KvN Phase Construction

The issue

The quantum harmonic oscillator is a mainstay of physical modeling, notably in quantum optics. One would like to carry over, as much as possible, concepts that help us to understand the behavior of oscillators into this domain. One such concept is that of the phase of oscillations. Here, for the quantum oscillator, a notorious problem arises. In the Fock basis

$$|0\rangle, |1\rangle, |2\rangle, \ldots,$$

the natural candidate for a number-changing lowering operator conjugate to phase is the partial isometry

$$E|n\rangle = |n-1\rangle, \qquad n \geq 1,$$

with

$$E|0\rangle = 0.$$

Though it preserves the norm on it range, E is not unitary. The obstruction is entirely localized at the vacuum, and arises because the oscillator ladder has a lower boundary.

Relatedly, if one defines

$$c = \frac{E + E^\dagger}{2}, \qquad s = \frac{E - E^\dagger}{2i},$$

then the trigonometric identities, including commutativity, fail by vacuum projectors. With the above convention,

$$c^2 + s^2 = I - \frac{1}{2}|0\rangle\langle 0|,$$

and

$$[c, s] = \frac{i}{2}|0\rangle\langle 0|.$$

These highlight the problematic role of the spectral boundary.

KvN doubling

In the minimal Koopman-von Neumann (KvN) description of the classical harmonic oscillator, one has commuting classical variables (Q, P), together with their conjugate generators (p, q):

$$[q, P] = i, \qquad [Q, p] = i,$$

with the other elementary commutators vanishing. The KvN Hamiltonian may be written schematically as

$$H_{\mathrm{KvN}} = pP + qQ.$$

This four-operator structure naturally contains two independent quantum oscillator pairs

$$(\frac{P+p}{\sqrt{2}}, \frac{Q+q}{\sqrt{2}}),$$

and

$$(\frac{P-p}{\sqrt{2}}, \frac{Q-q}{\sqrt{2}}),$$

with opposite energy orientations. Indeed,

$$H_{\mathrm{KvN}} \;=\; \frac{1}{2}\big((\frac{P+p}{\sqrt{2}})^2 + (\frac{Q+q}{\sqrt{2}})^2\big) \;-\; \frac{1}{2}\big((\frac{P-p}{\sqrt{2}})^2 + (\frac{Q-q}{\sqrt{2}})^2\big)$$

and the commutation relations are appropriate. Thus the minimal KvN construction already contains a doubled oscillator structure: one positive-energy sector and one negative-energy "anti-oscillator" sector, both with positive Hilbert-space norm. The negative-energy sector need not be given a negative Hilbert space metric.

Smooth classical angular operators

In the Koopman-von Neumann Hilbert space realization for the classical trigonometic operators this obstruction does not arise. In the minimal KvN representation,

$$P = -i\partial_q, \qquad Q = i\partial_p,$$

so

$$P^2 + Q^2$$

has continuous spectrum

$$[0, \infty).$$

Alternatively, since P and Q are a complete set of commuting observables, we can represent states as wave functions in phase space, whereupon P and Q appear as simple multiplication. The normalized operators

$$\mathcal{C}_{\mathrm{cl}} = \frac{P}{\sqrt{P^2+Q^2}}, \qquad \mathcal{S}_{\mathrm{cl}} = \frac{Q}{\sqrt{P^2+Q^2}}$$

are then just multiplication by $\cos\theta$ and $\sin\theta$ in the joint spectral plane of (P, Q), i.e. phase space. They are undefined only at the origin $P = Q = 0$, which is a measure-zero point and not a normalizable eigenspace. Hence $\mathcal{C}_{\mathrm{cl}}$and $\mathcal{S}_{\mathrm{cl}}$ are well-defined bounded operators, with

$$\mathcal{C}_{\mathrm{cl}}^2 + \mathcal{S}_{\mathrm{cl}}^2 = I.$$

Thus the minimal KvN formalism already contains nonsingular classical (periodic) angular variables. The quantum oscillator phase obstruction is not caused by a catastrophic radial singularity in the enlarged KvN space; rather, it appears when one compresses the doubled oscillator structure down to a single one-sided quantum ladder.

The bilateral phase shift

The doubled structure suggests replacing the one-sided Fock ladder by a two-sided ladder,

$$\ldots, |-2\rangle, |-1\rangle, |0\rangle, |1\rangle, |2\rangle, \ldots.$$

On this enlarged space one may define the bilateral shift

$$U|n\rangle = |n-1\rangle, \qquad n \in \mathbb{Z}.$$

Unlike E, U is unitary:

$$U^\dagger U = UU^\dagger = I.$$

One then defines exact phase cosine and sine operators upstairs:

$$C = \frac{U+U^\dagger}{2}, \qquad S = \frac{U-U^\dagger}{2i}.$$

They are bounded, self-adjoint, and obey

$$C^2 + S^2 = I, \qquad [C,S] = 0.$$

The ordinary oscillator phase operators are recovered by compression. Let

$$\Pi_+ = \sum_{n=0}^{\infty} |n\rangle\langle n|$$

project onto the positive-energy oscillator sector. Then

$$E = \Pi_+ U \Pi_+,$$

and

$$c = \Pi_+ C \Pi_+, \qquad s = \Pi_+ S \Pi_+.$$

Thus E is the compressed shadow of a unitary phase operator on the doubled KvN space.

Interpretation

This is precisely the structure of a unitary dilation. The nonunitary oscillator phase step on the physical Hilbert space becomes unitary after adjoining the minimal missing half-ladder. Abstractly, this is the Sz.-Nagy dilation of the unilateral shift. The notable feature here is that the required dilation space is not an arbitrary auxiliary construction: it is already present in the minimal KvN representation of the classical oscillator.

In this view, the phase problem arises only after one projects away half of the natural KvN structure. The perfect phase algebra exists upstairs; the familiar defects appear downstairs as projection artifacts:

$$c^2 + s^2 = I - \frac{1}{2}|0\rangle\langle 0|,$$

$$[c,s] = \frac{i}{2}|0\rangle\langle 0|.$$

The entire failure of the projected phase algebra is localized at the oscillator vacuum.

Because of their simple algebraic properties, it may be convenient to work with the full (doubled) phase operators during intermediate steps of calculations that involve initial and final states that reside in the positive-energy sector. Projection can be performed as a final step. It only affects states near the bottom of the spectrum, and for reasonably simple expressions it is easy to do.

Summary

The minimal KvN oscillator contains two oscillator sectors with opposite energy orientation. These sectors may be organized into a bilateral ladder supporting a unitary shift operator. Projection onto the positive-energy sector recovers the usual quantum oscillator and its Susskind–Glogower phase operator, including its vacuum-localized defects.

Thus oscillator phase is perfectly regular in the minimal KvN/dilated space; the standard, familiar obstruction arises only upon compressing to a one-sided energy sector.