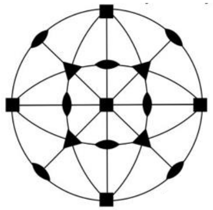
M. L. Frankenheim
1826





J. F. C. Hessel
1830

32



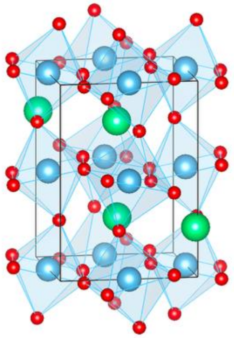
m-3m

Symmetries of geometric figures

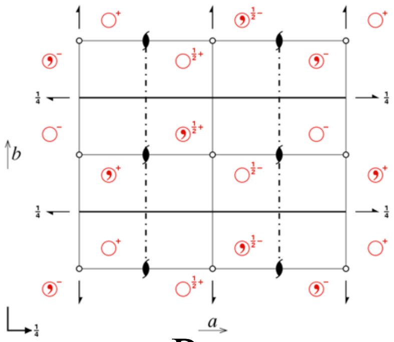
32 groups

Space groups

E. S. Fedorov
1891



A. M. Schoenflies
1891



Pnma

Symmetries of 3D lattices

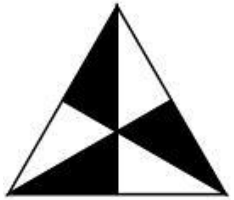
230 groups

Magnetic groups

H. Heesch
1930

A. V. Shubnikov
1951

A. Zamorzaev
1953

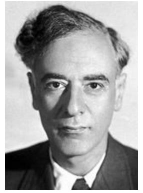


3m'

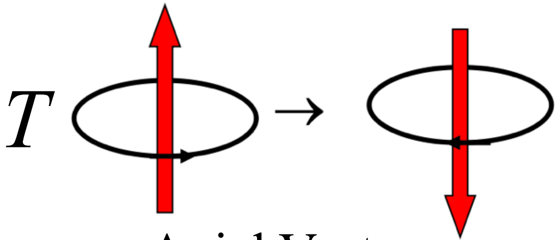
Symmetries of antisymmetric figures / 3D lattices

122/1651 groups

B. Tavger
1956



L. Landau
1958

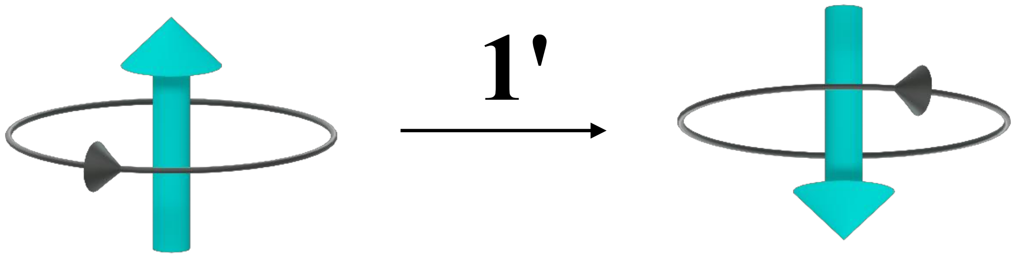


Axial Vector

Time reversal = spin reversal

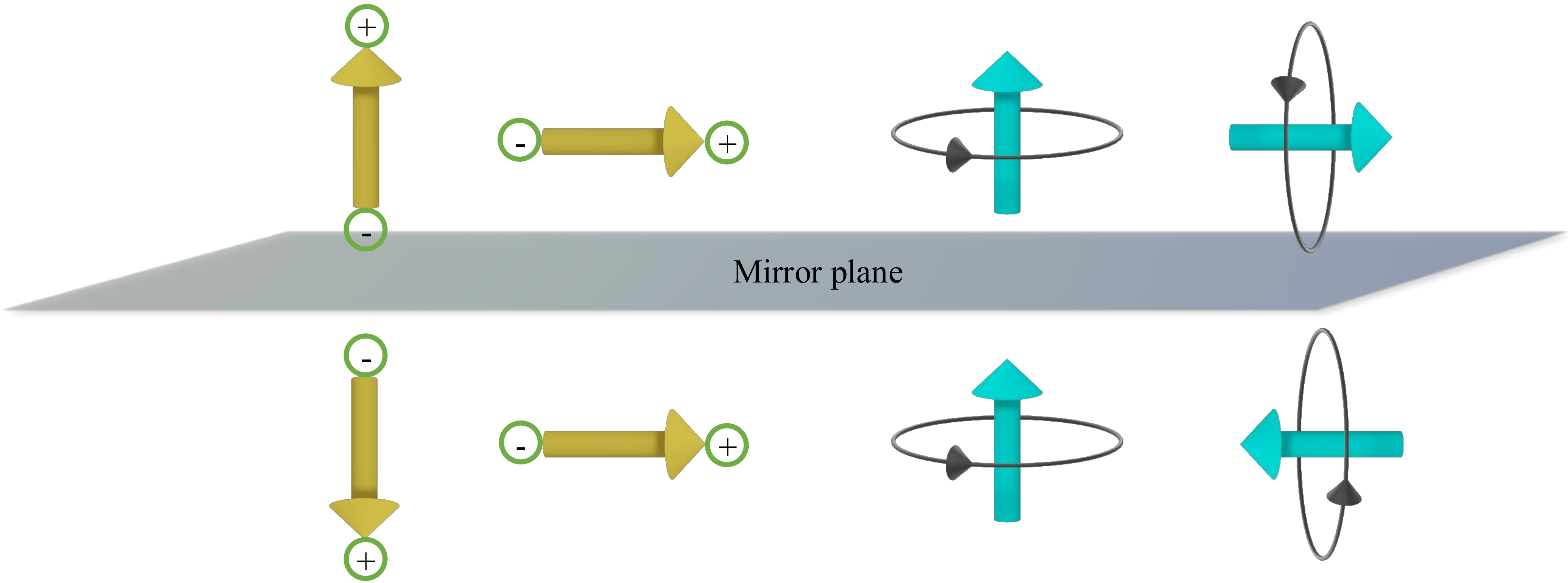
Time reversal

Time reversal = Spin reversal



Electrical dipole (Polar vector)

Magnetic dipole (Axial vector)



Symmetry Theory of Magnetism: Overview

Landau Phase Transition Theory

Symmetry breaking: Spin SU(2) + Time-Reversal

Magnetic Group

Finer description: Lattice Symmetry

1 Magnetic Structure

Representation Theory

- Enumerate all possible magnetic phases
- Neutron diffraction experiments

2 Magnetic Excitations

Symmetry-Protected Modes

- Broken SU(2) → gapless Goldstone magnons
- Magnetic group protected magnon modes

3 Physical Properties

Allowed / Forbidden + Topology

- Dzyaloshinskii-Moriya interaction, anomalous Hall effect
- Magnetic topological quantum chemistry

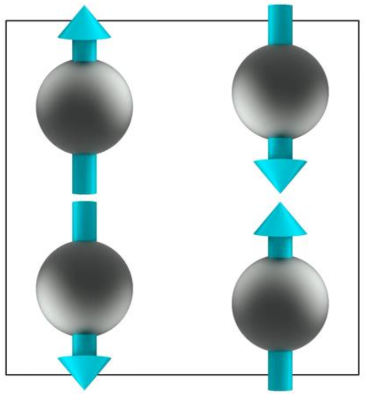
However, **magnetic groups face fundamental bottlenecks** in all these applications ...

Magnetic group: Binary spin operation

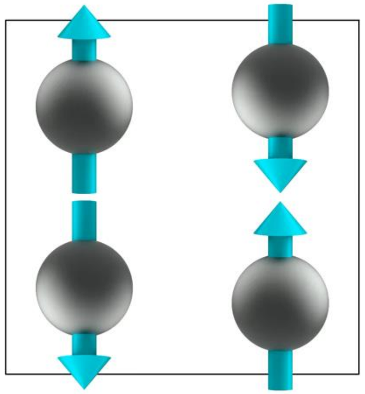
- Crystallographic group operations: C_n, P, M
- Magnetic group operations: C_n, P, M, C_nT, PT, MT

What does C_n mean in a magnetic group?

C_2 for lattice only

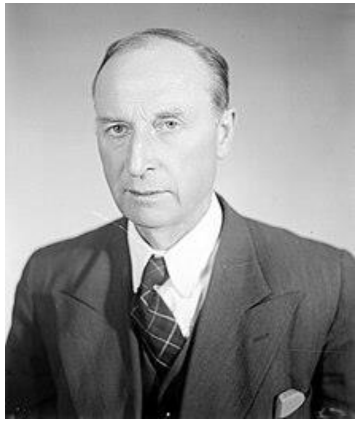


C_2 for lattice & spin



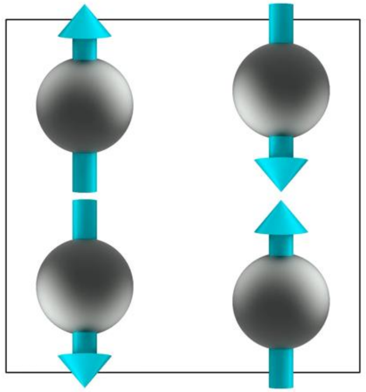
By each definition, you can get a "magnetic group" anyway…

Shubnikov group vs. Magnetic group



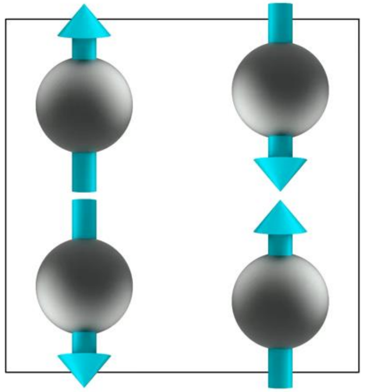
A. V. Shubnikov

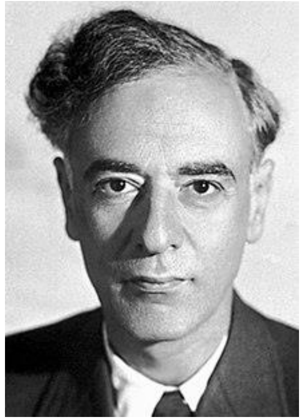
C_2 for lattice only



Shubnikov group

C_2 for lattice & spin



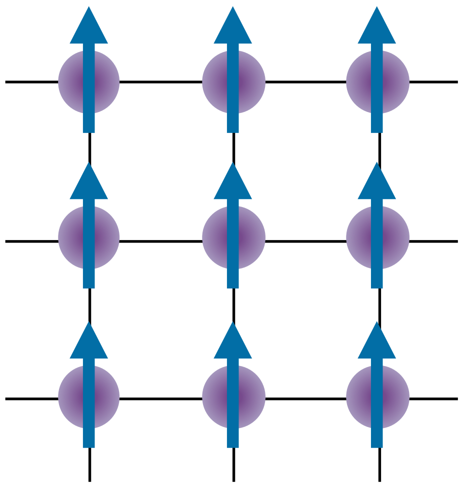


L. Landau

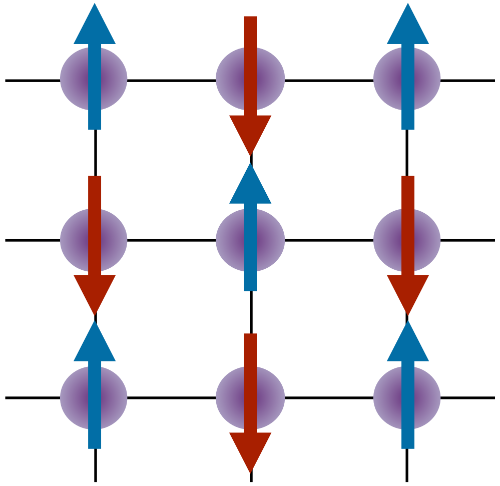
Magnetic group

Can magnetic group describe the magnetic geometry completely?

How to define and classify FM/AFM?

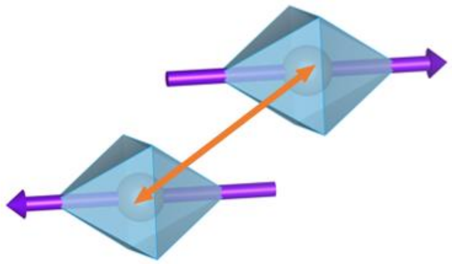
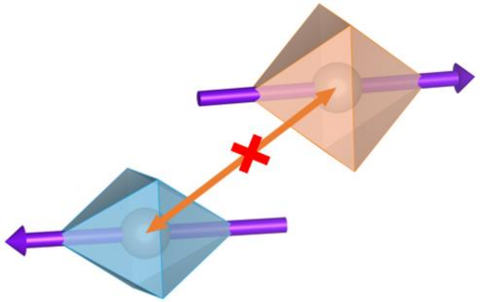


Ferromagnet (FM)



Antiferromagnet (AFM)

$$\boxed{M = 0}$$

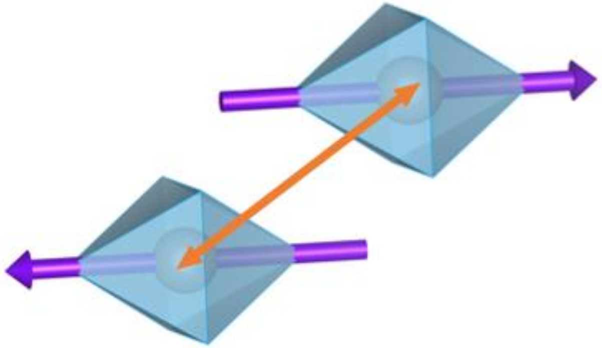
Antiferromagnetism	Ferrimagnetism (Compensated ferrimagnetism)
	

Classification of magnetism is based on geometry

"In AFM, two (or more) magnetic sublattices are **crystallographically equivalent**".

L. Néel, Ann. Phys. (Paris, Fr.) 3, 137 (1948)

Phenomenologically, but rooted in symmetry!

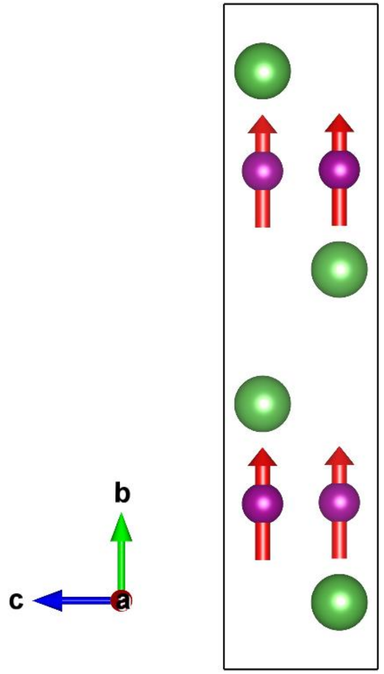


$M = 0$ enforced by symmetry !

Which symmetry?

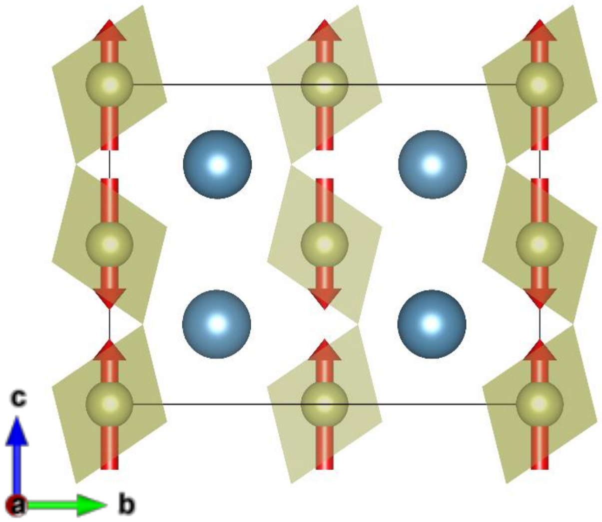
Magnetic group is insufficient to distinguish FM/AFM

MSG: $Cm'cm'$



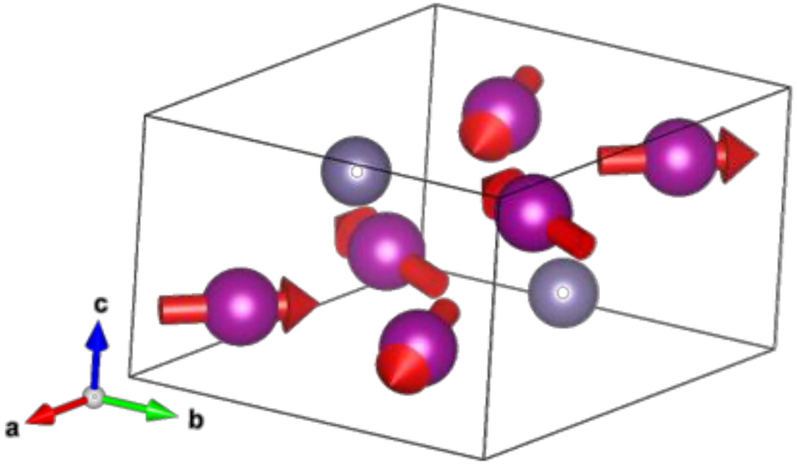
$LaMnSi_2$

Collinear FM



$CaIrO_3$

Collinear AFM



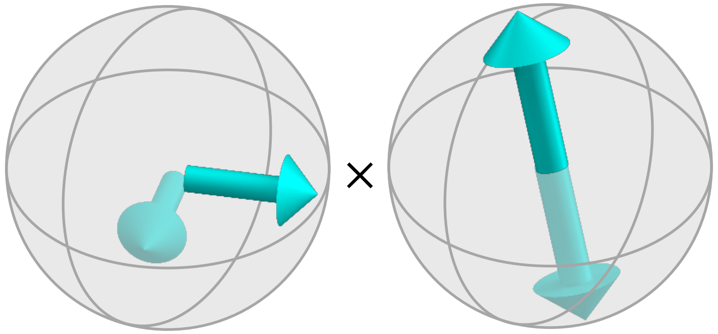
Mn_3Ge

Coplanar AFM

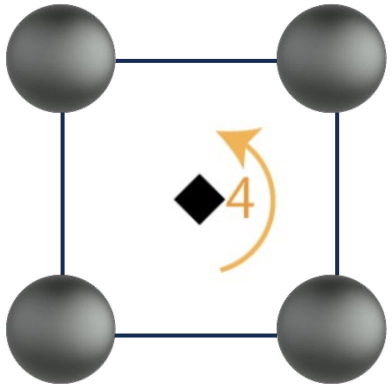
Let's use two rotation operators

Solution: separate spin and lattice rotations

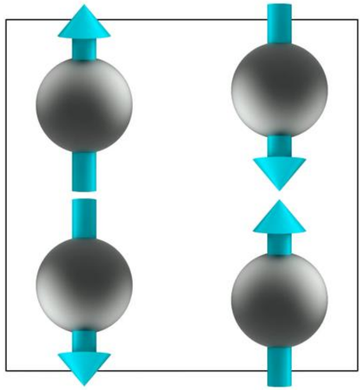
Spin group



$\times$



Lattice C_4 + Spin C_2 rotation



$SO(3) \times \{E, \mathcal{T}\} \times G_0$

$\{ \quad g_s \quad || \quad g_r \quad \}$

$\{2||4\}$

Separated spin & lattice operation

Spin group: All spin symmetry $\{g_S \| g_R\}$ that leaves the magnetic structure invariant.

$$\{g_S \| g_R\} = \{U_{\boldsymbol{m}}(\varphi), TU_{\boldsymbol{m}}(\varphi) || C_{\boldsymbol{n}}(\theta), IC_{\boldsymbol{n}}(\theta)\} \qquad \{g_S \| g_R\}S(r) = g_S S(g_R{}^{-1} r)$$

Pure lattice operation

$\{E \| g_R\}$

Only atoms transform

Spin group operation

$\{g_S \| g_R\}$

Spins & atoms can transform differently

Pure spin operation

$\{g_S \| E\}$

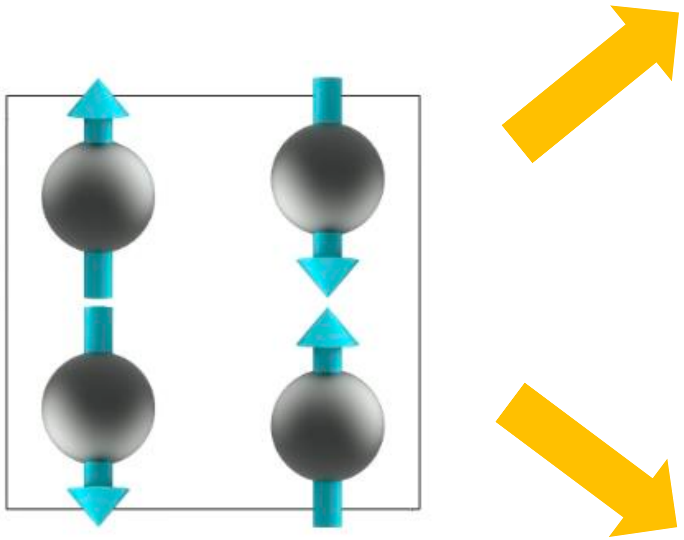
Only spins transform

Nontrivial spin group

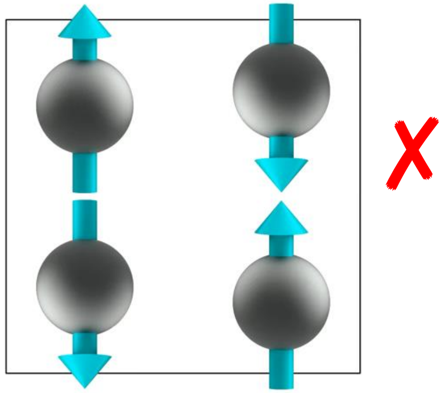
Spin-only group

$$G = G_{\mathrm{NS}} \times G_{\mathrm{SO}}$$

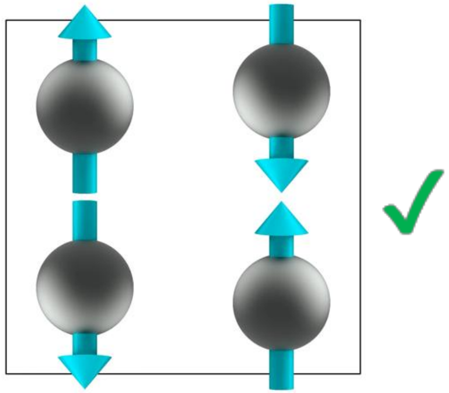
Spin group can completely describe magnetic geometry



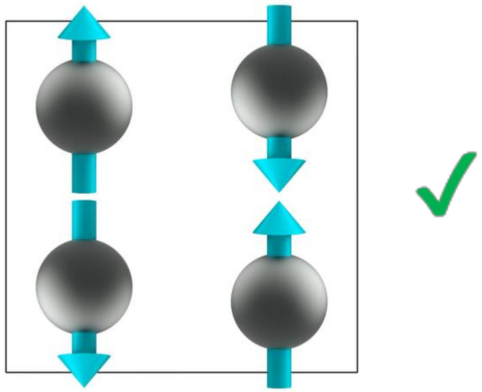
Consider an AFM spin arrangement



C_2 rotation in magnetic group

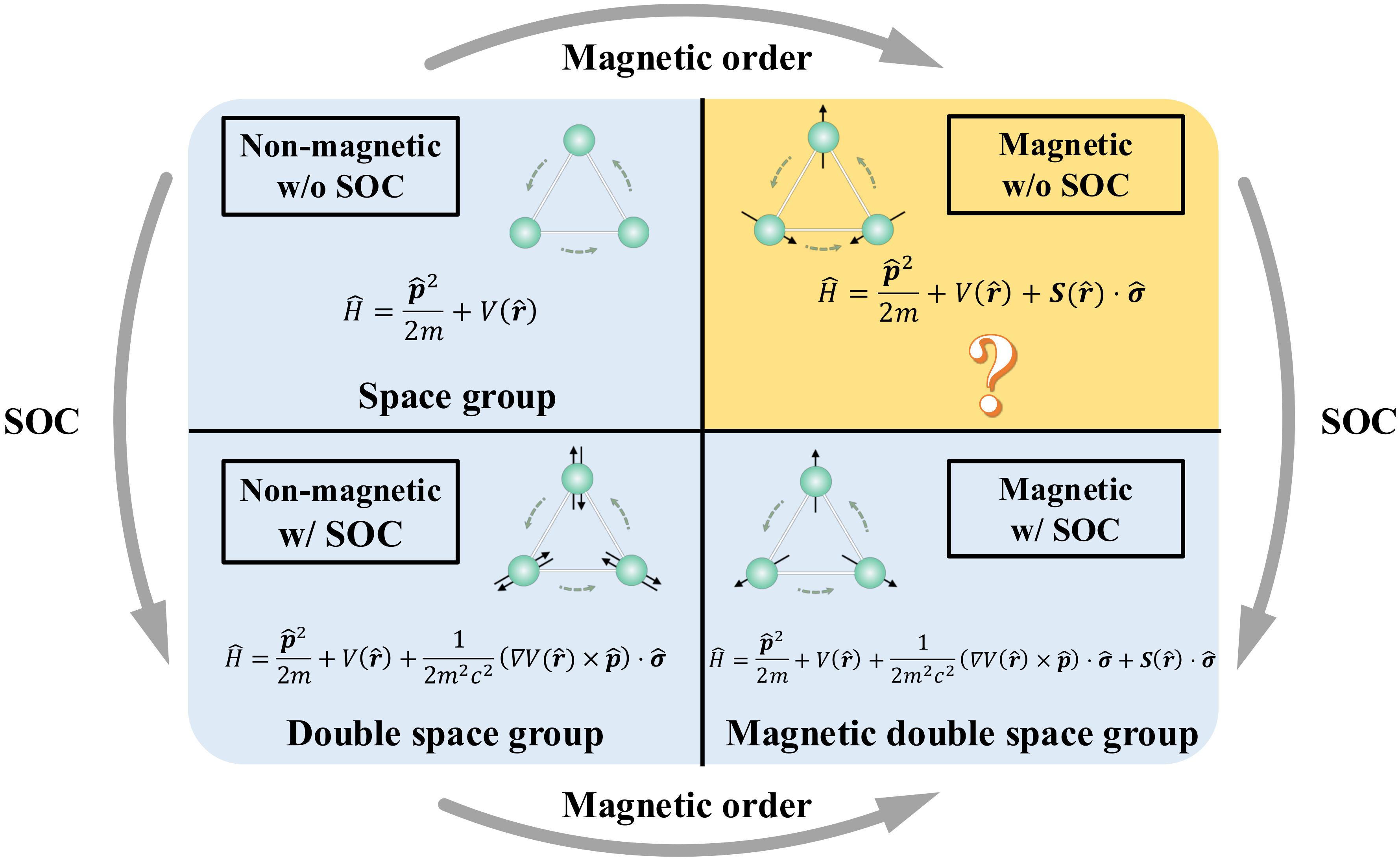


Lattice C_2 rotation (spin group)



Lattice C_4 + Spin C_2 rotation (spin group)

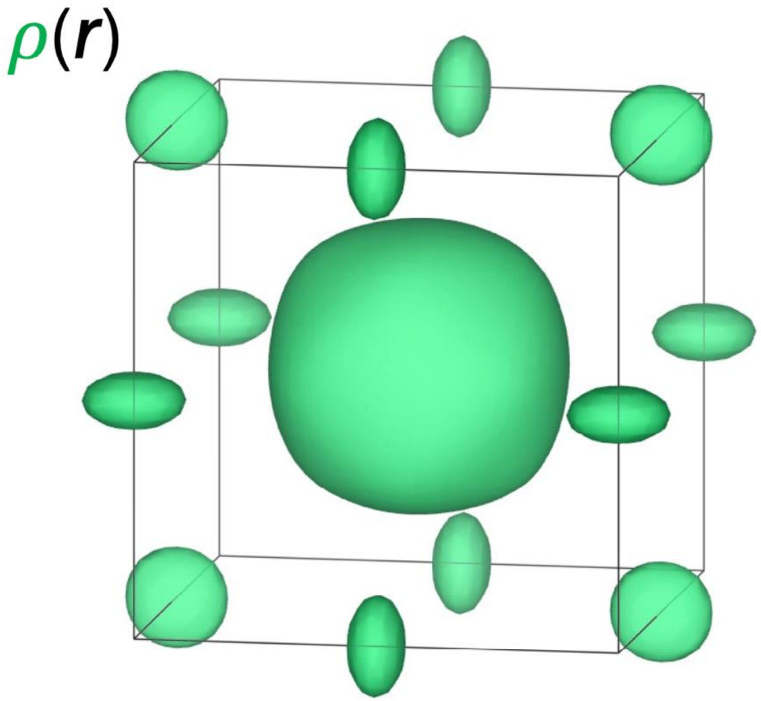
Another way to consider: symmetry of Hamiltonian



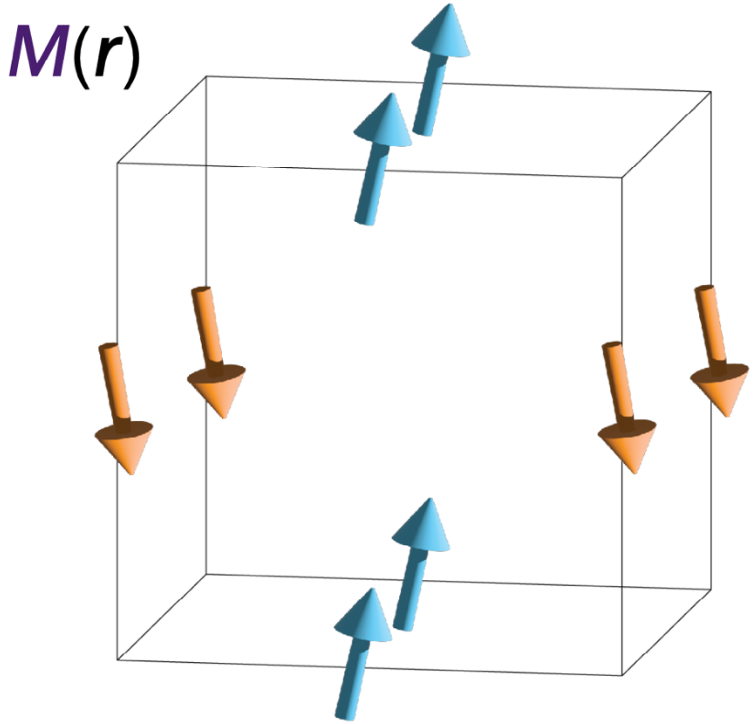
P. Liu, …, X. Wan*, **QL***, ***PRX*** 12, 021016 (2022)

How to describe the symmetry of a crystal

Magnetic-ordered crystals



+



Electron density (scalar field) spin arrangement (axial vector field)

Symmetry of magnetic and SOC Hamiltonian

w/ SOC	w/o SOC
$\hat{H}=\frac{\hat{\boldsymbol{p}}^2}{2m}+V(\hat{\mathbf{r}})+\frac{1}{2m^2c^2}(\nabla V(\hat{\mathbf{r}})\times\hat{\boldsymbol{p}})\cdot\hat{\boldsymbol{\sigma}}+\boldsymbol{S}(\hat{\mathbf{r}})\cdot\hat{\boldsymbol{\sigma}}$	$\hat{H}=\frac{\hat{\boldsymbol{p}}^2}{2m}+V(\hat{\mathbf{r}})+\boldsymbol{S}(\hat{\mathbf{r}})\cdot\hat{\boldsymbol{\sigma}}$

$$\{g_S\|g_R\}\left(\frac{1}{2m^2c^2}(\nabla V(\hat{\mathbf{r}})\times\hat{\mathbf{p}})\cdot\hat{\boldsymbol{\sigma}}\right)\{g_S\|g_R\}^{-1}$$
$$=\frac{1}{2m^2c^2}(g_R\nabla V(g_R{}^{-1}\hat{\mathbf{r}})\times g_R\hat{\mathbf{p}})\cdot g_S\hat{\boldsymbol{\sigma}}$$
$$=\frac{1}{2m^2c^2}g_R(\nabla V(\hat{\mathbf{r}})\times\hat{\mathbf{p}})\cdot g_S\hat{\boldsymbol{\sigma}}$$
$$=\frac{1}{2m^2c^2}(\nabla V(\hat{\mathbf{r}})\times\hat{\mathbf{p}})\cdot\hat{\boldsymbol{\sigma}}\qquad\Rightarrow\ g_S=g_R$$

Spin rotation g_S should be equal to spatial rotation g_R

$$\{g_S\|g_R\}\boldsymbol{S}(\hat{\mathbf{r}})\cdot\hat{\boldsymbol{\sigma}}\{g_S\|g_R\}^{-1}$$
$$=\boldsymbol{S}(g_R{}^{-1}\hat{\mathbf{r}})\cdot g_S{}^{-1}\hat{\boldsymbol{\sigma}}$$
$$=g_S\boldsymbol{S}(g_R{}^{-1}\hat{\mathbf{r}})\cdot\hat{\boldsymbol{\sigma}}$$
$$=\boldsymbol{S}(\hat{\mathbf{r}})\cdot\hat{\boldsymbol{\sigma}}\qquad\Rightarrow\ g_S\boldsymbol{S}(g_R{}^{-1}\hat{\mathbf{r}})=\boldsymbol{S}(\hat{\mathbf{r}})$$

Spin rotation g_S should be an integer multiple of spatial rotation g_R

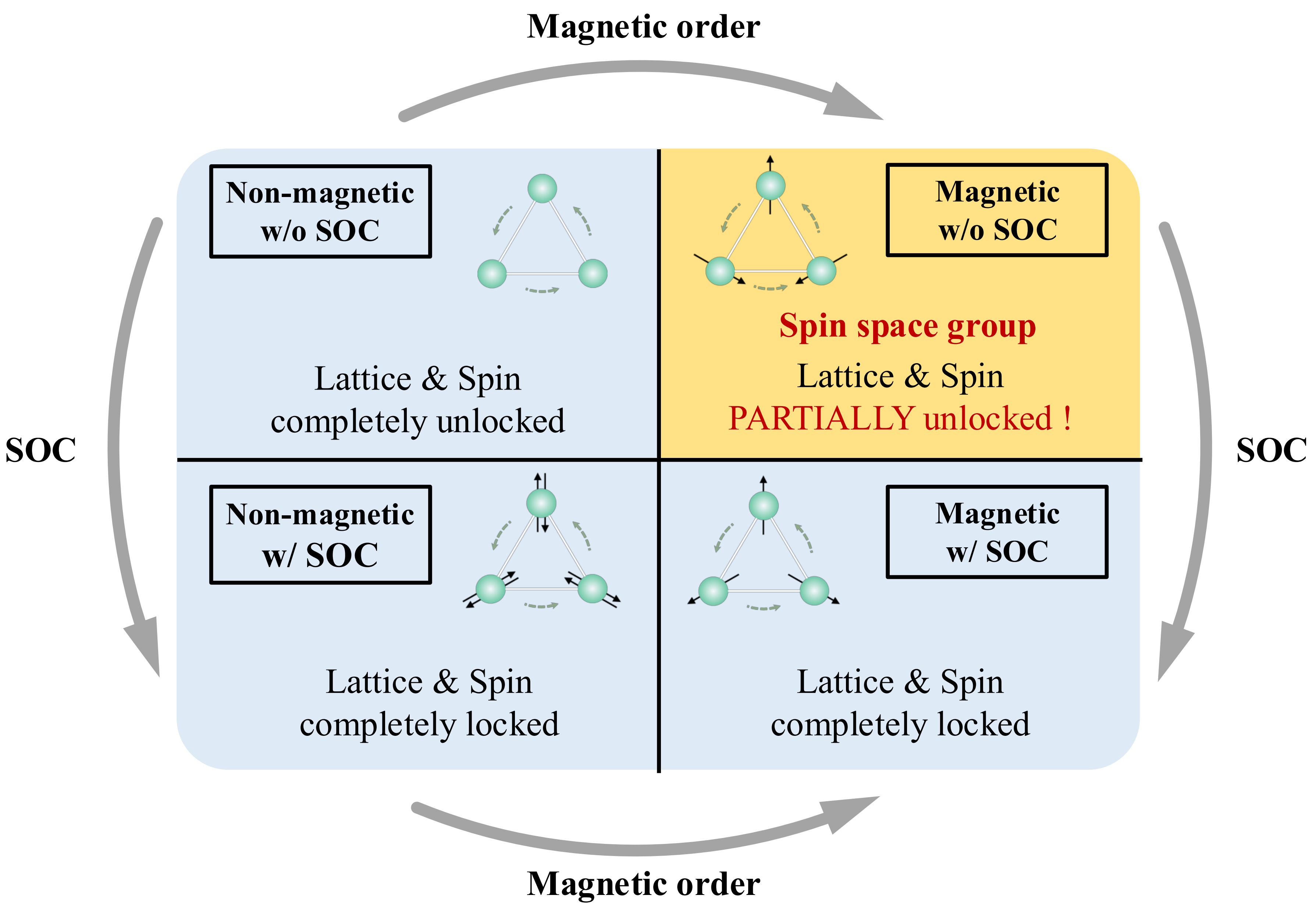
Magnetic group

Each spatial rotation attached to ONE spin rotation

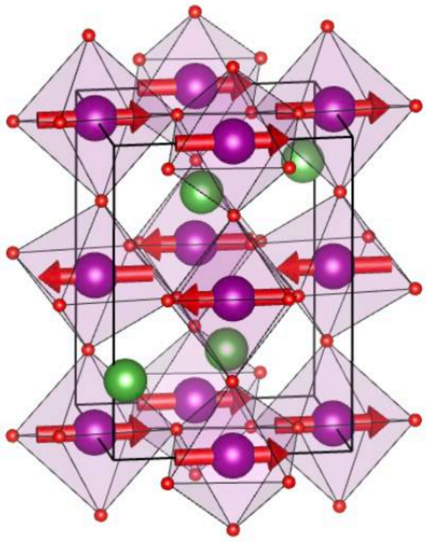
Spin group

Each spatial rotation allows MULTIPLE spin rotations

Locking between lattice & spin degrees of freedom



New insight of symmetry description for magnetic crystals



Magnetic order:

Spin point/space groups

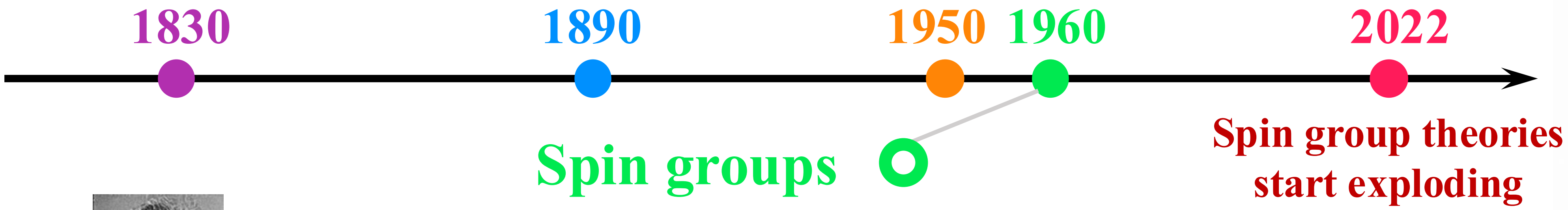
Symmetry breaking

If spin can feel lattice:

Magnetic point/space groups

Spin group describes geometry
Magnetic group describes SOC

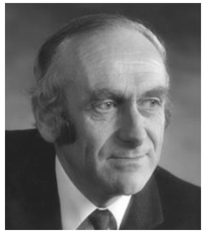
Original works about spin group theories



V. E. Naish

- **Geometry (1963):** More "colors" are needed to completely describe the magnetic symmetry.

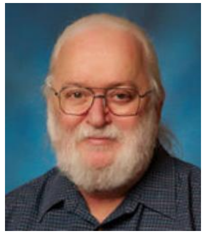
V. E. Naish, *Izv. Akad. Nauk SSSR, Ser. Fiz.* **27**, 1496 (1963).



W. Brinkman E. Elliott

- **Hamiltonian (1966):** Multiple-fold degeneracies of spin wave spectrum require an enhanced symmetry group called "spin group", which consider spin rotation and spatial rotation separately.

W. F. Brinkman and R. J. Elliott, *J. Appl. Phys.* **37**, 1457 (1966); *Proc. R. Soc. A* **294**, 343 (1966).



D. B. Litvin

- **Structure (1974):** The spin group is the symmetry group that leaves the magnetic structure invariant, which can be expressed as

$$G = \underbrace{G_{\mathrm{NS}}}_{\text{Nontrivial spin group}} \times \underbrace{G_{\mathrm{SO}}}_{\text{Spin-only group}}$$

598 spin point groups

D. B. Litvin and W. Opechowski, Physica **76**, 538 (1974); D. B. Litvin, Acta Crystallogr. A **33**, 279 (1977)

New development of spin group theories

- Spin point groups for specific magnetic configuration?
  PRX 12, 021016 (2022)

- Enumeration of spin space groups?
  PRX 14, 031038 (2024)

- Representation theory of spin space groups?
  Collinear: Nature 640, 349 (2025)
  Noncollinear: PRX 14, 031038 (2024)

- Unifying SSG and MSG?
  Oriented spin space groups: Nature 652, 869 (2026)

Pengfei Liu

Xiaobing Chen

Yuntian Liu

- Parallel works on spin group theory:

P. McClarty's group: PRB 105, 064430 (2022), SciPost Phys. 18, 109 (2025)

L. Šmejkal et al.: PRX 12, 031042 (2022)

R. Arita's group: Acta Cryst. A80, 94 (2024), PRB 109, 094438 (2024)

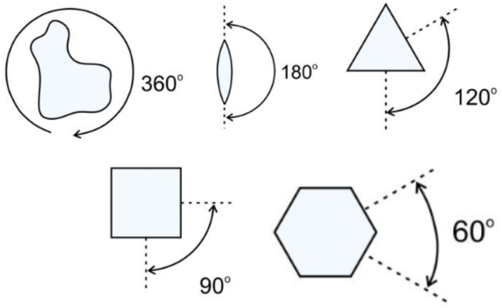
Z.-X. Liu and C. Fang's group: PRX 14, 031039 (2024), Nat. Commun. 15, 10203 (2024)

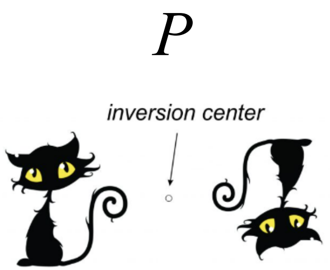
Z. Song's group: PRX 14, 031037 (2024)

Lecture I: Spin group theory

- Introduction
  - Magnetic groups and spin groups
- Theories of spin groups
  - Group extension
  - Enumeration of spin groups
  - Representation of spin groups
- Classification of magnetic orders
  - Dichotomy of ferromagnets/antiferromagnets
  - Unifying spin group and magnetic group
  - FINDSPINGROUP

Group extension methods





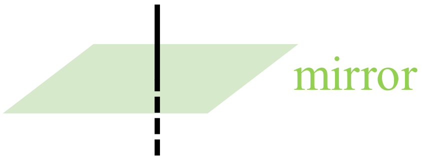
Type I: $G_I = H$

Type II: $G_{II} = H + PH$

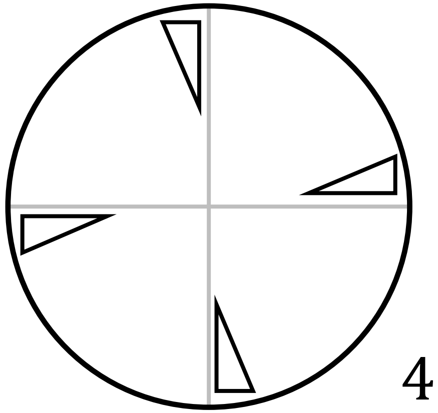
Type III: $G_{III} = H + PAH$

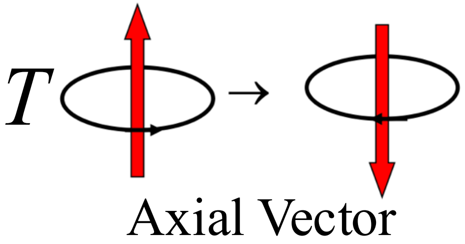
P: inversion, A: rotation

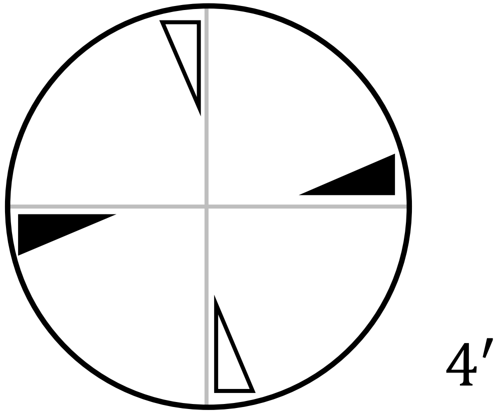
Type I point group (11) → Point group (32)



Wallpaper group (17) → Layer group (80)







Point group (32) → Magnetic point group (122)

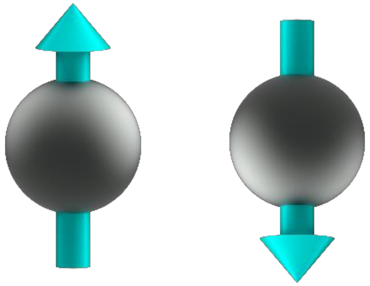
Antisymmetry
Dichromatic extension

Magnetic point group

Type I (32): Colorless group **G**

Type II (32): Grey group **G + TG**

Type III (58): "black-white" group **H + T(G - H)**



Constructive lemma: *the magnetic groups derived from the crystallographic group* **G** *can be constructed considering the index 2 subgroups* **H** *of* **G** *as constituting the "unprimed" elements and the rest of operators,* **G – H**, *those that are multiplied by the time reversal operator.*

Magnetic group: Dichromatic extension

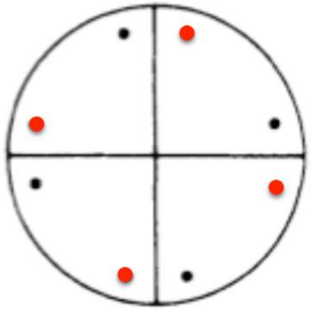
Magnetic point group (32 + 32 + 58 = 122)

1. Take a nonmagnetic crystallographic point group G,
2. Find a halves normal subgroup H,
3. Construct the type-III magnetic group $\{H, T(G-H)\}$.

$$G = 4mm \qquad \{1, 4^{+}_{001}, 4^{-}_{001}, 2_{001}, m_{100}, m_{110}, m_{010}, m_{1-10}\}$$

$$H = 4 \qquad \{1, 4^{+}_{001}, 4^{-}_{001}, 2_{001}\}$$

$$M = 4m'm' \qquad \{1, 4^{+}_{001}, 4^{-}_{001}, 2_{001}, m'_{100}, m'_{110}, m'_{010}, m'_{1-10}\}$$



Spin group: Polychromatic extension

Nontrivial spin point group (598)

1. Take a nonmagnetic crystallographic point group G.
2. Find all normal subgroups $L \trianglelefteq G$
3. Determine the quotient group $G/L = \{g_1 L, \ldots, g_n L\}$
4. Find a point group (a subgroup of the spin-part operations $\mathrm{SO}(3)\otimes \mathrm{Z}_2^{\mathrm{T}}$) that is isomorphic to G/L, denoted as $G^s = \{g_1^s, \ldots, g_n^s\}$.
5. Choose an isomorphism relation $\varphi: g_i^s \to g_i L$, then we have a nontrivial spin point group:

$$G_{\mathrm{NSP}} = \{g_1^s || g_1 L\} + \ldots + \{g_n^s || g_n L\}$$

6. Deduplication of nontrivial spin point group using normalizer:

$$\exists a \in N_{O(3)}(G) \times N_{O(3)}(G^S):\ a^{-1} G_{\mathrm{NSP}}^1 a = G_{\mathrm{NSP}}^2$$

Nontrivial spin point group: Derivation

2. Find all normal subgroups $L \trianglelefteq G$

Definition: A subgroup L of a group G is called a normal subgroup of G if it is invariant under conjugation.
For all operation $g \in G$, the left and right cosets gL and Lg are equal.

3. Determine the quotient group $G/L = \{g_1 L, \ldots, g_n L\}$

For example:	$G = 32$	$\{1, 3^+_{001}, 3^-_{001}, 2_{100}, 2_{010}, 2_{110}\}$	$gL = Lg$?
subgroup of G:	1	$\{1\}$	yes
	2	$\{1, 2_{100}\}$	no
	3	$\{1, 3^+_{001}, 3^-_{001}\}$	yes
	32	$\{1, 3^+_{001}, 3^-_{001}, 2_{100}, 2_{010}, 2_{110}\}$	yes

Results:

G	L	G/L
32	1	32
	3	2
	32	1

Nontrivial spin point group: Derivation

4. Find a point group (a subgroup of the spin-part operations $\mathrm{SO(3)}\otimes \mathrm{Z}_2^{\mathrm{T}}$) that is isomorphic to G/L, denoted as $G^s = \{g_1^s, \ldots, g_n^s\}$.

G	L	G/L	G^s
32	1	32	32
			$3m$
	3	2	2
			m
			-1
	32	1	1

Nontrivial spin point group: Derivation

5. Choose an isomorphism relation $\varphi\colon g_i^s \to g_i L$, then we have a nontrivial spin point group

$$G_{\mathrm{NSP}} = \{g_1^s||g_1 L\}+\ldots+\{g_n^s||g_n L\}$$

G	L	G/L	G^s	G_{NSP}
32	1	32	32	${}^{3^+_{001}}3^{2_{100}}2$
			$3m$	${}^{3^+_{001}}3^{m_{100}}2$
	3	2	2	${}^{1}3^{2_{001}}2$
			m	${}^{1}3^{m_{001}}2$
			-1	${}^{1}3^{-1}2$
	32	1	1	${}^{1}3^{1}2$

D. B. Litvin, Acta Crystallogr. A **33**, 279 (1977)

Spin-only groups for collinear and coplanar configurations

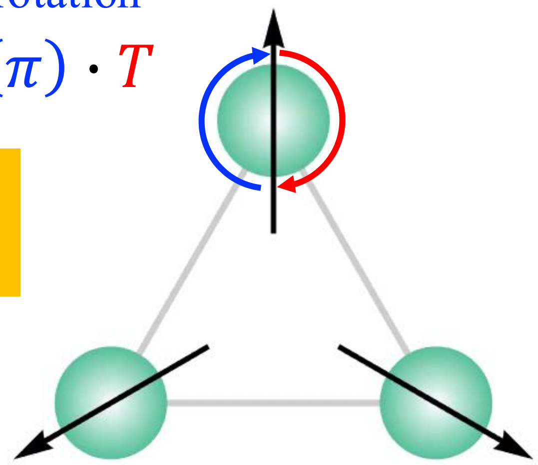
Coplanar: $G_{SO} = Z_2^K$ $\quad {}^{m}1$

$Z_2^K = \{\{E||E\},\{TU_{\boldsymbol{n}}(\pi)||E\}\}$

Spin rotation

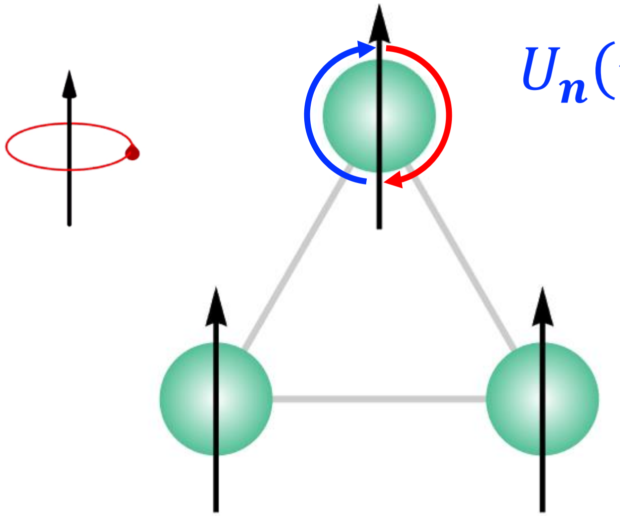
$m = U_{\boldsymbol{n}}(\pi) \cdot T$

253



Collinear: $G_{SO} = SO(2) \rtimes Z_2^K$ $\quad {}^{\infty m}1$

$SO(2) = \{\{E||E\},\{U_z(\phi)||E\}\}, \phi \in [0, 2\pi)$



$U_{\boldsymbol{n}}(\pi) \cdot T$

90

$U_{\boldsymbol{n}}(\pi)$: two-fold rotation about any axis perpendicular to spin axis

P. Liu, …, X. Wan*, **QL*** et al. ***PRX*** 12, 021016 (2022)

Symbol of Spin Space Group (Chen-Liu symbol)

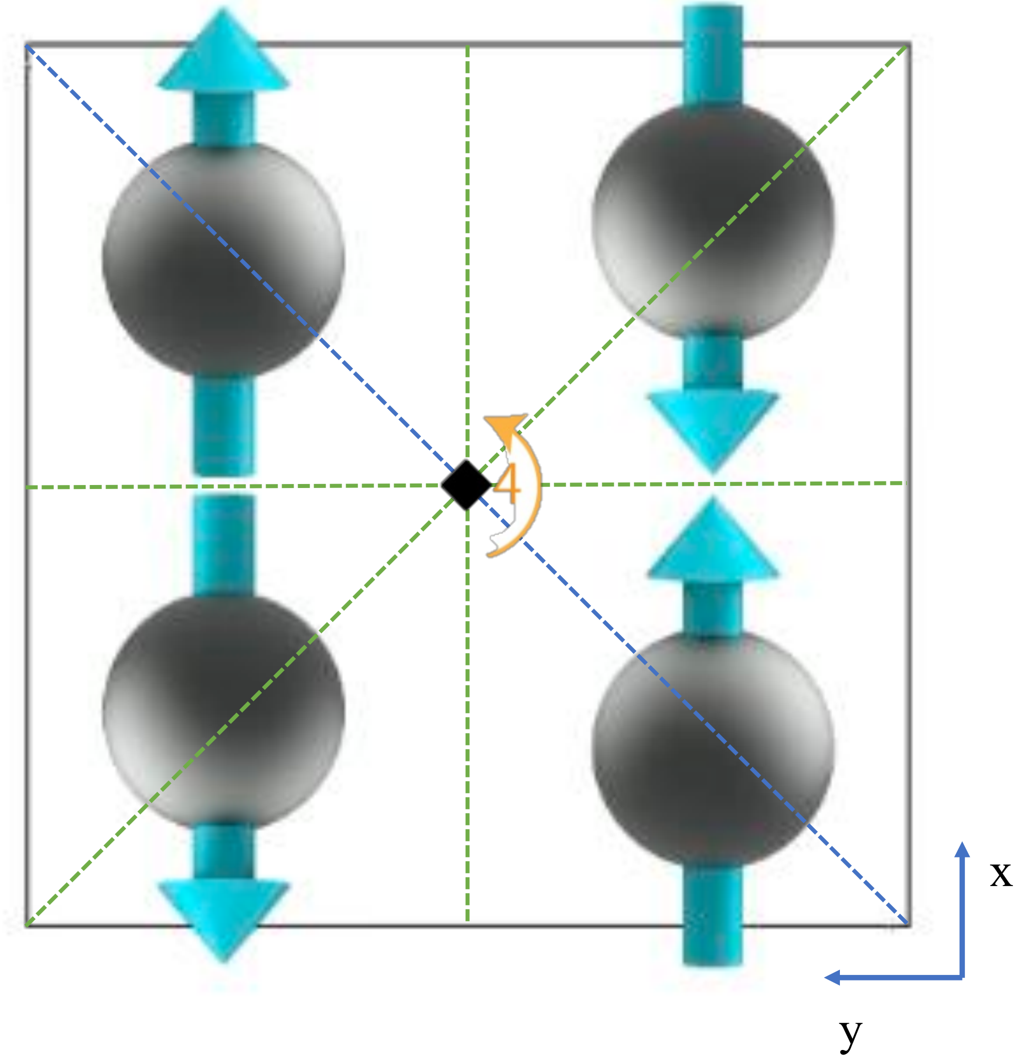
$\{1||m_{001}\}$ $\{1||m_{110}\}$

$$P\,{}^{2}4\;/\;{}^{1}m\;{}^{2}m\;{}^{1}m\;{}^{\infty m}1$$

$\{2_{001}||4_{001}\}$ $\{2_{100}||m_{100}\}$ $\{m_{001}||1\}$

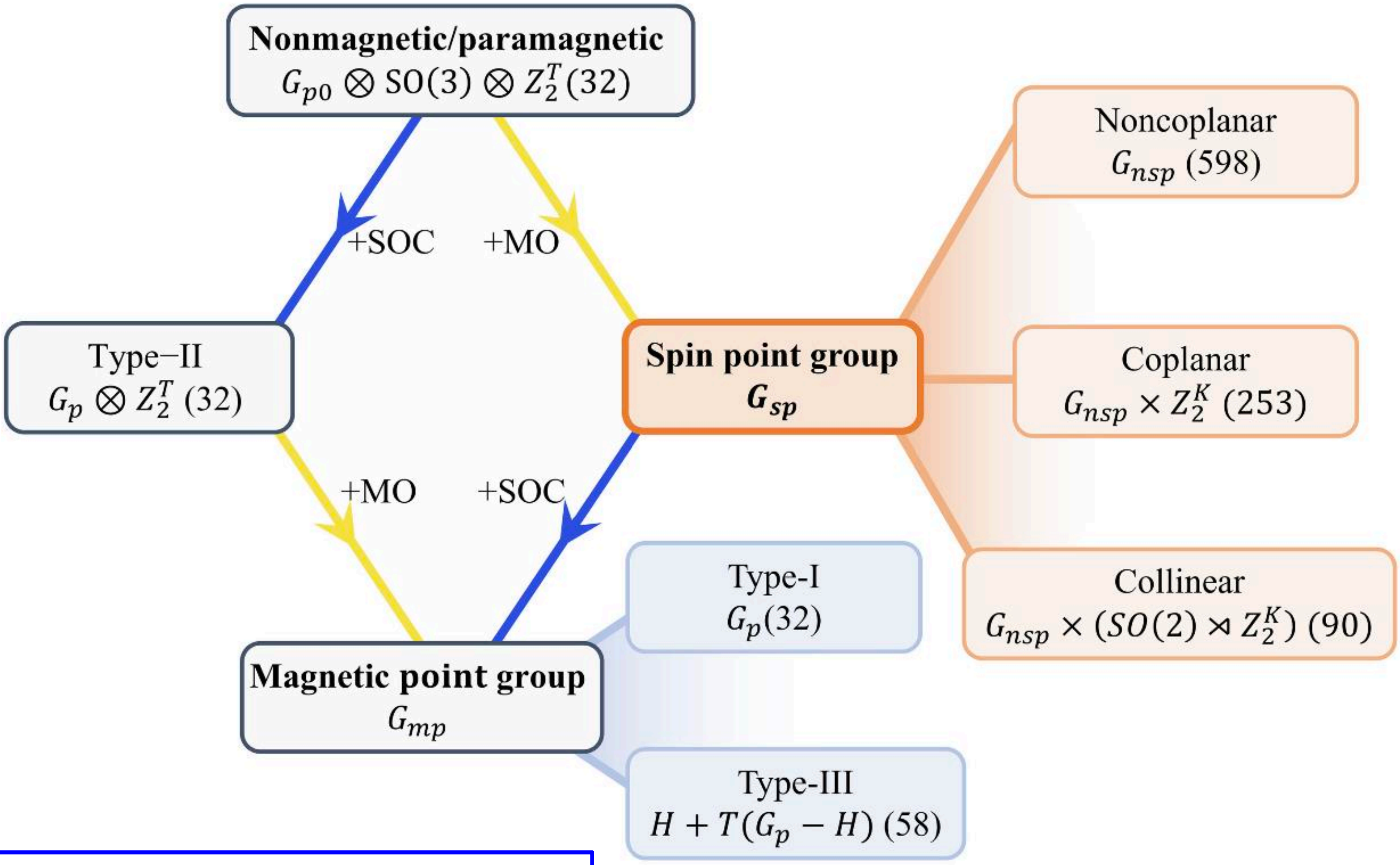
Superscript:
Spin space point group operations

Subscript:
real space point/space group operations



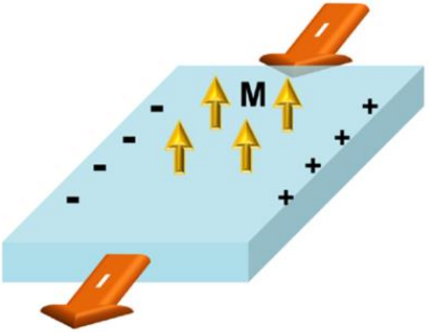
X.Chen,…, **Q. Liu***, ***PRX*** 14, 031038 (2024)

Symmetry hierarchy from spin to magnetic point groups



P. Liu, …, X. Wan*, **QL*** et al. ***PRX*** 12, 021016 (2022)

Anomalous Hall effect: time-reversal breaking vs. spin-group breaking



Empirical law:

$$\rho_{xy} = \rho_M M_z$$

SOC expansion : multipolar anisotropy

$$\sigma_i = p_{ij}\widehat{M}_j + o_{ijk\ell}\widehat{M}_j\widehat{M}_k\widehat{M}_\ell + \cdots$$

dipole　　　octupole

Z. Liu, Y. Gao, Q. Niu et al. ***PRX*** 15, 031006 (2025)

$$\text{Double Group} \xrightarrow[\text{T breaking}]{\cancel{\text{Magnetic order}}} \text{Magnetic Group}$$

$$E_{SOC} \ll E_{\text{exchange}}$$

$$\text{Spin Group} \xrightarrow[\text{Spin-group breaking}]{\text{SOC}} \text{Magnetic Group} \checkmark$$

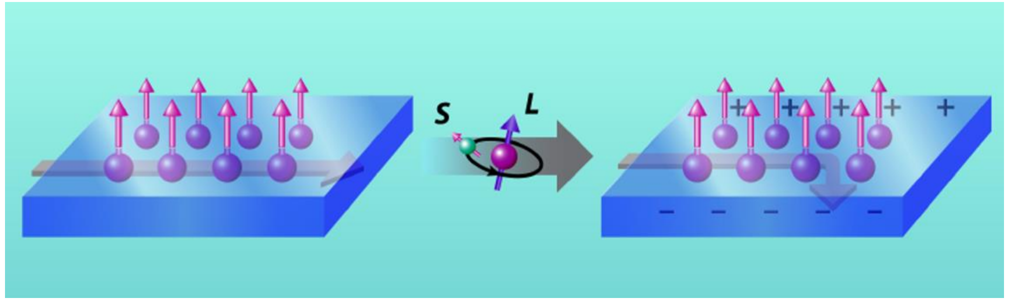
Q. Liu, ***Physics*** 18, 127 (2025)

Physics

VIEWPOINT

Rethinking the Anomalous Hall Effect: A Symmetry Revolution

Qihang Liu
Department of Physics, Southern University of Science and Technology, Shenzhen, China



Enumeration of spin space groups

Nontrivial spin space group (infinite in principle)

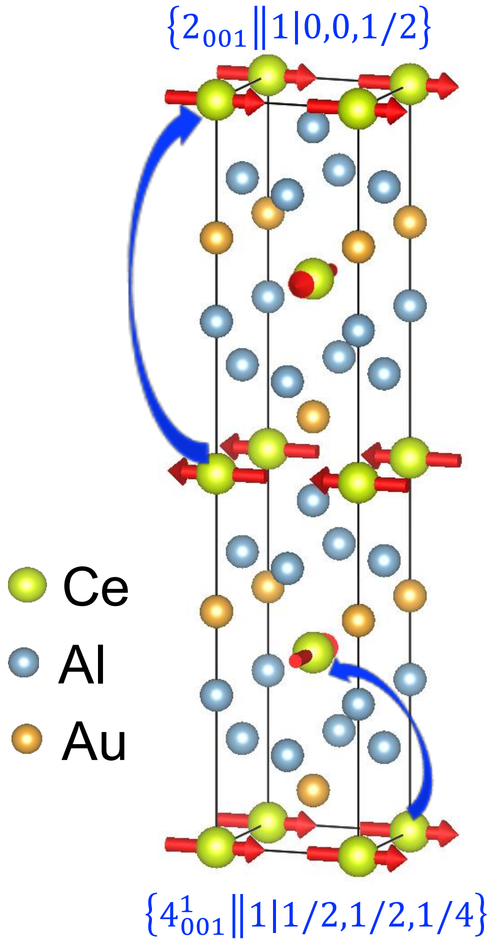
1. Take a nonmagnetic crystallographic space group G_0 (including cell expansion).
2. Find all normal subgroups $L_0 \trianglelefteq G_0$
3. Determine the quotient group $G_0/L_0 = \{g_1 L_0, \dots, g_n L_0\}$
4. Find a point group (a subgroup of the spin-part operations $\mathrm{SO}(3) \otimes \mathrm{Z}_2^{\mathrm{T}}$) that is isomorphic to G_0/L_0, denoted as $G^s = \{g_1^s, \dots, g_n^s\}$.
5. Choose an isomorphism relation φ: $g_n^s \rightarrow g_n L_0$, then we have a nontrivial spin space group:

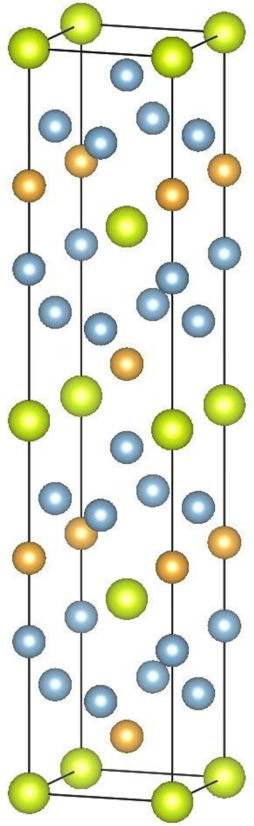
$$G_{\mathrm{NSS}} = \{g_1^s || g_1 L_0\} + \dots + \{g_n^s || g_n L_0\}$$

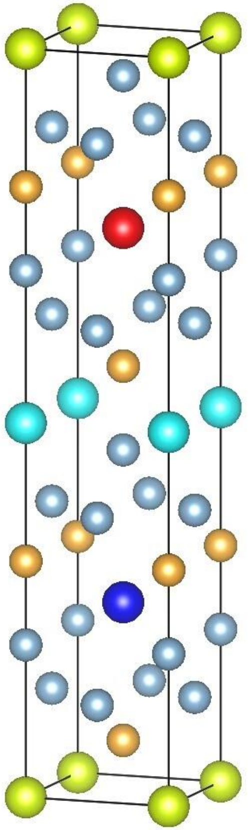
6. Deduplication of nontrivial spin space group using chirality-preserved affine normalizer:

$$\exists a \in [N_{A^+}(L) \cap N_{A^+}(G)] \times N_{O(3)}(G^S):\ a^{-1} G_{\mathrm{NSP}}^1 a = G_{\mathrm{NSP}}^2$$

Constructing spin space group: spiral AFM $CeAuAl_3$







$G_0 = I4mm$ (No. 107)

Cell expansion $i_k = 4$

$L_0 = P4mm$ (No. 99)

Sublattice group

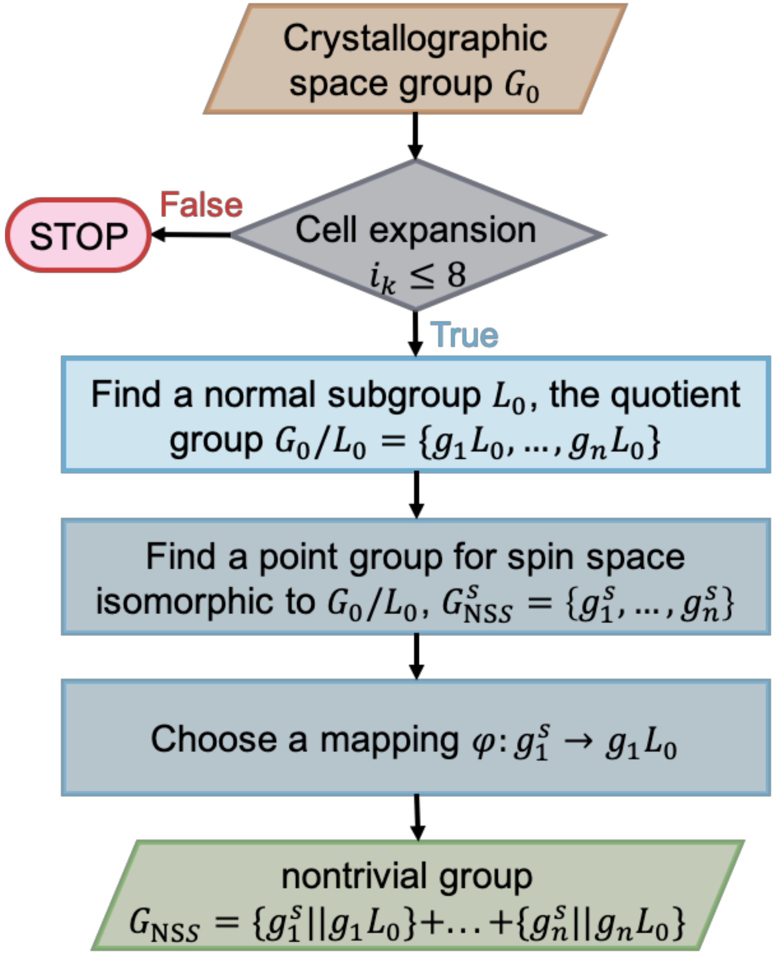
Notation: (99.107.4.1)

$P^14^1m^1m^{4^1_{001}(1/2\ 1/2\ 1/4)^m}1$

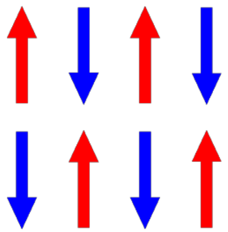
64 operations

MSG: P_c4_1 (76.10) 8 operations

Enumeration of spin space groups

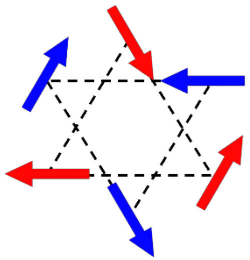


Crystal system	t-type	k-type	g-type	Total
Triclinic (2)	5	25	30	60
Monoclinic (13)	148	666	3005	3819
Orthorhombic (59)	2685	3987	54457	61129
Tetragonal (68)	3752	1456	28311	33519
Trigonal (25)	270	277	2090	2637
Hexagonal (27)	1128	243	6720	8091
Cubic (36)	517	84	538	1139
Total (230)	8505	6738	95151	110394



Collinear (1421)

$G^l_{SO} = Z^K_2 \ltimes SO(2)$

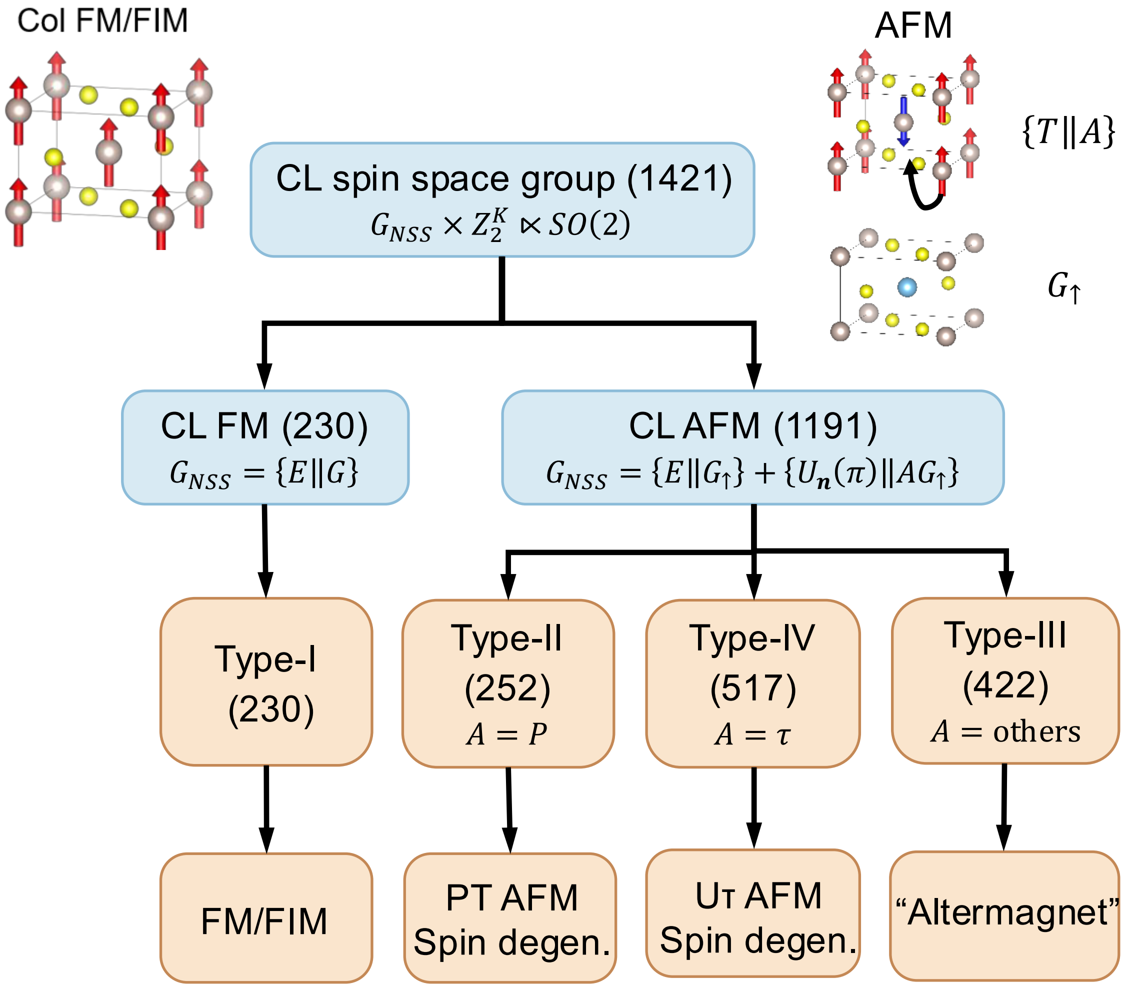


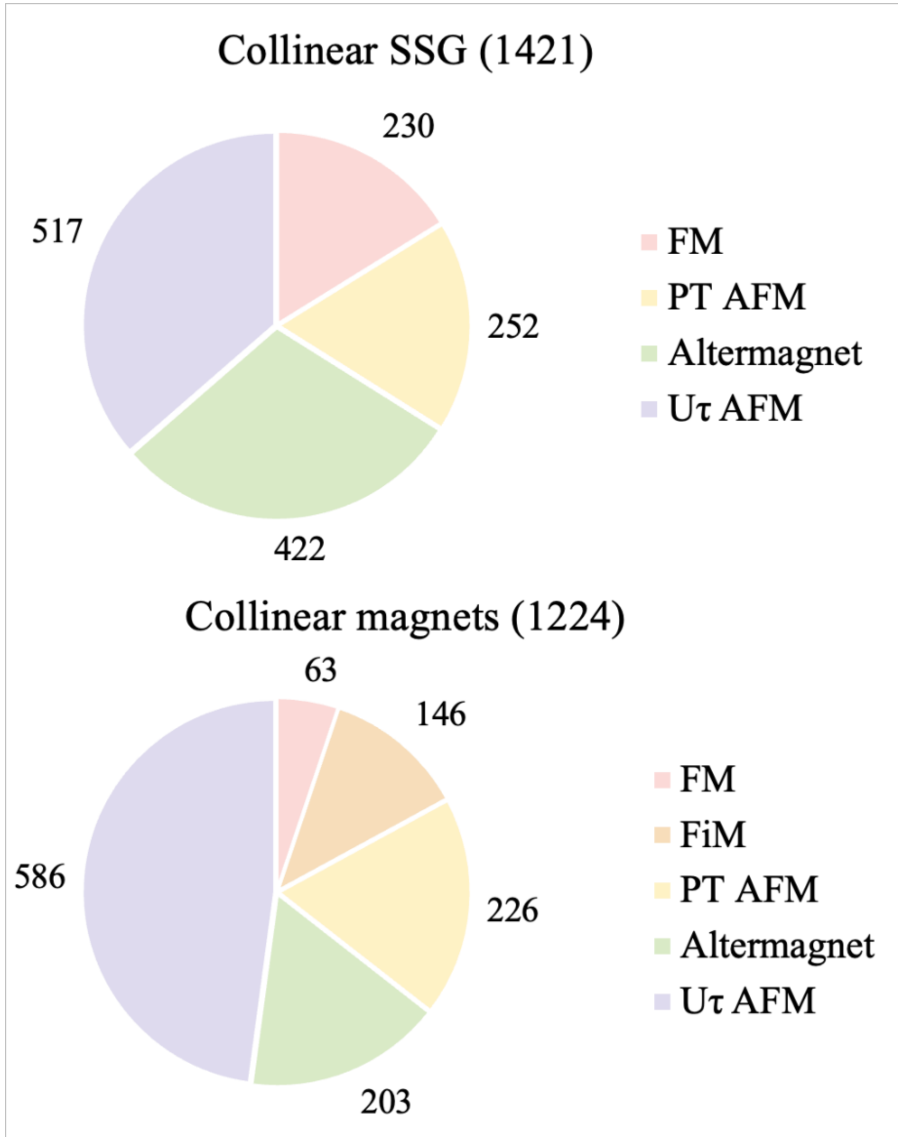
Coplanar (18250)

$G^p_{SO} = Z^K_2$

X.Chen,…, **Q. Liu***, ***PRX*** 14, 031038 (2024)

Enumeration of collinear spin space groups





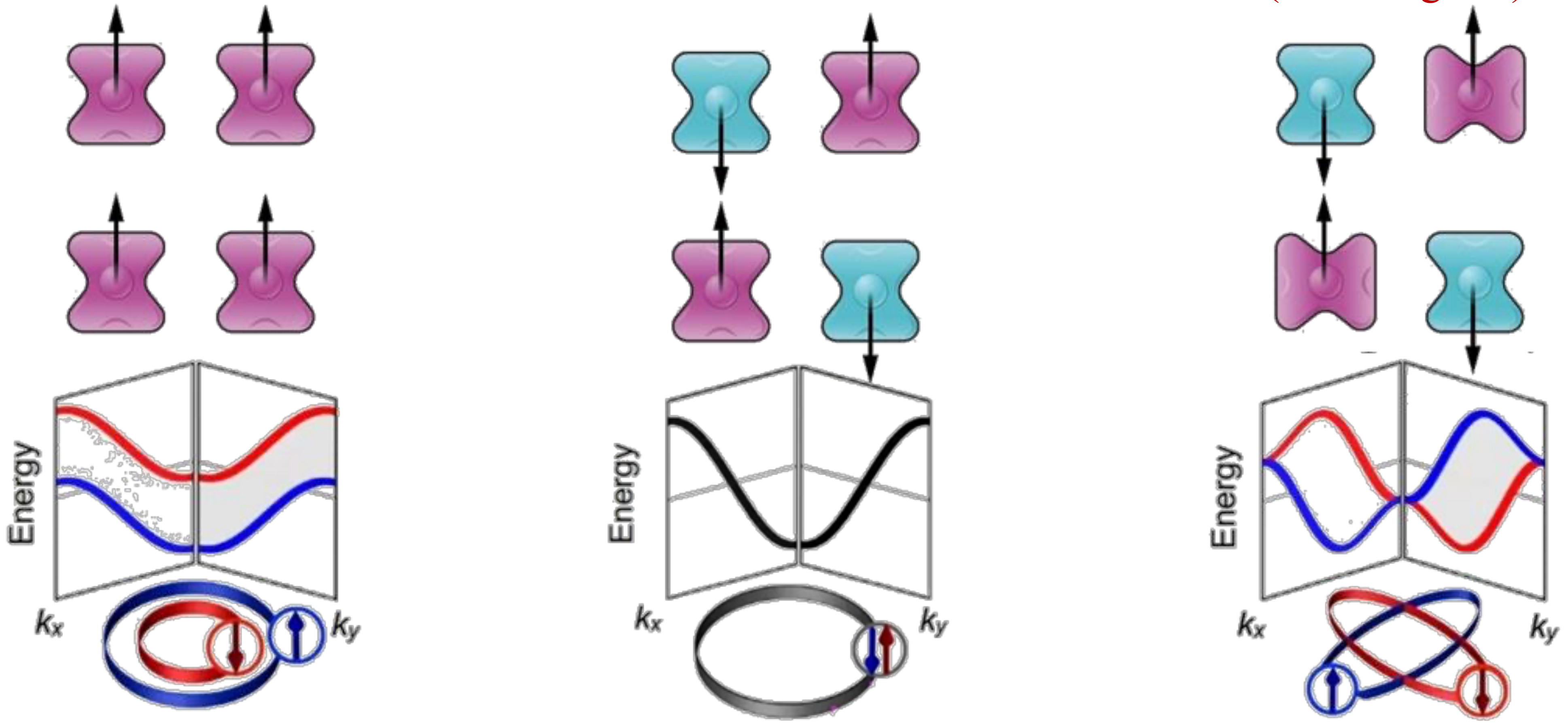
X.Chen,…, **Q. Liu***, ***Nature*** 640, 349 (2025)

Altermagnetism: Collinear spin-split AFM

FM

Spin-degenerate collinear AFM

Spin-split collinear AFM (Altermagnets)



Early works: JSPJ 88, 123702 (2019); PRB 102, 014422 (2020); Sci. Adv. 6, eaaz8809 (2020) …

Theory: Šmekjal et al. PRX 12, 031042 (2022); PRX 12, 040501 (2022)…

Experiment: Krempaský et al. Nature 626, 517 (2024); Lee et al. PRL 132, 036702 (2024)…

Band representations for noncollinear SSGs

Space groups

little group: $G_k = \{g \in G_k | gk \equiv k\}$

if G is a symmorphic SG or
G_k is a symmorphic group or
k lies inside the first BZ → point group

otherwise → central extension

Magnetic space groups

Dimmock-Wheeler sum rule:
(anti-unitary elements)

$$\sum_{\beta \in AG} \chi(\beta^2) = \begin{cases} g & Type\ (a) & no \\ -g & Type\ (b) & double \\ 0 & Type\ (c) & double \end{cases}$$

Regular representation contains all the inequivalent irreps of G

C. J. Bradley, A. P. Cracknell. The Mathematica theory of symmetry in solids (1972)

Band representations for noncollinear SSGs

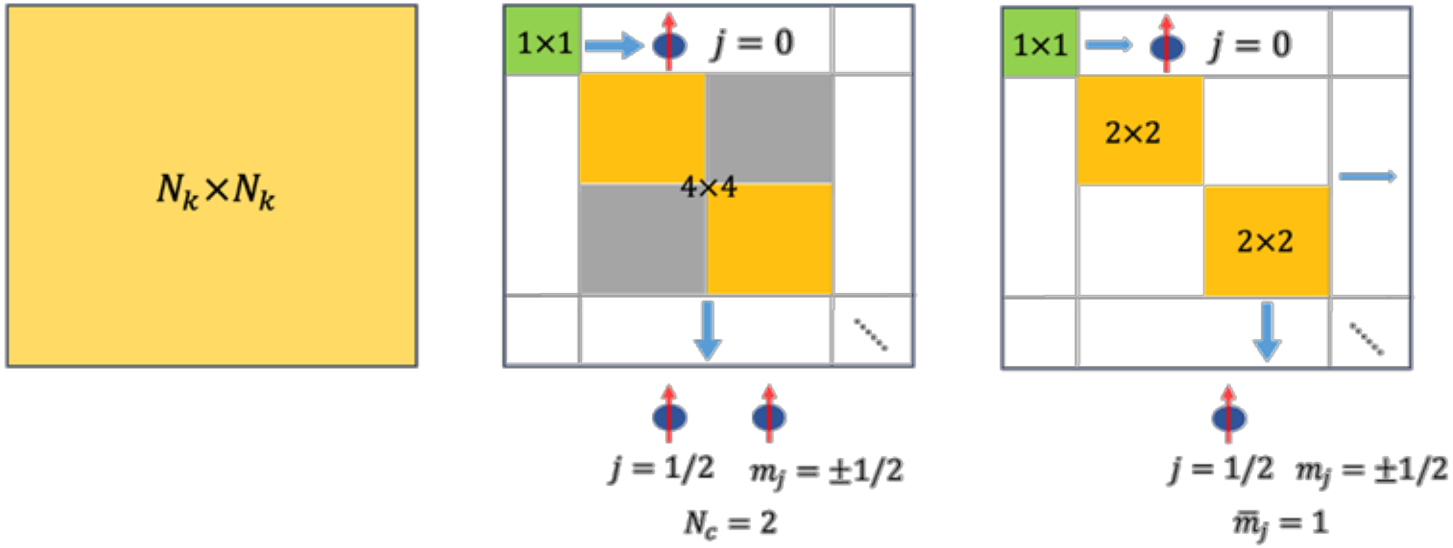
complete set of commuting operators (CSCO)

CSCO: $\{\hat{J}^2, \hat{J}_z\}$ quantum state: $|j, m_j\rangle$ $SO(3) \supset SO(2)$

$$\begin{pmatrix} \hat{J}^2 \\ \hat{J}_z \end{pmatrix} |j, m_j\rangle = \begin{pmatrix} j(j+1) \\ m_j \end{pmatrix} |j, m_j\rangle$$

Step 1: CSCO-I $C \rightarrow$ irrep space
Step 2: CSCO-II $(C, C(s)) \rightarrow$ label irrep space
Step 3: CSCO-III $(C, C(s), \bar{C}(s)) \rightarrow$ irreducible basis

C: class operator of G
$C(s)$: canonical subgroup chain of G
$\bar{C}(s)$: canonical subgroup chain of intrinsic group $\bar{G}$

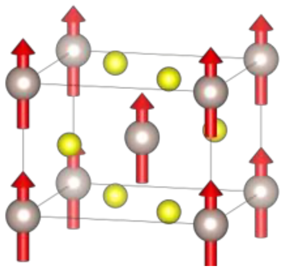


X.Chen,…, **Q. Liu***, ***PRX*** 14, 031038 (2024)

J. Q. Chen, et al., Rev. Mod. Phys. 57, 211 (1985)

Band representation for collinear SSGs

Col FM/FIM

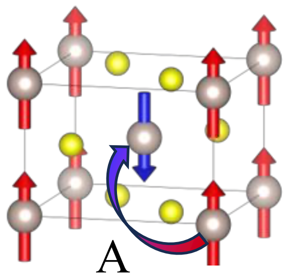


$$G_{SSG}^{FM} = \{E||G\} \times G_{SO}$$

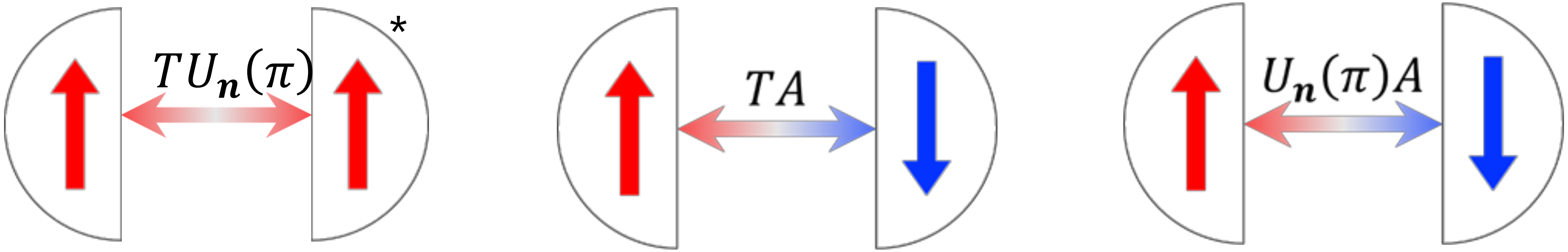
$$G_{SSG}^{AFM} = (\{E||G_{\uparrow}\} + \{U_{\boldsymbol{n}}(\pi)||AG_{\uparrow}\}) \times G_{SO}$$

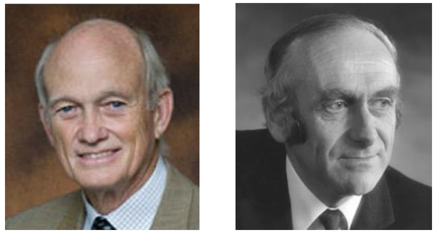
$$G_{SO} = Z_2^K \ltimes SO(2),\ Z_2^K = \{\{E||E\}, \{TU_{\boldsymbol{n}}(\pi)||E\}\}$$

Col AFM



New symmetry: $SO(2), \{TU_{\boldsymbol{n}}(\pi)||E\}, \{U_{\boldsymbol{n}}(\pi)||A\}, \{T||A\}$



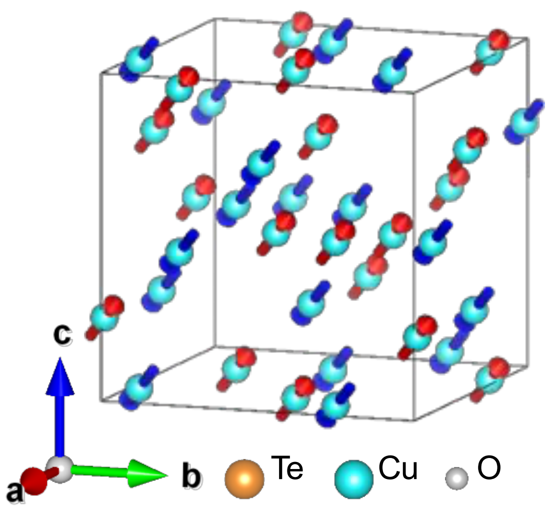


Multiple-fold degeneracies of spin wave spectrum require an enhanced symmetry group called “spin group”,

W. F. Brinkman and R. J. Elliott, *J. Appl. Phys.* **37**, 1457 (1966); *Proc. R. Soc. A* **294**, 343 (1966).

Collinear PT AFM: Cu_3TeO_6

Cubic + Collinear

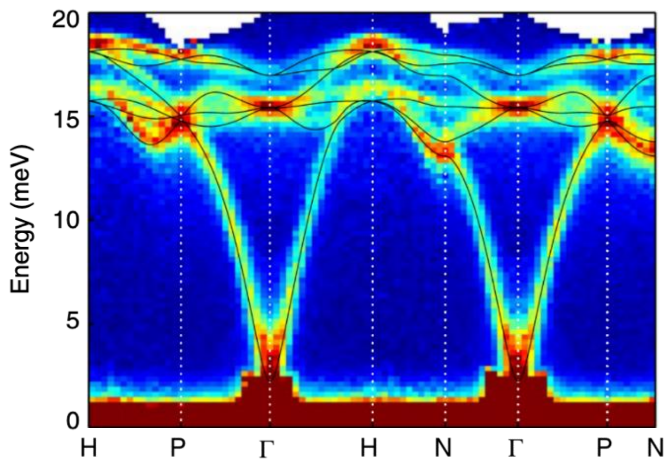


Space group :$Ia\bar{3}$ (Cubic)

Magnetic space group: $R\bar{3}'$ (Rhombohedral)

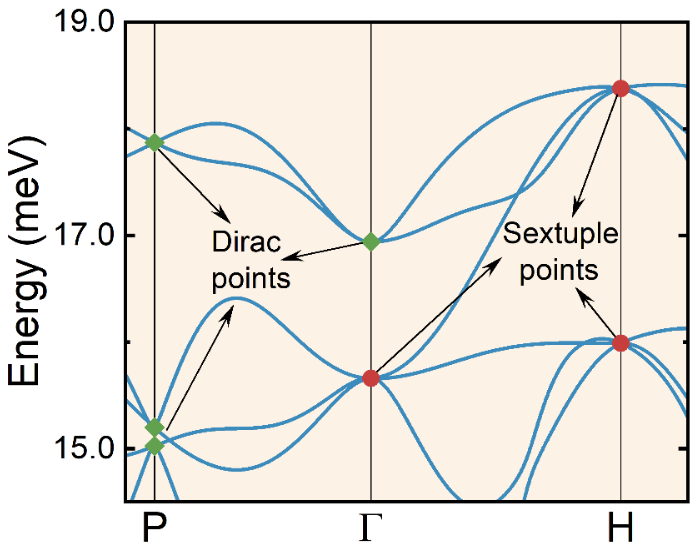
Spin space group: $I^{\bar{1}}a^{\bar{1}}\bar{3}^{\infty m}1$ (Cubic)

- ✓ Six-fold degenerate points at Γ、H
- ✓ Dirac magnons at Γ、P
- ✓ Doubly-degenerate at arbitrary *k*



Yao, et al. Nat. Phys. 14, 1011 (2018)
Bao, et al. Nat. Commun. 9, 2591 (2018)

	Magnetic space group $R\bar{3}'$	Spin space group $I^{\bar{1}}a^{\bar{1}}\bar{3}^{\infty m}1$
Band rep	**2A ↑ G (12)**	**A^S ↑ G (12)**
Γ (0, 0, 0)	$4\Gamma_1(1) + 4\Gamma_2\Gamma_3(2)$	$\Gamma_1^S(2) + \Gamma_2^S\Gamma_3^S(4) + \Gamma_4^S(6)$
H (0, 1, 0)	$4H_1(1) + 4H_2H_3(2)$	$2H_4^S(6)$
N (1/2, 1/2, 0)	$12N_1(1)$	$3N_1^S(2) + 3N_2^S(2)$
P (1/2, 1/2, 1/2)	$4P_1(1) + 4P_2P_3(2)$	$P_1^S(4) + P_2^S(4) + P_3^S(4)$
GP (u, v, w)	$12GP_1(1)$	$6GP_1^S(2)$



X. Chen …, **QL*** et al. *Nature* 640, 349 (2025)

Lecture I: Spin group theory

- Introduction
  - Magnetic groups and spin groups
- Theories of spin groups
  - Group extension
  - Enumeration of spin groups
  - Representation of spin groups
- Classification of magnetic orders
  - Dichotomy of ferromagnets/antiferromagnets
  - Unifying spin group and magnetic group
  - FINDSPINGROUP

Applications of spin group theory

1) Classification of magnetic orders

   Nature 652, 869 (2026)

2) Unconventional magnetism

   Overview: Nat. Phys. 21, 329 (2025); arXiv:2603.27505 (2026)

   Spin splitting: Nature 626, 523 (2024)

   Topological magnons: Nature 640, 349 (2025)

   Quantum geometry: Nat. Commun. 16, 4882 (2025)

   Multiferroicity: PRL 134, 106802 (2025); PRL 135, 056801 (2025)

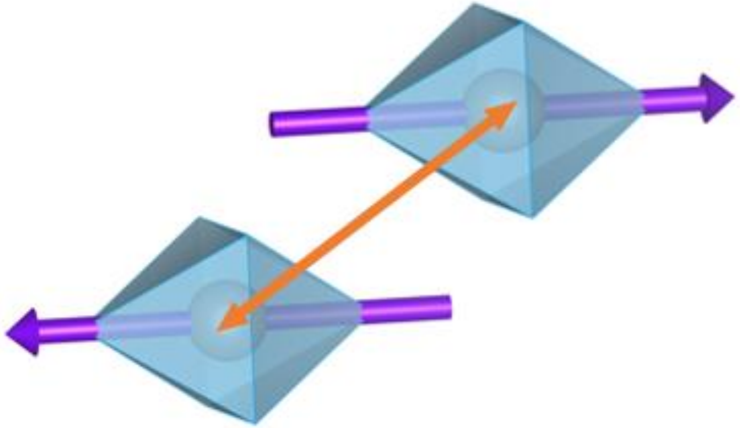
3) Prediction of magnetic configurations

   arXiv:2512.21672 (2025)

Classification of magnetism is based on geometry

"In AFM, two (or more) magnetic sublattices are **crystallographically equivalent**".

L. Néel, Ann. Phys. (Paris, Fr.) 3, 137 (1948)



$M = 0$ enforced by symmetry !

Separate spin and lattice rotations

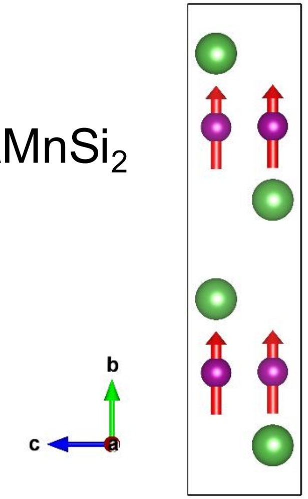
Spin group symmetry

completely describe magnetic geometry

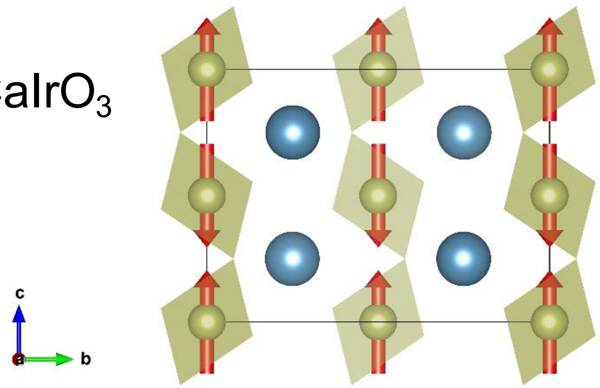
Rigorous classification of FM/AFM based on geometry

MSG: $Cm'cm'$

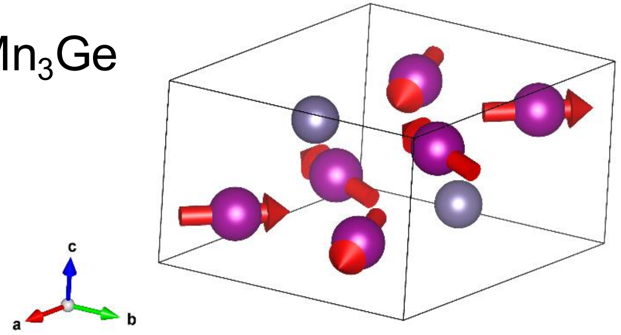
LaMnSi$_2$



CaIrO$_3$



Mn$_3$Ge



Collinear FM

SSG: $C^{1}m^{1}c^{1}m^{\infty m}1$

Collinear AFM

SSG: $C^{1}m^{\bar{1}}c^{\bar{1}}m^{\infty m}1$

Coplanar AFM

SSG: $P^{3^{1}_{001}}6_3/^{1}m^{2_{100}}m^{2_{\pi/3}}c^{m}1$

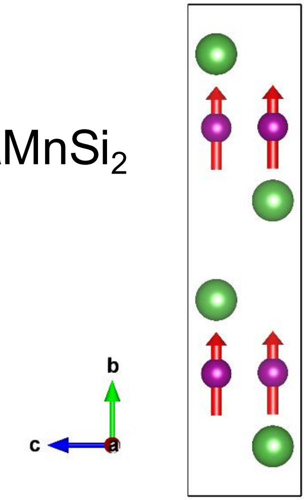
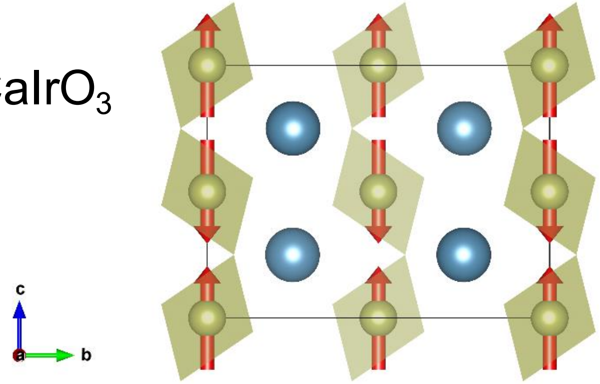
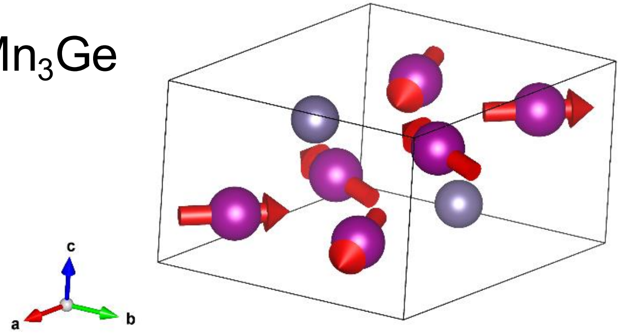
Y. Liu, X. Chen …, **QL*** et al. ***Nature*** 652, 869 (2026)

Rigorous classification of FM/AFM based on geometry

Criteria: The polarity of spin-space point groups P_{spin}: $\{g_s\}$

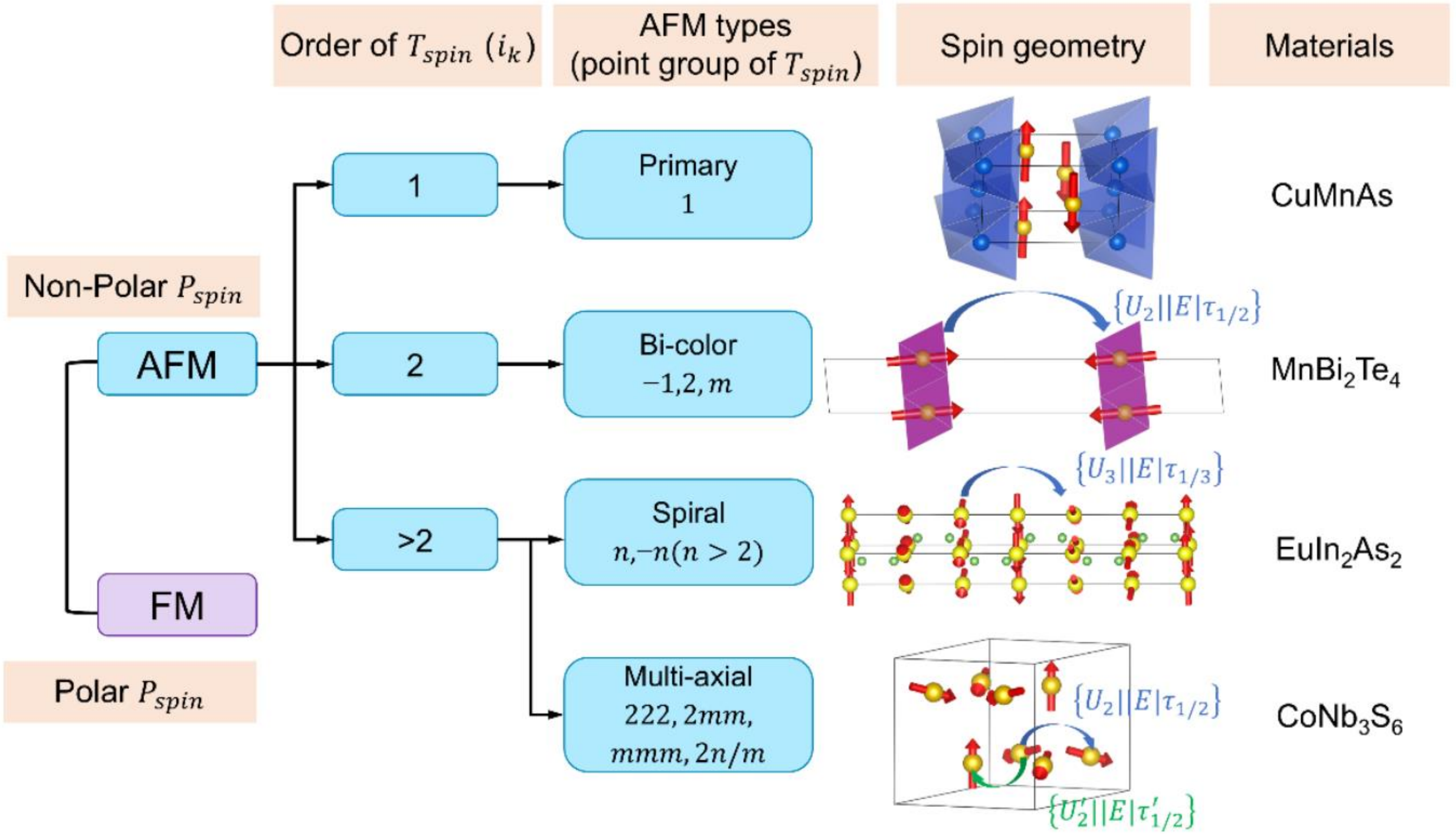
FM: SSG allows nonzero M_s $\longrightarrow$ polar P_{spin}

AFM: SSG enforces zero M_s $\longrightarrow$ non-polar P_{spin}

	$LaMnSi_2$	$CaIrO_3$	Mn_3Ge
			
SSG:	$C^{1}m^{1}c^{1}m^{\infty m}1$	$C^{1}m^{\bar{1}}c^{\bar{1}}m^{\infty m}1$	$P^{3^{1}_{001}}6_3/^{1}m^{2_{100}}m^{2\pi/3}c^{m}1$
P_{spin}:	∞m (Polar)	∞/mm (non-polar)	$\bar{6}2m$ (nonpolar)
	FM	AFM	AFM

Finer classification of AFM orders

Spin translation group T_{spin} $\{g_s \,||\, E|\, \tau\}$: Describing propagations



Magnetic classification in MAGNDATA

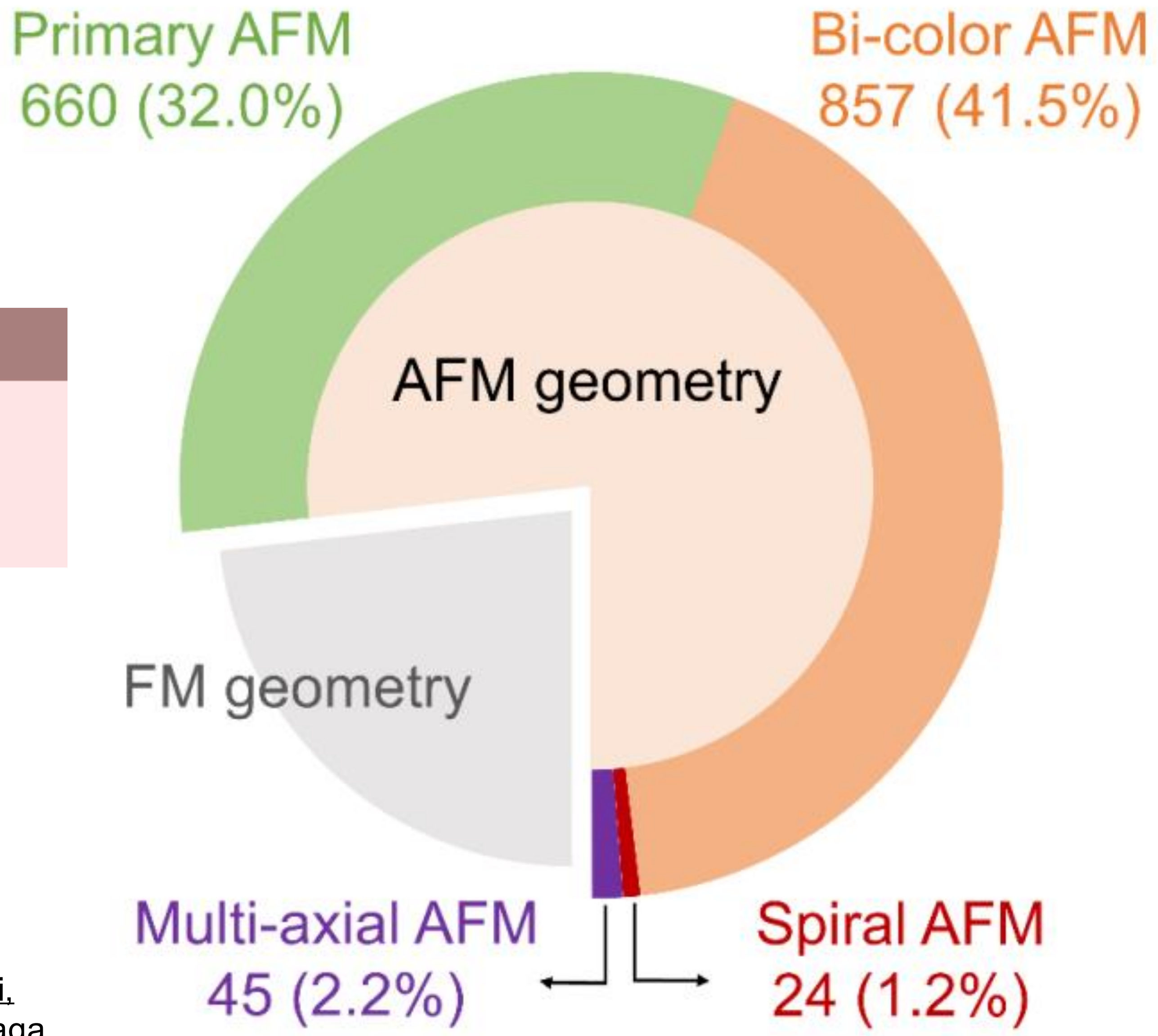
UPV EHU

https://cryst.ehu.es/magndata/

bilbao crystallographic server

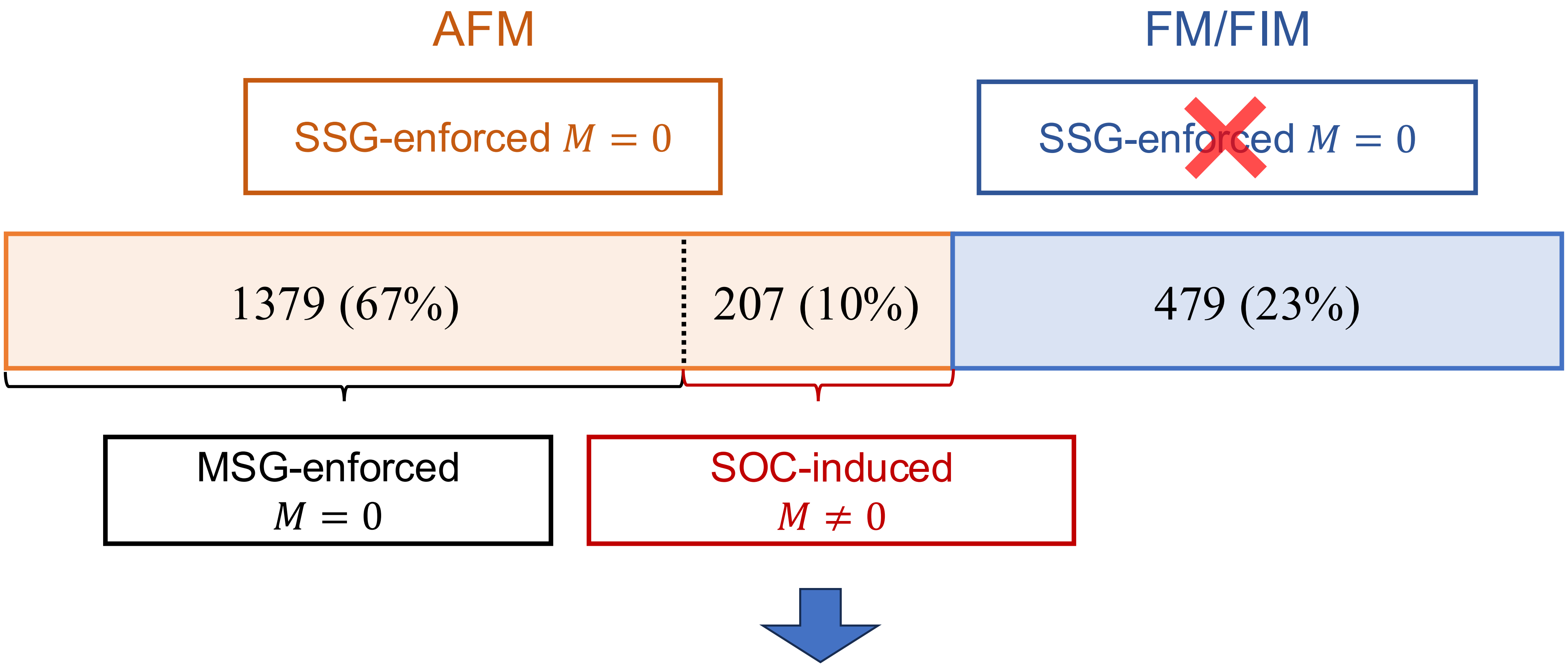
Structure Databases	
B-IncStrDB	The Bilbao Incommensurate Crystal Structures Database
MAGNDATA	A collection of magnetic structures with portable cif-type files
Topological Materials	The Topological Materials Database

2000+ magnetic structures with neutron scattering measurements



Samuel V. Gallego, J. Manuel Perez-Mato, Luis Elcoro, Emre S. Tasci, Robert M. Hanson, Koichi Momma, Mois I. Aroyo and Gotzon Madariaga
J. Appl. Cryst. (2016). **49**, 1750-1776

Role of magnetic group: Spin-orbit magnetism

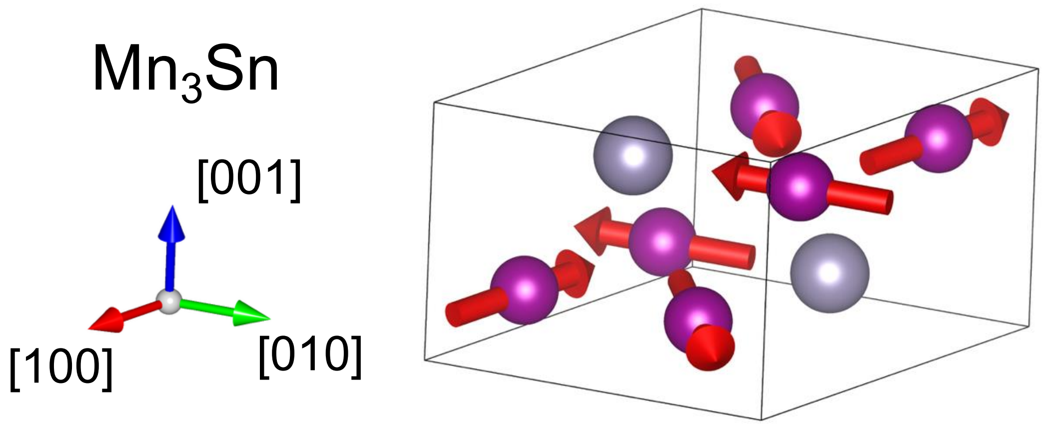


Spin-orbit magnetism: canting AFM, weak FM, anomalous Hall, Nernst, magneto-optical Kerr effects, etc.

Y. Liu, X. Chen …, **QL*** et al. *Nature* 652, 869 (2026)

How to unify SSG (exchange) and MSG (SOC)?

Mn_3Sn

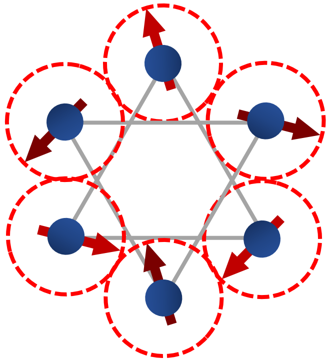


Spin group describes geometry

Magnetic group describes SOC

SSG: $P^{3^1}6_3/{}^1m^2m^2c^m1$

Isotropic exchange

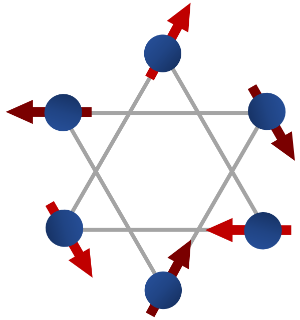


SOC

Symmetry breaking path?

MSG: $Cmc'm'$

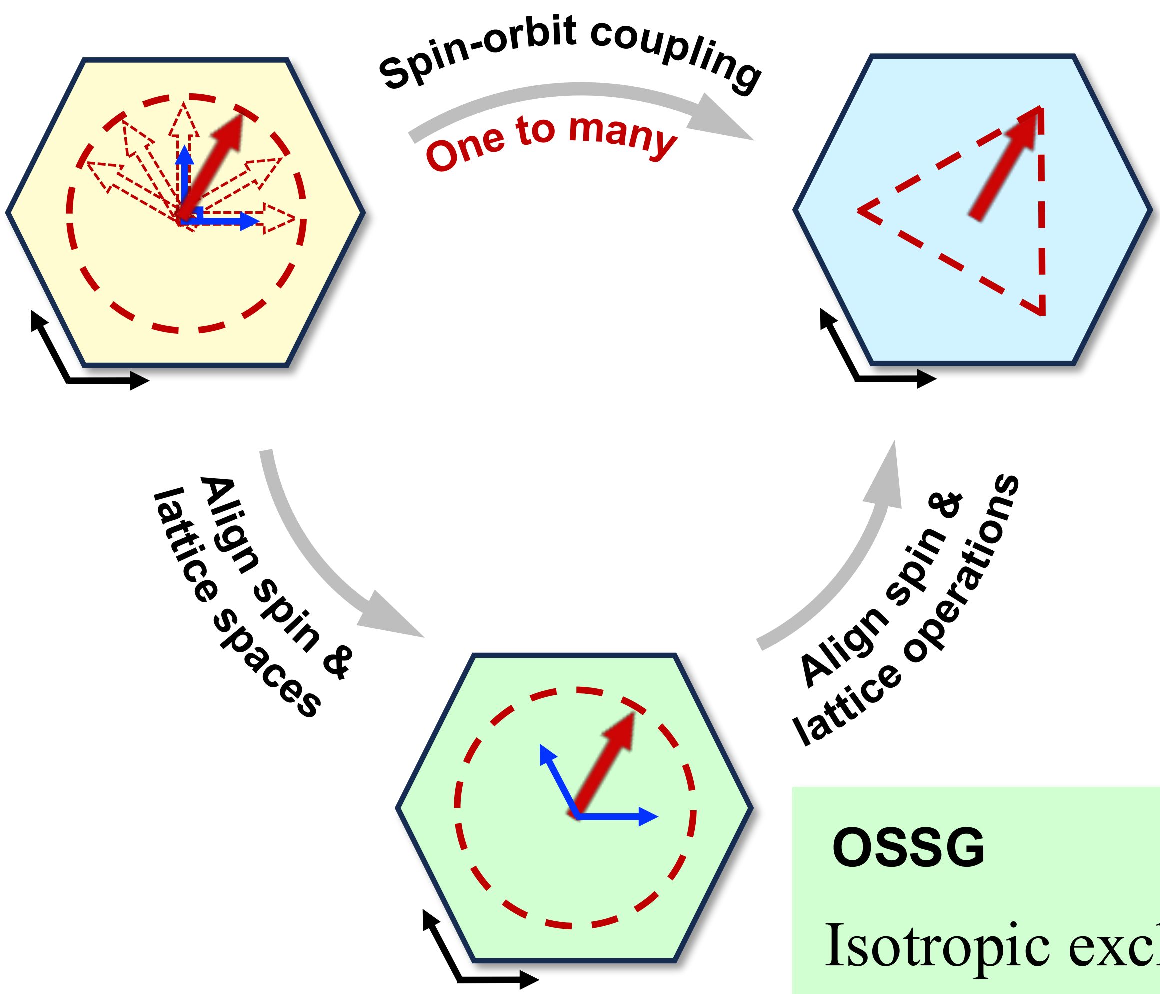
Anisotropic SOC



The information of spin orientation!

SSG + MSG = Oriented spin space group (OSSG)

SSG
Isotropic exchange

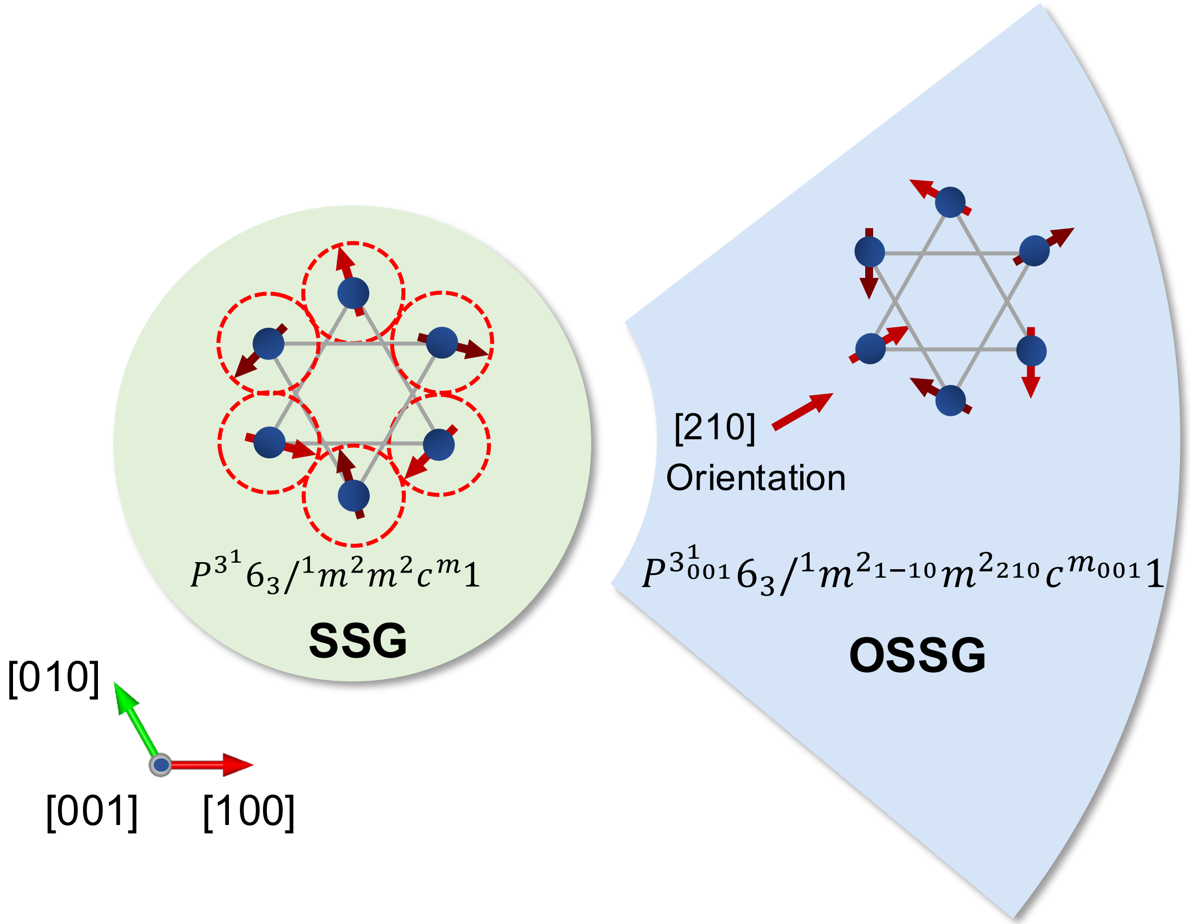


MSG
Anisotropic SOC

OSSG
Isotropic exchange +
Spin orientation in real space

OSSG completes the underlying symmetry logic of magnetism

Correspondence between OSSG and MSG for Mn_3Sn



OSSG generators

$\{-6^5_{001}||6^1_{001}\}$, $\{m_{001}||m_{001}\}$,

$\{m_{001}||1\}$, $\{m_{210}||m_{210}\}$, $\{1||-1\}$

MSG generators

MSG $Cm'cm'$

Spin-orbit magnetism

Y. Liu, X. Chen …, **QL*** et al. ***Nature*** 652, 869 (2026)

More applications of spin space groups?

➢ Lecture II: Unconventional magnetism

- Introduction: conventional magnets and spintronics.
- Unconventional magnetism: spin-split AFM and anomalous Hall AFM.
- Other facets: topological magnons, quantum geometry, multiferroicity, etc.
- Prediction of new magnetic structures

Thank you for your attention !