Analog Quantum Simulation with Ultracold Atoms

Quantum Connections School 2026 · student handout (companion to the lectures)

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> **One number, three faces.** A single energy scale, the super-exchange $J = 4t^2/U$, runs through the whole course: it is the antiferromagnetic spin coupling that orders the Fermi-Hubbard antiferromagnet, the string tension that confines a magnetic polaron, and the clock rate of a two-qubit gate. The logical chain that builds it is
>
> $$\text{real potential} \;\to\; a \;\to\; U \;\to\; \{t, U\} \;\to\; J.$$
>
> Read this handout once for the chain, once for the four boxed results, and once for the numbers.

Analog quantum simulation in one minute

The goal of analog quantum simulation is to *engineer* a many-body Hamiltonian that we can write down but not solve, realise it in hardware with full control over its parameters, and then measure observables that are hard to compute. Ultracold atoms are an almost ideal platform: the microscopic Hamiltonian is known exactly, every coupling (t, U, filling, dimensionality, geometry) is tunable in real time, the energy scales are slow enough ($\sim$ kHz) to resolve in time, and a quantum-gas microscope reads out individual atoms on individual sites in a single shot. The four topics below are the rungs of one ladder: how interactions arise (§1), how to *see* a quantity the detector cannot directly measure (§2), what a single dopant does to an ordered spin background (§3), and how the same interaction becomes a quantum gate (§4).

Symbols. a scattering length; $g = 4\pi\hbar^2 a/m$ contact coupling; t tunneling (hopping); U on-site interaction; $J = 4t^2/U$ super-exchange; $w(\mathbf{r})$ lowest-band Wannier orbital; $\hat{n}$ number operator; $\hat{\mathbf{S}}$ spin operator; ℓ^* polaron size; E_r lattice recoil energy.

1. Collisional interactions: from a potential to a single number

The true interatomic potential $V(R)$ is complicated: a deep molecular (Born-Oppenheimer) well supporting many bound states, joined to a van der Waals tail $V(R) \to -C_6/R^6$ with range $r_0 \sim (mC_6/\hbar^2)^{1/4}$ (a few nm for alkalis). The remarkable fact of the ultracold regime is that none of this detail survives. Two *separate* inequalities hold,

$$r_0 \ll \lambda_{\mathrm{dB}} \;\; (\text{cold}), \qquad r_0 \ll n^{-1/3} \;\; (\text{dilute}),$$

where λ_{dB} is the thermal de Broglie wavelength (a few hundred nm at $\sim 100\,\mathrm{nK}$) and $n^{-1/3}$ the mean interparticle spacing. The first makes the collision *unable to resolve the shape* of $V(R)$: higher partial waves $\ell > 0$ are blocked by the centrifugal barrier $\sim \ell(\ell+1)\hbar^2/mR^2$, so only the s-wave ($\ell = 0$) survives. The second makes collisions binary. Together they collapse the whole potential to one length, the s-wave scattering length a, and we may replace V by the contact pseudopotential

$$\boxed{V_{\mathrm{pp}}(\mathbf{r}) = \frac{4\pi\hbar^2 a}{m}\,\delta(\mathbf{r}) \equiv g\,\delta(\mathbf{r})}$$

The factor 4π (rather than 2π) reflects the two-body reduced mass $m/2$. The sign of g follows the sign of a: $a > 0$ is repulsive, $a < 0$ attractive. The depth of the real V never enters. The *same* g is the mean-field nonlinearity in the Gross-Pitaevskii equation, the on-site U below, and, through U, the exchange J: this is why a is the master knob of the field, not a scattering detail.

Statistics gate the collision. The total two-body wavefunction is symmetric for bosons and antisymmetric for fermions. The s-wave spatial part is symmetric, so two *identical* (same-spin) fermions cannot s-wave scatter and are effectively non-interacting at low temperature. Fermions therefore need two spin components ($\uparrow\downarrow$) to interact at all, which is exactly what makes the two-component Fermi-Hubbard model the natural fermionic platform.

Tuning a. The scattering length is not fixed by nature. Near a Feshbach resonance a closed-channel molecular state is tuned through the collision threshold with a magnetic field B, and

$$\boxed{a(B) = a_{\mathrm{bg}}\left(1 + \frac{\Delta}{B - B_0}\right)}$$

so a can be swept from large positive, through zero, to large negative, and even across the unitary point $|a| \to \infty$.

On a lattice or in a tweezer. Load the atoms into the lowest band of an optical lattice (or a tightly focused tweezer). Two atoms sharing one site occupy the lowest-band Wannier orbital $w(\mathbf{r})$ and feel $g\,\delta(\mathbf{r})$, giving the on-site interaction

$$\boxed{U = g\int |w(\mathbf{r})|^4\, d^3r = \frac{4\pi\hbar^2 a}{m}\int |w(\mathbf{r})|^4\, d^3r}$$

A deeper lattice narrows w, raises $\int |w|^4$, and so raises U while *lowering* the tunneling t (the overlap of neighbouring Wannier orbitals). Crucially t and U are independently tunable: the lattice depth sets t, while depth together with $a(B)$ sets U. For typical depths of a few to $\sim 20\,E_r$, both t/h and U/h fall in the $\sim 100\,\mathrm{Hz}$ to few-kHz range, slow enough to resolve dynamics in real time.

These two numbers define the Hubbard models,

$$\hat{H}_{\mathrm{BH}} = -t\sum_{\langle i,j\rangle} \hat{a}_i^\dagger \hat{a}_j + \frac{U}{2}\sum_i \hat{n}_i(\hat{n}_i - 1), \qquad \hat{H}_{\mathrm{FH}} = -t\sum_{\langle i,j\rangle,\sigma} \hat{c}_{i\sigma}^\dagger \hat{c}_{j\sigma} + U\sum_i \hat{n}_{i\uparrow}\hat{n}_{i\downarrow}.$$

The opposite signs are physical: $-t$ is a kinetic amplitude that *lowers* energy by delocalising, while $+U$ is an energy *penalty* for double occupancy. At half filling with $U \gg t$, double occupancy is forbidden, charge fluctuations freeze, and only the spins move:

$$\boxed{\hat{H} = J\sum_{\langle i,j\rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j, \qquad J = \frac{4t^2}{U}} \qquad \text{(derived in §4).}$$

The Hubbard model has thus reduced, in this corner of its phase diagram, to the Heisenberg quantum antiferromagnet, the starting point for both §3 and §4.

2. Orbital currents and how to measure them

A current is a phase coherence between sites, whereas a microscope counts atoms. How, then, do we ever measure one? The trick is general and reused throughout the course: rotate the operator you want onto the one axis the detector can read.

Consider one particle in a double well, modes $|L\rangle, |R\rangle$, mapped to an orbital pseudospin with $|L\rangle, |R\rangle$ at the poles of a Bloch sphere:

$$\hat{\sigma}_z = \hat{n}_L - \hat{n}_R \text{ (density)}, \quad \hat{\sigma}_x = \hat{a}_L^\dagger \hat{a}_R + \hat{a}_R^\dagger \hat{a}_L \text{ (kinetic } \hat{T}\text{)}, \quad \hat{\sigma}_y = i(\hat{a}_L^\dagger \hat{a}_R - \hat{a}_R^\dagger \hat{a}_L) \text{ (current)}.$$

The three axes are density, kinetic energy, and current. Note that the *real* part of the bond coherence is the kinetic energy and the *imaginary* part is the current: the factor i is exactly what makes the current operator Hermitian, and it is why a static density snapshot can never reveal a flow. From the continuity equation with $\hat{H} = -J\hat{\sigma}_x$ (here J is the bare tunneling),

$$\boxed{\hat{\jmath}_{L\to R} = \dot{\hat{n}}_R = \frac{i}{\hbar}[\hat{H}, \hat{n}_R] = \frac{iJ}{\hbar}(\hat{a}_R^\dagger \hat{a}_L - \hat{a}_L^\dagger \hat{a}_R)}$$

which sits on the $\hat{\sigma}_y$ axis. A quantum-gas microscope reads only $\hat{\sigma}_z$ (atoms per site), so the current is invisible directly.

The Ramsey trick. Let the double well tunnel freely: $\hat{H} = -J\hat{\sigma}_x$ makes the pseudospin precess about the x-axis at rate $2J/\hbar$. Solving the Heisenberg equation of motion, a balanced initial state ($\hat{n}_R(0) = \hat{n}_L(0)$) evolves as

$$\hat{n}_R(t) - \hat{n}_L(t) = i(\hat{a}_R^\dagger \hat{a}_L - \hat{a}_L^\dagger \hat{a}_R)\sin\frac{2Jt}{\hbar},$$

i.e. the initial current is rotated onto the density axis. Choosing the quarter-period pulse $\tilde{t} = h/8J$ gives $2J\tilde{t}/\hbar = \pi/2$, $\sin = 1$, and

$$\boxed{\hat{n}_R(\tilde{t}) - \hat{n}_L(\tilde{t}) = i(\hat{a}_R^\dagger \hat{a}_L - \hat{a}_L^\dagger \hat{a}_R) = \hat{\jmath}_{L\to R}/(J/\hbar)}$$

The current has become a density imbalance the microscope can read, site by site. The very same idea with an additional tilt pulse (a $Z_{\pi/2}$ followed by $X_{\pi/2}$) rotates the kinetic energy $\hat{T}$ ($\hat{\sigma}_x$) onto the density axis, so the local kinetic energy is measurable too. Because the read-out is single-shot and site-resolved, one obtains not just averages but full current-current correlations $\langle \hat{\jmath}\hat{\jmath}\rangle$, the real payoff for diagnosing topological and loop-current states. None of this is "watching atoms flow": it is operator rotation followed by a density read-out, the same logic as Ramsey interferometry, spin-echo, and quantum-state tomography.

Notation: in this one-particle double-well section J denotes the bare tunneling, following the lecture; everywhere else $J = 4t^2/U$ is super-exchange.

3. The magnetic polaron

Take the half-filled Fermi-Hubbard antiferromagnet of §1 and remove a single atom. The resulting hole would like to delocalise and lower its kinetic energy, but it cannot move through an ordered spin background without disturbing it. That tension dresses the bare hole into a composite particle, the magnetic polaron, and we can estimate its size with two competing energies.

Confinement (potential energy). On a 2D square lattice with Néel order, removing one spin creates a hole (a holon). Each time the hole hops by one site it drags a spin onto the vacated site and leaves behind a *wrong* (ferromagnetic) bond; after ℓ steps it trails a string of $\sim \ell$ frustrated bonds, each costing of order J. The string energy is therefore *linear* in length,

$$V(\ell) \approx \alpha J \ell \qquad (\text{string tension } \sim J).$$

A linear potential confines: pull the holon and the spinon apart and the cost grows without bound, so they stay bound. This is the same mechanism as quark confinement, and the bound object is aptly called a "parton".

Kinetic energy. Confining the hole to a region of ℓ sites gives it a momentum spread $\sim \hbar/(\ell a)$. With the band mass $m^* \sim \hbar^2/2ta^2$ (read off from the tight-binding dispersion $\varepsilon_k = -2t\cos ka$, whose curvature at the band bottom is $2ta^2/\hbar^2$),

$$E_{\text{kin}}(\ell) \sim \frac{\hbar^2}{2m^*(\ell a)^2} \sim \frac{t}{\ell^2}.$$

Balance. Minimising $E(\ell) \approx t/\ell^2 + \alpha J \ell$ ($dE/d\ell = -2t/\ell^3 + \alpha J = 0$) gives

$$\boxed{\ell^* \sim a\,(t/J)^{1/3}, \qquad E^* \sim t^{1/3} J^{2/3}}$$

The exponent, not the prefactor, is the lesson: the size grows only as the *cube root* of t/J, so to double the polaron one must change t/J by a factor of eight. Plugging in cold-atom numbers ($U/t \approx 8$ to 10, so $t/J = U/4t \approx 2$ to 2.5) gives $\ell^* \approx 1.3$ sites: a *small* polaron, one to two sites across, in agreement with the measured image. Experimentally the object is reconstructed from the connected three-point correlator

$$\langle \hat{n}^h_{\mathbf{r}}\, \hat{S}^z_{\mathbf{r}+\mathbf{i}}\, \hat{S}^z_{\mathbf{r}+\mathbf{j}} \rangle^c,$$

which correlates the hole position with the surrounding spin texture (Koepsell 2019). A sharper treatment quantises the hole in the linear potential: the Schrödinger equation $-(\hbar^2/2m^*)\psi'' + (\alpha J/a)\,x\,\psi = E\psi$ is the Airy equation, with discrete string levels $E_n \sim t^{1/3}(\alpha J)^{2/3}\xi_n$ (the ξ_n are zeros of the Airy function) representing internal spinon-holon excitations; the same length ℓ^* sets the size.

2D versus 1D. This confinement is special to two (and higher) dimensions. In 1D the hole can move *without* leaving a trail of broken bonds, so the string costs nothing, spinon and holon separate freely, and one has spin-charge separation rather than a bound polaron. Do not carry the 1D intuition into 2D.

4. Super-exchange and the collisional SWAP gate

Now build J from scratch and turn it into a gate. The minimal unit is two fermions $(\uparrow,\downarrow)$ in a single double well,

$$\hat{H} = -t\sum_{\sigma}(\hat{c}^\dagger_{L\sigma}\hat{c}_{R\sigma} + \text{h.c.}) + U\sum_{i=L,R}\hat{n}_{i\uparrow}\hat{n}_{i\downarrow}.$$

In the $S_z = 0$ sector there are four states: two singly-occupied, combining into a singlet and a triplet,

$$|s\rangle = \tfrac{1}{\sqrt{2}}\big(\,|\uparrow,\downarrow\rangle - |\downarrow,\uparrow\rangle\,\big), \qquad |t_0\rangle = \tfrac{1}{\sqrt{2}}\big(\,|\uparrow,\downarrow\rangle + |\downarrow,\uparrow\rangle\,\big),$$

and two doublons $|{\uparrow\downarrow},0\rangle$, $|0,{\uparrow\downarrow}\rangle$ at energy U. Tunneling connects singly-occupied to doublon states, but by antisymmetry *only the singlet couples*: making a doublon out of a triplet would put two identical fermions on one site, which is Pauli-forbidden, so the triplet is dark. Second-order perturbation theory for $t \ll U$ (matrix element $\langle d|\hat{H}_t|s\rangle = \sqrt{2}\,t$, summed over the two virtual doublons with energy denominator $-U$) gives

$$E_s^{(2)} = \sum_{\text{doublons}} \frac{|\langle d|\hat{H}_t|s\rangle|^2}{0-U} = -\frac{4t^2}{U}, \qquad E_{t_0}^{(2)} = 0 \;\Longrightarrow\; \boxed{\hat{H}_{\text{eff}} = J\big(\hat{\mathbf{S}}_L\cdot\hat{\mathbf{S}}_R - \tfrac{1}{4}\big), \quad J = \frac{4t^2}{U}}$$

The factor of four is worth tracing: the matrix element is $\sqrt{2}\,t$ (the two singlet terms add), squared gives $2t^2$, and the two doublon paths double it to $4t^2$, over U. The singlet is pulled below the triplet by J, so $J > 0$ is antiferromagnetic; this is the Heisenberg bond of §1, now derived. (The $-\frac{1}{4}$ keeps the singlet at exactly $-J$ and the triplet at 0; on a lattice it is a constant shift and is dropped.)

Spin exchange. Writing $|\uparrow,\downarrow\rangle = \frac{1}{\sqrt{2}}(|s\rangle + |t_0\rangle)$, the singlet and triplet acquire a relative phase at the rate set by their gap J, so the two spins swap back and forth,

$$P_{\downarrow\uparrow}(\tau) = \sin^2\!\Big(\frac{J\tau}{2\hbar}\Big), \qquad |\uparrow,\downarrow\rangle \;\leftrightarrow\; |\downarrow,\uparrow\rangle \;\text{ at }\; J/\hbar.$$

This has been measured directly: $J/h = 3.32(3)$ kHz with a single-spin-exchange (π-pulse) fidelity above 99.9%.

The gate. This exchange *is* a two-qubit gate. On the logical states $\{|\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$, the evolution $e^{-i\hat{H}_{\text{eff}}\tau/\hbar}$ with $\alpha = J\tau/\hbar$ gives

$$\boxed{R_{\text{SWAP}}(\alpha) = \begin{pmatrix} \dfrac{1+e^{i\alpha}}{2} & \dfrac{1-e^{i\alpha}}{2} \\ \dfrac{1-e^{i\alpha}}{2} & \dfrac{1+e^{i\alpha}}{2} \end{pmatrix}}$$

$\alpha = \pi$ gives **SWAP** (full exchange); $\alpha = \pi/2$ gives $\sqrt{\text{SWAP}}$, a maximally entangling, universal two-qubit primitive that turns a product state into a Bell state.

Killing the gate error. The only error is leakage into doublons, and there are two strategies. (1) Work at $U/t \gg 1$: the dynamics is slow and the residual doublon admixture is only $\sim (t/U)^2$. (2) Use the *magic ratio* $U/t = 4/\sqrt{3}$. Beyond perturbation theory the singlet couples to one doublon combination $|D\rangle$ with strength $2t$, forming a two-level system with level splitting $\sqrt{U^2 + 16t^2}$; the doublon population returns exactly to zero after one full Rabi cycle. Demanding that this return coincide with the $\sqrt{\text{SWAP}}$ phase $\alpha = \pi/2$ fixes $U/t = 4/\sqrt{3}$, giving a gate that is simultaneously fast and (in principle) leakage-free. Experiments reach $\sqrt{\text{SWAP}}$ fidelities of 99.8%; encoding the qubit in a nuclear spin (99% nuclear character, hence weak magnetic-field sensitivity) pushes singlet-triplet coherence beyond 10 s.

Key results at a glance

Quantity	Expression
Contact coupling	$g = 4\pi\hbar^2 a/m$
Pseudopotential	$V_{\text{pp}}(\mathbf{r}) = g\,\delta(\mathbf{r})$
Feshbach tuning	$a(B) = a_{\text{bg}}(1 + \Delta/(B - B_0))$
On-site interaction	$U = g\int \lvert w\rvert^4\, d^3r$
Heisenberg limit	$\hat{H} = J\sum_{\langle i,j\rangle} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$
Super-exchange	$J = 4t^2/U$
Current operator	$\hat{j}_{L\to R} = (iJ/\hbar)(\hat{a}_R^\dagger\hat{a}_L - \hat{a}_L^\dagger\hat{a}_R)$
Current read-out pulse	$\tilde{t} = h/8J$ (quarter tunneling period)
Spin-exchange	$P_{\downarrow\uparrow}(\tau) = \sin^2(J\tau/2\hbar)$
Polaron size / energy	$\ell^* \sim a(t/J)^{1/3},\ E^* \sim t^{1/3}J^{2/3}$
Exchange gate	$\alpha = \pi$: SWAP; $\alpha = \pi/2$: $\sqrt{\text{SWAP}}$
Magic ratio	$U/t = 4/\sqrt{3}$

Check your understanding

1. **Statistics.** Explain why two identical spin-polarised fermions cannot interact via s-wave collisions, and why a two-component Fermi gas can. (*Hint: which symmetry does the spatial s-wave part have, and what does that force on the spin part?*)
2. **Lattice knobs.** A lattice is made deeper at fixed magnetic field. State what happens, qualitatively, to the tunneling t, the Wannier width, and the on-site U, and explain why t and U can nevertheless be tuned independently.
3. **The read-out pulse.** Verify that $\tilde{t} = h/8J$ gives a $\pi/2$ rotation, and explain in one sentence why this maps the current onto a measurable density imbalance.
4. **The factor of four.** Reproduce $J = 4t^2/U$ from second-order perturbation theory, accounting for the matrix element $\sqrt{2}\,t$ and the two virtual-doublon paths.
5. **Polaron size.** Using $\ell^* \sim (t/J)^{1/3}$ and $J = 4t^2/U$, write ℓ^* in terms of U/t, and evaluate it for $U/t = 8$. Is the polaron large or small?

6. **The entangling gate.** Which α makes $R_{\mathrm{SWAP}}(\alpha)$ maximally entangling? Apply that gate to $|\uparrow\downarrow\rangle$ and show the output is a (maximally entangled) Bell-type state.

Brief answers. (1) The s-wave spatial part is symmetric, so fermionic antisymmetry forces an antisymmetric (singlet) spin state, impossible for identical spins; two components restore it. (2) Deeper lattice: t decreases, w narrows, U increases; independence comes from also tuning $a(B)$. (3) $2J\tilde{t}/\hbar = 2J(h/8J)/\hbar = \pi/2$; the precession about x rotates $\hat{\sigma}_y$ (current) onto $\hat{\sigma}_z$ (density). (4) $|\sqrt{2}\,t|^2 = 2t^2$, times two doublons, over U, gives $4t^2/U$. (5) $\ell^* \sim (U/4t)^{1/3} = ((U/t)/4)^{1/3}$; for $U/t = 8$, $\ell^* \sim 2^{1/3} \approx 1.26$ sites, small. (6) $\alpha = \pi/2$; $R_{\mathrm{SWAP}}(\pi/2)\,|\uparrow\downarrow\rangle = \frac{1+i}{2}\,|\uparrow\downarrow\rangle + \frac{1-i}{2}\,|\downarrow\uparrow\rangle$, with equal weights and a relative phase i, hence maximally entangled.

References and further reading

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