Transition Flaps

Rules for an Instanton Flap in a Multichannel Problem

The purpose of the "instanton flap" construction is to represent a localized complex-time or Euclidean segment of a semiclassical history that carries an amplitude from one effective propagation branch to another. It is not merely a decorative connector between curves in a branch diagram; it is the stationary complex saddle that realizes the branch transfer.

1. Definition of the Flap

Let the real-time WKB histories propagate on two effective branches,

$$\mathcal{B}_1, \qquad \mathcal{B}_2.$$

These branches may represent adiabatic or diabatic levels, flavor eigenstates, band branches, meson channels, or other effective semiclassical sheets.

The instanton flap is a complexified trajectory

$$X^\mu(\ell) \in \mathcal{C}$$

which connects the two real-time branches over an imaginary-time or complex-time interval

$$\tau \in [-\tau_*, \tau_*].$$

Schematically,

$$\text{real-time branch } \mathcal{B}_1 \longrightarrow \text{Euclidean/complex flap} \longrightarrow \text{real-time branch } \mathcal{B}_2.$$

For a Landau–Zener problem, $\mathcal{B}_1$ and $\mathcal{B}_2$ may be the adiabatic or diabatic sheets. For an MSW problem, they may be the matter-dependent propagation eigenstates. For a world-line decay problem, they may be distinct classical world-line channels.

2. Equations on the Real-Time Branches

On each real-time branch, the motion is ordinary WKB or classical motion:

$$\dot{X}^\mu = \frac{\partial H_i}{\partial p_\mu}, \qquad \dot{p}_\mu = -\frac{\partial H_i}{\partial X^\mu},$$

where i labels the branch or channel.

In a one-coordinate channel problem this reduces to

$$p_i(q) = \partial_q S_i(q), \qquad H_i(q, p_i) = E.$$

For a simple two-level problem, one may think of the effective branch Hamiltonians as

$$H_\pm(q,p) = \frac{p^2}{2M} + E_\pm(q),$$

where $E_\pm(q)$ are the adiabatic or diabatic eigenvalue branches.

3. Equations Inside the Flap

Inside the flap, time is rotated or complexified,

$$t \to t + i\tau,$$

or, more generally, the trajectory is continued onto a complex contour.

The same Hamilton equations are solved on this complex contour:

$$\frac{dX^\mu}{d\ell} = \frac{\partial H}{\partial p_\mu}, \qquad \frac{dp_\mu}{d\ell} = -\frac{\partial H}{\partial X^\mu},$$

but now X^μ and p_μ may be complex.

For a simple nonrelativistic one-coordinate model, the Euclidean equation is

$$M\frac{d^2q}{d\tau^2} = +\frac{\partial V_{\text{eff}}}{\partial q},$$

rather than the Lorentzian equation

$$M\frac{d^2q}{dt^2} = -\frac{\partial V_{\text{eff}}}{\partial q}.$$

In a multichannel problem, the flap should not necessarily be interpreted as motion through a forbidden region in ordinary space. More generally, it is a complex trajectory through the branch structure.

4. Junction Rules

Let J_1 and J_2 denote the endpoints where the flap attaches to real-time WKB histories. At each junction one imposes matching and stationarity conditions.

First, the configuration must be continuous:

$$\boxed{X^\mu_{\text{real}}(J) = X^\mu_{\text{flap}}(J).}$$

Second, the canonical momentum must match, with the understanding that the flap momentum is analytically continued:

$$\boxed{p^\mu_{\text{real}}(J) = p^\mu_{\text{flap}}(J).}$$

Third, the full action must be stationary with respect to variations of the junction location:

$$\boxed{\frac{\delta S_{\text{total}}}{\delta J} = 0.}$$

Equivalently, if

$$S_{\text{total}} = S^{(1)}_{\text{real}} + S_{\text{flap}} + S^{(2)}_{\text{real}},$$

then J_1 and J_2 are determined by

$$\delta_{J_1,J_2} S_{\text{total}} = 0.$$

This is the multichannel analogue of force balance at a world-line junction. The flap is therefore not inserted by hand at an arbitrary point; its attachment points are fixed by stationary action.

5. Branch-Changing Rule

The flap is allowed only where the complexified branch geometry permits analytic continuation from one sheet to another. In a two-level problem this usually means that the complex trajectory passes near a complex degeneracy, turning point, or avoided-crossing point:

$$E_+(z_*) = E_-(z_*),$$

where z is a complexified control coordinate.

Thus the rule is

$$\boxed{\text{the flap connects sheets through a complex branch point or avoided-crossing region.}}$$

It is not an arbitrary tunnel drawn between curves in a real branch diagram.

For the Landau–Zener problem, the relevant saddles are the familiar complex turning points. For the MSW problem, the analogous objects are complex positions or densities at which the matter eigenvalue branches meet.

6. Action of the Flap

The flap contributes the semiclassical transition factor

$$\mathcal{A}_{1\to 2} \sim A_{\text{prefactor}} \exp\left(-S_{\text{flap}} + i\Phi\right).$$

In Hamiltonian form,

$$S_{\text{flap}} = \int_{\text{flap}} \left[p_\mu \, dX^\mu - H \, dt\right]_{\text{complex}}.$$

In a Euclidean Lagrangian form,

$$S^E_{\text{flap}} = \int d\tau \left[\frac{M}{2}\left(\frac{dq}{d\tau}\right)^2 + V_{\text{eff}}(q)\right].$$

Thus the flap is the saddle contribution controlling the exponential weight of the branch transition.

7. Algorithmic Construction

A practical construction of the flap proceeds as follows:

1. Choose incoming and outgoing branches $\mathcal{B}_1, \mathcal{B}_2$.
2. Complexify time or the relevant control coordinate.
3. Find the complex saddle connecting the two sheets.
4. Impose continuity of X^μ at J_1, J_2.
5. Match canonical momenta at J_1, J_2.
6. Vary J_1, J_2 to enforce stationarity or force balance.
7. Evaluate S_{flap}.
8. Attach the resulting flap amplitude to the real WKB branches.

8. Interpretation for a Two-Panel Figure

In a two-panel schematic, it is useful to separate the branch picture from the history picture.

Panel A should represent the effective multichannel branch diagram. Its connector should be labeled neutrally, for example as a

$$\text{complex branch connection}$$

or

$$\text{transition saddle region.}$$

This panel indicates where the branch structure supports a nonadiabatic or complex-time transition, but it does not itself resolve the instanton history.

Panel B should represent the semiclassical history. There the same transition region is resolved into an actual instanton flap:

$$J_1,\ J_2, \qquad \text{continuity of } X^\mu, \qquad \text{momentum matching}, \qquad \text{stationarity/force balance}, \qquad S_{\text{flap}}.$$

A concise caption would be:

> **(A)** Effective multichannel branch diagram. The transition is supported near a complex branch-connection region between two propagation sheets.
>
> **(B)** Semiclassical resolution of that region. The branch connection is represented by an instanton flap: a complex-time saddle joined to real WKB histories at J_1 and J_2, with continuity, momentum matching, and stationarity conditions determining the junctions.

The most important conceptual rule is:

$$\boxed{\text{The flap is the stationary complex-time history that realizes branch transfer.}}$$