Unified Semiclassical Framework for Transport Bifurcation Geometry and the Larmor Tunneling Clock

Abstract

We construct a rigorous derivation framework for the Larmor tunneling clock, bridging exact analytic scattering with semiclassical instanton geometry. We strictly enforce a transition to canonically normalized variables to preserve dynamical mass scales. The weak T-odd measurement field is introduced not as a perturbative correction, but as a dynamical bias on the Euclidean trajectory. We subsequently regulate the fluctuation geometry coefficient χ via structural normalization, and benchmark the formalism against the exact hypergeometric solutions of the Pöschl-Teller barrier, expressing the fundamental traversal times exactly via the complex digamma function.

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1 Semiclassical Tunneling Geometry

1.1 Canonical Normalization of the Euclidean Action

We begin with the physical metastable cubic potential representing the tunneling landscape:

$$V(x) = \frac{1}{2}m\omega_0^2 x^2 - \frac{\lambda}{3}x^3 \tag{1}$$

To evaluate tunneling from the local ground state ($E \simeq 0$), we perform a Wick rotation to Euclidean time, $\tau = it$. The Euclidean action is:

$$S_E[x] = \int d\tau \left[\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right] \tag{2}$$

To ensure consistency in subsequent dynamical and fluctuation analyses, we must strictly isolate the mass dependency. We define the canonically normalized coordinate field $\phi_c \equiv \sqrt{m}x$. The interaction strength transforms to a canonically preserved coupling $g \equiv \lambda/m^{3/2}$. The canonical Euclidean action becomes:

$$S_E[\phi_c] = \int d\tau \left[\frac{1}{2}\left(\frac{d\phi_c}{d\tau}\right)^2 + V_c(\phi_c)\right], \quad \text{where} \quad V_c(\phi_c) = \frac{1}{2}\omega_0^2\phi_c^2 - \frac{g}{3}\phi_c^3 \tag{3}$$

1.2 Exact Euclidean Bounce

The canonical Euclidean equation of motion for zero energy is $d\phi_c/d\tau = \sqrt{2V_c(\phi_c)}$. Direct integration yields the closed-form exact canonical bounce:

$$\bar{\phi}_c(\tau) = \frac{3\omega_0^2}{2g}\mathrm{sech}^2\left(\frac{\omega_0\tau}{2}\right) \tag{4}$$

1.3 Fluctuation Operator and Structural Normalization

The second variation of the action is governed by the canonical fluctuation operator $\mathcal{L}_0$:

$$\mathcal{L}_0 = -\frac{d^2}{d\tau^2} + \omega_0^2\left[1 - 3\mathrm{sech}^2\left(\frac{\omega_0\tau}{2}\right)\right] \tag{5}$$

The exact zero mode, arising from translational invariance, is $\psi_1(\tau) = \partial_\tau\bar{\phi}_c(\tau)$. As $|\tau| \to \infty$, $\psi_1(\tau) \sim e^{-\omega_0|\tau|}$. The second linearly independent solution, $\psi_2(\tau)$, obtained via Wronskian construction, diverges asymptotically as $\psi_2(\tau) \sim e^{+\omega_0|\tau|}$.

To ensure globally convergent overlap integration for the fluctuation geometry coefficient χ, we modify the integration approach. We align it with the asymptotic limits of the fundamental solutions to $\mathcal{L}_0$ by defining a structural normalization factor $\mathcal{N} \propto (2\omega_0 AB)^{-1}$, where A and B are the asymptotic amplitudes of ψ_1 and ψ_2. The structurally stabilized coefficient is:

$$\chi = \mathcal{N}\int_{-\infty}^{\infty} d\tau \left[F[\bar{\phi}_c(\tau)]\psi_1(\tau)\tilde{\psi}_2(\tau)\right] \tag{6}$$

where $\tilde{\psi}_2(\tau)$ is the regularized second solution.

2 Operational Tunneling Clocks

2.1 The T-Odd Invariant as a Dynamical Bias

We couple a localized magnetic measurement field strictly within the canonical classically forbidden region, $\phi_c \in \left[0, \frac{3\omega_0^2}{2g}\right]$. As the dynamics become exponentially sensitive, the T-odd invariant acts as a dynamical bias rather than a simple perturbative correction. The biased canonical action is:

$$S_{E,\pm}[\phi_c] = \int d\tau \left[\frac{1}{2}\left(\frac{d\phi_c}{d\tau}\right)^2 + V_c(\phi_c) \mp \frac{\hbar\omega_L}{2}\chi_B(\phi_c)\right] \tag{7}$$

This shifts the local inversion curvature, splitting the bounce into spin-dependent paths $\bar{\phi}_{c,\pm}(\tau)$.

2.2 Larmor Filtering Time and Cutoff Regularization

The operational spin-filtering time τ_z measures the differential decay. By the envelope theorem, it is the integral of the field profile over the unperturbed trajectory:

$$\tau_z \simeq \int_{-\infty}^{\infty} d\tau \, \chi_B(\bar{\phi}_c(\tau)) \tag{8}$$

Because the canonical bounce asymptotically approaches the well bottom ($\tau \to \pm\infty$), this yields a logarithmic divergence. To isolate the region of finite Euclidean velocity, we introduce a canonical spatial cutoff $\epsilon > 0$:

$$\chi_{B,\epsilon}(\phi_c) = \Theta(\phi_c - \epsilon)\Theta\left(\frac{3\omega_0^2}{2g} - \phi_c\right) \tag{9}$$

Mapping this to Euclidean time bounds gives $\tau_\epsilon = \frac{2}{\omega_0}\text{arccosh}\left(\sqrt{\frac{3\omega_0^2}{2g\epsilon}}\right)$. The finite regulated filtering time is precisely the width of this time window:

$$\tau_{z,\epsilon} = \int_{-\tau_\epsilon}^{\tau_\epsilon} d\tau = \frac{4}{\omega_0}\text{arccosh}\left(\sqrt{\frac{3\omega_0^2}{2g\epsilon}}\right) \xrightarrow{\epsilon\to 0} \frac{2}{\omega_0}\ln\left(\frac{6\omega_0^2}{g\epsilon}\right) \tag{10}$$

3 Exact Solvable Benchmark: Pöschl-Teller Barrier

3.1 Hypergeometric Coordinate Transformation

To provide an exact benchmark free of semiclassical artifacts, we solve the Pauli equation for the exact Pöschl-Teller barrier $V(x) = V_0\text{sech}^2(ax)$. The zero-field scalar equation is normalized to:

$$\frac{d^2\psi}{dx^2} + \left[k^2 - \kappa_0^2\text{sech}^2(ax)\right]\psi = 0 \tag{11}$$

We execute the substitution $z = \frac{1-\tanh(ax)}{2}$. Peeling off the asymptotic phase factors with $\alpha = -i\frac{k}{2a}$, we define $\psi(z) = z^\alpha(1-z)^\alpha w(z)$. The system precisely reduces to the standard hypergeometric form:

$$z(1-z)w'' + (2\alpha+1)(1-2z)w' - \left(4\alpha^2 + 2\alpha + \frac{\kappa_0^2}{a^2}\right)w = 0 \tag{12}$$

3.2 Exact Transmission Amplitude

Applying Kummer's connection formula to match the $z = 0$ (transmission) and $z = 1$ (incidence) basis limits exactly yields the transmission amplitude $t(E)$. Defining the geometric parameter $\mu = \frac{1}{2}\sqrt{\frac{4\kappa_0^2}{a^2} - 1}$, the amplitude is:

$$t(E) = \frac{\Gamma\left(\frac{1}{2} - i\frac{k}{a} + i\mu\right)\Gamma\left(\frac{1}{2} - i\frac{k}{a} - i\mu\right)}{\Gamma\left(1 - i\frac{k}{a}\right)\Gamma\left(-i\frac{k}{a}\right)} \tag{13}$$

3.3 Analytic Larmor Observables via Digamma Operators

We combine the Larmor definitions $\tau_y = \hbar\partial_E \arg t$ and $\tau_z = \hbar\partial_E \ln|t|$ into a complex operator with respect to the dimensionless momentum $q = k/a$:

$$\tau_z + i\tau_y = \frac{m}{\hbar ka}\frac{\partial \ln t(q)}{\partial q} \tag{14}$$

Operating on the Gamma functions analytically reduces the derivative strictly to the complex digamma function, $\psi_0(z) \equiv \frac{d}{dz}\ln\Gamma(z)$. Defining the complex digamma difference operator $\Delta\psi_0(q,\mu)$:

$$\Delta\psi_0(q,\mu) \equiv \psi_0\left(\frac{1}{2} - iq + i\mu\right) + \psi_0\left(\frac{1}{2} - iq - i\mu\right) - \psi_0(1-iq) - \psi_0(-iq) \tag{15}$$

The final exact Larmor observables separate natively into bounded, analytic forms:

$$\tau_z = \frac{m}{\hbar ka}\Im\left[\Delta\psi_0\left(\frac{k}{a},\mu\right)\right] \tag{16}$$

$$\tau_y = -\frac{m}{\hbar ka}\Re\left[\Delta\psi_0\left(\frac{k}{a},\mu\right)\right] \tag{17}$$

These exact expressions seamlessly isolate resonance structures via the poles of $\psi_0(z)$ and cap the logarithmic divergence seen in the semiclassical limit.