Fractional Angular Momentum on Rings

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Abstract

We consider a circle of ideas centering on fractionalization of angular momentum. We begin by analyzing the elementary but fundamental example of a charged particle on a ring. This affords perhaps the simplest and most transparent example of an "Aharonov-Bohm" type effect. It is also the beginning of a central chapter in mesoscopic physics. We make a careful distinction between kinetic and canonical angular momentum. The former is gauge invariant, generally fractional, and has direct dynamical significance; the latter is gauge dependent, quantized, and significant in the Hamiltonian formalism of dynamics. We revisit the Dirac quantization condition for magnetic monopoles, and connect it to quantization of angular momentum.

1 Aharonov-Bohm Made Simple

Consider a particle with charge q that moves on a ring of unit radius. We are interested in the quantum theory of such a particle. The governing Lagrangian is

$$\mathcal{L} = \frac{m}{2}\dot{\theta}^2 - qA_\theta(\theta, t)\dot{\theta} \tag{1}$$

To set up the quantum theory in the standard way we need to introduce the canonical (Hamiltonian) formulation. The canonical momentum conjugate to θ is

$$p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\dot{\theta} - qA_\theta \tag{2}$$

and the Hamiltonian is

$$\mathcal{H} = p_\theta \dot{\theta} - \mathcal{L} = \frac{(p + qA_\theta)^2}{2m} \tag{3}$$

Note that in forming $\mathcal{H}$ the second term in $\mathcal{L}$, which is linear in $\dot{\theta}$, cancels out. But that second term plays a role in defining the relationship between the coordinate and its associated canonical momentum. If A_θ is time-independent, that second term does not contribute to the Lagrangian equation of motion. Nevertheless, as we shall see, it has physical effects in the quantum theory, even if $A_\theta(\theta, t)$ reduces to a constant. This phenomenon, that vector potentials (= parallel transport) can lead to physical effects even in the absence of explicit field strength (= curvature) is the distilled essence of the Aharonov-Bohm effect.

One might be tempted to argue that a time-independent A_θ, which after all depends on only one variable, can be removed by a gauge transformation, in the form

$$A_\theta \;\rightarrow\; A_\theta - \partial_\theta \Lambda \tag{4}$$

with

$$\Lambda(\theta, t) \;=\; \int_0^\theta du A_\theta(u) \tag{5}$$

This proposal runs into the difficulty that $\Lambda(2\pi)$ may not be equal to $\Lambda(0)$, so the definition is not consistent. We *can* however use the gauge freedom to reduce A_θ to a constant

$$A_\theta(\theta) \;\rightarrow\; \frac{1}{2\pi} \int_0^{2\pi} du A_\theta(u) \tag{6}$$

by using

$$\Lambda(\theta, t) \;=\; \int_0^\theta du A_\theta(u) - \frac{\theta}{2\pi} \int_0^{2\pi} du A_\theta(u) \tag{7}$$

Let us assume that that has been done, so that we are dealing with a constant vector potential, which we'll call simply A. This residual A has a simple physical interpretation. Applying Stokes' law, we see that

$$A \;=\; \frac{\Phi}{2\pi} \tag{8}$$

where Φ is the magnetic flux through any surface bounded by our ring. It is entirely possible to have Φ nonzero while the magnetic field vanishes on the ring itself. For example, if we imagine that the flux is created by a long thin solenoid that threads the ring, we will have that situation: enclosed flux but no magnetic field on the ring (where the particle is).

Having set up the canonical formalism, we go to the quantum theory by considering wave functions $\psi(\theta)$ and operators that act on them. Specifically, we promote

$$\begin{aligned} p_\theta &\rightarrow -i\frac{\partial}{\partial\theta} \\ \mathcal{H} &\rightarrow \frac{(-i\partial_\theta + qA)^2}{2m} \end{aligned} \tag{9}$$

It is easy to see that the regular harmonics

$$\psi_l(\theta) = e^{il\theta} \tag{10}$$

are eigenfunctions of this Hamiltonian, with eigenvalues

$$E_l = \frac{(l + qA)^2}{2m} \tag{11}$$

From one point of view, Eqn. (11) is a remarkable result. The energy spectrum depends quite explicitly on the vector potential A, even though the particle sees no field. On the other hand, it is a very simple result to derive, as we've seen. It is also very easy to understand, physically, if we combine two basic observations. First, we recognize that the energy is nothing but the primitive kinetic energy:

$$E_l = \frac{m}{2}\dot{\theta}^2 \tag{12}$$

where we define $\dot{\theta}$ by inverting Eqn. (2):

$$\dot{\theta} = \frac{p_\theta + qA}{m} \tag{13}$$

We can call $m\dot{\theta}$, which directly reflects the motion of our particle's coordinate θ, the *kinetic* angular momentum (as opposed to the *canonical* angular momentum p_θ). It is the kinetic momentum that appears in the energy. In the quantum theory the kinetic momentum is implemented by a gauge covariant derivative, while the canonical angular momentum is implemented by an ordinary derivative. Second, we recognize that it is the canonical momentum that obeys a simple commutation relation with the coordinate, and is naturally quantized[1].

[1] Since θ is not a globally defined variable, there is some subtlety involved in using it to set up the dynamical formalism. One possibility is to divide the ring into patches, with appropriate conditions on the overlaps, as we have done in other contexts earlier. I will not pursue that – not uninteresting – digression further here now.

Another perspective: Let us suppose that our vector potential is indeed caused by a thin solenoid that threads the ring along its central axis, and consider what happens as flux Φ is turned on, or varied, by altering the current through the solenoid. According to Faraday's law, the changing flux through the ring induces an azimuthal electric field

$$E_\theta \, = \, \frac{1}{2\pi}\dot{\Phi} \tag{14}$$

This exerts a torque, which changes the kinetic angular momentum, and also can do work. We have

$$\Delta(m\dot{\theta}) \, = \, \int dt \frac{1}{2\pi} q\dot{\Phi} \, = \, \Delta \frac{1}{2\pi} q\Phi \, = \, \Delta q A \tag{15}$$

So for fixed p – appropriate here, because a rotationally symmetric perturbation cannot induce transitions between different harmonics – we recover the kinetic momentum that we previously derived using a different argument.

Now the wave function of a charged particle is a gauge-dependent quantity, and so one might question why we insisted that it should be smooth. After all, a non-smooth gauge transformation will take smooth wave functions into wave functions that are not smooth, so we should ask why we choose smooth wave functions in some particular gauge. To answer this question, we should consult the Hamiltonian, which governs the dynamics. The Hamiltonian will give infinities if there are jumps in $(-i\partial_\theta + qA)\psi$. The criterion we adopted, that ψ should be smooth, appropriate in a gauge where A is smooth, . It is sometimes useful to work with so-called *singular* gauges, where we allow jumps in A. We could, for example, have used the gauge transformation of Eqn. (5) to transform A away entirely, except for a discontinuity at $\theta = 2\pi$ versus 0. In that gauge, the wave functions $\psi(\theta)$, or more precisely their logarithms, will need to have a compensating discontinuity. Indeed, the allowed wave functions will be of the form $\psi_\alpha = e^{i\alpha\theta}$ where the allowed values of α track the *kinetic* angular momentum.

The energy levels in Eqn. (11) flow continuously as qA varies, as of course does the spectrum of allowed kinetic angular momentum. But while for each individual eigenvalue l of the canonical angular momentum p_θ the kinetic angular momentum $l+qA$ varies monotonically with qA, the transformation

$$\begin{aligned} \delta(qA) \quad &= \quad k \\ \delta l \quad &= \quad -k \end{aligned} \tag{16}$$

merely relabels the levels if k is an integer. Thus the *entire spectrum* is periodic under changes in the enclosed flux that correspond to $\delta(qA) = k$, or in other words

$$\delta\Phi = \frac{2\pi}{q} \times \text{integer} \tag{17}$$

We have shown the periodicity in flux explicitly for a few quantities in an idealized geometry, but the core argument can be formulated abstractly, and is extremely general. Let us return to Eqn. (5). The ambiguity in $\Lambda(\theta)$, which led us to discard it, was a jump by Φ at $\theta = 2\pi$. But the gauge transformation acts by

$$\psi(\theta) \rightarrow e^{iq\Lambda(\theta)}\psi(\theta) \tag{18}$$

on fields, or wave functions, describing particles of charge q. Thus if all the relevant charges are integer multiples of q, a change $\delta\Lambda = k\theta\frac{2\pi}{q}$, with k integral, has no effect. In the present context, this means that fluxes which are integral multiples of

$$\Phi_q = \frac{2\pi}{q} \tag{19}$$

can be transformed away using superficially, though not genuinely, singular gauge transformations.

Our idealized ring geometry is a recognizable caricature of something one might realize experimentally, and indeed "Aharonov-Bohm" effects of this type have inspired a lot of experimental activity in recent years. There are several complications, including that one generally one has many electrons to worry about (i.e. a fermi sea); that the ring is actually an annulus of finite size; and, especially, that practically attainable magnetic fields are not strictly segregated from the annulus. But the most fundamentally interesting and fruitful complication, perhaps, is that the electrons in the ring scatter off impurities and interact with other electrons. Thus they might *decohere* before making it all the way around, and in that case, we might expect that they will not be sensitive to the total flux.

One can have observable oscillations at period Φ_e in the flux, in quantities like electrical resistance or thermal conductivity at low temperatures for samples of high purity (though the periodicity won't be exact, due to leakage in the field). Turning it around, the magnitude of observed oscillations in those properties, as flux is varied, can be used to probe the properties of material samples. This has become a major technique of mesoscopic physics.

2 Path Integrals Formulation: The Simplest Topological Interaction

It is informative to consider the effect of our term

$$\Delta\mathcal{L} = \frac{q\Phi}{2\pi}\dot{\theta} \tag{20}$$

in the path integral formulation of quantum mechanics. (We are supposing Φ constant.) In that formulation, we compute a transition amplitude, say from (θ_I, t_I) to (θ_F, t_F), by summing over paths, with each path weighted by a factor

$$W_\gamma = \exp i \int_{\gamma\ (\theta_I, t_I)}^{(\theta_F, t_F)} \mathcal{L} dt \tag{21}$$

Now the effect of Eqn. (20) is easy to dope out: it multiplies the contribution of a given path by the factor $\exp iq\Phi\frac{\Delta\theta}{2\pi}$. The value of $\Delta\theta$ is equal to $\theta_F - \theta_I$ plus some multiple k of 2π; k measures how many times the path winds around the circle! So now manifestly:

- This is perhaps the simplest example of a *topological interaction*; it depends only on the topology of the paths, not on the speed with which they are traversed, nor any metric quantity (e.g. overall size).
- Its effects come from interference between paths that wind around the ring different numbers of times, which have different values of k. As a consequence, if the coherence length is smaller than the radius those effects will disappear, and as the coherence length grows features that are periodic as functions of Φ will sharpen up.
- All effects of this added term are periodic in $\delta\Phi = \frac{2\pi}{q}$.

3 Dirac Quantization Revisited

Not coincidentally, the flux quantum we found here is the same as the Dirac flux quantum for magnetic monopoles. The reason is very closely related to Dirac's his original discussion of monopoles and their quantization, where he introduced what we now call the Dirac string. Dirac imagined that a monopole could be realized with a regular vector potential A for its uniform flux together with a very thin tube of cancelling flux – the Dirac string. Then he demanded that the string should be invisible, so that the monopole would act like a proper localized particle, with finite energy, well-defined angular momentum, and so forth (all of which an endless physical string precludes). What we've illustrated here is that the Dirac string is indeed invisible, both by direct calculation in a simple geometry and by an abstract argument very close to Dirac's own.

To draw the sharpest consequence from this connection, let us set things up as symmetrically as possible. We put the monopole at the origin, and imagine drawing the total flux of the monopole out in two equal thin strings, aligned say along the $\hat{z}$ axis. Thus if we count flux in the $+\hat{z}$ direction, and the total monopole flux is Φ_{pole}, then the flux through the north-pointing string is $+\Phi_{\text{pole}}/2$ and the flux through the south-pointing string is $-\Phi_{\text{pole}}/2$. We imagine a charged particle the charge q whose motion is confined a ring threaded by these strings, with the ring at first well below $z = 0$ and gradually moving up. When it is well below $z = 0$ its orbital L_z angular momentum spectrum is displaced by $-q\Phi_{\text{pole}}/4\pi$; when it is well above $z = 0$ that spectrum is displaced by $+q\Phi_{\text{pole}}/4\pi$. But the symmetry of the problem requires that these belong these equivalent (after rotation) situations support the same physical content, so we must have

$$-q\Phi_{\text{pole}}/4\pi \;=\; +\,q\Phi_{\text{pole}}/4\pi + \text{integer} \tag{22}$$

or equivalently

$$q\Phi_{\text{pole}} \;=\; 2\pi \times \text{integer} \tag{23}$$

Thus, as expected, we recover the Dirac quantization condition. As a bonus, we see from this argument that when the condition is satisfied minimally, or by any odd integer, then the orbital angular momentum spectrum is displaced by *half* an odd integer.