we arrived at an effective potential of the form - E_1 \cos \alpha \theta - E_2 \cos \beta \theta . now we want to analyze its local minima. when \alpha and \beta have a rational ratio we can get both cosines to be 1 for the same \theta, but not when the ratio is irrational, so we have to be satisfied with getting close ... This problem certainly has the flavor of a Diophantine approximation problem, but is there a well-developed mathematical theory that applies directly?

This is a rich question because the answer is "yes," but not by one single theorem. The right framework mixes almost-periodic functions, Diophantine approximation, and quasicrystal/ergodic ideas.

Thought for 14s >

Yes. The directly relevant theory is **inhomogeneous Diophantine approximation on a torus**, with **Kronecker's theorem** as the qualitative backbone and refinements from **continued fractions**, **geometry of numbers**, and **quasi-periodic/Bohr almost-periodic functions**.

Your potential is

$$V(\theta) = -E_1 \cos(\alpha\theta) - E_2 \cos(\beta\theta).$$

Set

$$x = \alpha\theta, \qquad \rho = \frac{\beta}{\alpha}.$$

Then the problem is to make

$$x \approx 2\pi m, \qquad \rho x \approx 2\pi n,$$

or equivalently

$$\rho m \approx n.$$

So the near-minima are controlled by how well $\rho = \beta/\alpha$ is approximated by rationals n/m. This is exactly Diophantine approximation.

The clean geometric picture is: as θ varies, the point

$$(\alpha\theta, \beta\theta) \mod 2\pi$$

moves on the two-torus T^2. If α/β is rational, the orbit closes. If irrational, the orbit is dense. Thus it comes arbitrarily close to $(0,0)$, where both cosines are maximized, but only after a recurrence time governed by the Diophantine quality of α/β. This is precisely the torus-sampling viewpoint we had identified for the Kronecker regime: offset phases live on a torus, integer moves sample that torus, and observables depend on rational versus irrational recurrence structure. Kronecker_Calculations

The basic theorem is:

$$\boxed{\text{Kronecker approximation theorem.}}$$

For irrational ρ, the sequence

$$m\rho \mod 1, \qquad m \in \mathbb{Z}$$

is dense in $[0,1]$. More quantitatively, one wants integers m, n with

$$|m\rho - n| \ll 1.$$

The best such approximants are supplied by the continued fraction convergents

$$\frac{p_k}{q_k} \approx \rho,$$

which obey

$$\left|\rho - \frac{p_k}{q_k}\right| < \frac{1}{q_k q_{k+1}},$$

or

$$|q_k \rho - p_k| < \frac{1}{q_{k+1}}.$$

These q_k give the natural hierarchy of exceptionally good near-minima.

Near such a point, take

$$\alpha\theta = 2\pi q, \qquad \beta\theta = 2\pi p + \delta,$$

with

$$\delta = 2\pi(q\rho - p).$$

Then

$$V(\theta) \approx -E_1 - E_2 + \frac{E_2}{2}\delta^2$$

if the first cosine is exactly aligned. More generally, if neither phase is exactly aligned, the relevant local cost is the weighted squared distance to the lattice:

$$\Delta V \approx \frac{1}{2}\left[E_1\,\Delta x^2 + E_2\,\Delta y^2\right],$$

where

$$(\Delta x, \Delta y) = (\alpha\theta, \beta\theta) - 2\pi(m, n).$$

So the minima are governed by a **weighted closest-return problem** for a line of irrational slope on a torus.

The especially useful distinction is:

$$\boxed{\text{badly approximable } \rho \quad \text{versus} \quad \text{well/Liouville approximable } \rho.}$$

If ρ is badly approximable, as for the golden ratio, there is a constant $c > 0$ such that

$$|m\rho - n| > \frac{c}{m}$$

for all integers m, n. Then close simultaneous minima exist, but the improvement with search length is limited. Recurrences are relatively "stiff."

If ρ is very well approximable, the denominators q_k can give extraordinarily small mismatches. Then one gets rare, extremely deep near-minima at very long scales. This is where Diophantine quality becomes physically predictive rather than ornamental, exactly as in the earlier Kronecker program: compare badly approximable and well-approximable choices, estimate the smallest phase mismatch up to complexity M, and connect it to splittings, bottlenecks, or recurrence times. Kronecker_Calculations

For your potential, the most direct mathematical formulation is:

$$\boxed{\text{minimize } d_E^2(t, \mathbb{Z}^2) = E_1\,\|t\|^2 + E_2\,\|\rho t\|^2 \quad \text{over } t = \frac{\alpha\theta}{2\pi}.}$$

Here $\|\cdot\|$ means distance to the nearest integer. Thus

$$V(\theta) \approx -E_1 - E_2 + 2\pi^2\left[E_1\|t\|^2 + E_2\|\rho t\|^2\right].$$

This is a weighted Diophantine approximation problem.

There is also a complementary spectral/dynamical systems language. Since $V(\theta)$ is a finite sum of incommensurate cosines, it is a **quasi-periodic** or **Bohr almost-periodic** function. Its local minima form a non-periodic but recurrent hierarchy. Rational approximants $\rho \approx p/q$ give long-period surrogate potentials with period roughly

$$\Delta\theta \sim \frac{2\pi q}{\alpha}.$$

Taking better and better convergents gives a sequence of periodic approximations to the irrational problem. That is often the most practical way to analyze or plot the structure.

So the answer in physical terms is:

$$\boxed{\text{the deep local minima are organized by the continued-fraction convergents of } \alpha/\beta.}$$

For a lecture, I would phrase it this way:

The irrational double-cosine potential does not lose its minima; it loses exact periodic coincidence. The replacements are near-coincidences governed by Diophantine approximation. Rational approximants p/q to β/α identify long intervals in θ over which the two cosines nearly lock. The denominator q sets the recurrence length, while the approximation error $|q\beta/\alpha - p|$ sets the energy penalty above the ideal value $-E_1 - E_2$. Thus different irrational numbers give physically different hierarchies of near-vacua.

This is very much a developed mathematical theory, and it is directly the “Kronecker physics” we wanted: irrationality is not decoration, but a control parameter for recurrence scales, near-degeneracies, and fine spectral structure.