Case Studies for Lecture 4

Case Study I: Circuit Discharge as Engineered Schwinger Physics

1. From Field Discharge to Circuit Discharge

A central example from the previous lecture was the discharge of a spatially uniform electric field in $1+1$ dimensions. A pair of charged particles nucleates, separates, and cancels part of the electric flux between them. In the simplest semiclassical description, the Euclidean saddle is a circular worldline. Cutting the circle at a diameter gives the familiar Lorentzian interpretation: a particle and antiparticle appear at rest, separated by the critical distance

$$L_c = \frac{2m}{qE - q^2/2},$$

after which they accelerate apart, leaving behind a region of diminished electric field.

Superconducting circuits provide a natural laboratory for an analogous problem. The stored field energy is now the energy in circuit variables: capacitor charge, inductor flux, or a biased Josephson potential. Discharge takes place through quantum phase slips or Cooper-pair tunneling events. The circuit analogue of electric flux is a global variable, such as a node flux or winding number. The analogue of pair creation is a quantum transition between neighboring flux or charge sectors.

The conceptual replacement is

$$\boxed{\text{field discharge} \quad \longleftrightarrow \quad \text{relaxation among discrete circuit sectors.}}$$

In a continuum field theory the electric field can be changed locally by the creation and separation of charges. In a circuit, a global flux or phase variable changes by a discrete tunneling event. The circuit therefore turns Schwinger discharge into an engineered problem in global quantum mechanics.

2. The Minimal Washboard Discharge Model

The simplest circuit realization is a current-biased Josephson junction. Its phase difference ϕ behaves like the coordinate of a particle moving in a tilted periodic potential:

$$L_E = \frac{M}{2}\dot{\phi}^2 + U(\phi),$$

with

$$U(\phi) = -E_J \cos\phi - \frac{\hbar I}{2e}\phi.$$

Equivalently, up to an additive constant,

$$U(\phi) = E_J(1-\cos\phi) - E_J i\,\phi, \qquad i \equiv \frac{I}{I_c}.$$

Here

$$I_c = \frac{2e}{\hbar} E_J,$$

and the effective mass is set by the capacitance:

$$M = \left(\frac{\hbar}{2e}\right)^2 C.$$

For $0 < i < 1$, the potential has metastable wells. A phase trapped in one well can escape by thermal activation or by quantum tunneling. In the tunneling regime the escape rate has the semiclassical form

$$\Gamma \sim A e^{-S_B/\hbar},$$

where S_B is the Euclidean bounce action for a trajectory that leaves the metastable minimum and returns to it in imaginary time.

The stationary points satisfy

$$\frac{dU}{d\phi} = E_J(\sin\phi - i) = 0.$$

Thus

$$\phi_{\text{min}} = \arcsin i, \qquad \phi_{\text{top}} = \pi - \arcsin i.$$

The barrier height is

$$\Delta U = 2E_J \left[\sqrt{1 - i^2} - i \cos^{-1} i\right].$$

Near the critical current, $i \to 1^-$, the potential is well approximated by a cubic,

$$U(x) \simeq \frac{1}{2} a x^2 - \frac{1}{3} b x^3,$$

where $x = \phi - \phi_{\text{min}}$. The corresponding bounce is

$$x_B(\tau) = \frac{3a}{2b} \operatorname{sech}^2\left(\frac{\omega_p \tau}{2}\right),$$

with plasma frequency

$$\omega_p = \sqrt{\frac{1}{M} U''(\phi_{\text{min}})} = \sqrt{\frac{E_J}{M} \cos\phi_{\text{min}}} = \sqrt{\frac{E_J}{M}} (1 - i^2)^{1/4}.$$

A standard near-critical estimate gives

$$S_B \simeq \frac{36}{5} \frac{\Delta U}{\hbar \omega_p}.$$

This formula makes the circuit-discharge interpretation vivid: increasing the bias current lowers the barrier, accelerates escape, and eventually converts rare tunneling into frequent switching.

3. Circuit Discharge as Flux-Sector Relaxation

The washboard model is already a discharge model. The phase ϕ is not merely an angle on a circle; in a current-biased junction one must decompactify it:

$$\phi \in \mathbb{R}.$$

Each advance

$$\Delta\phi = 2\pi$$

corresponds to a voltage pulse:

$$V = \frac{\hbar}{2e}\dot{\phi}, \qquad \int dt\, V = \frac{\hbar}{2e}\Delta\phi = \frac{h}{2e}.$$

Thus a phase slip releases a quantized amount of flux:

$$\boxed{\Delta\phi = 2\pi \quad \Longleftrightarrow \quad \Delta\Phi = \Phi_0 = \frac{h}{2e}.}$$

This is the circuit analogue of changing the electric flux in $1+1$-dimensional QED by producing charged matter. The background drive stores energy in a global variable. A tunneling event moves the system to a neighboring sector, reducing the stored energy and producing real-time dynamical products: voltage pulses, quasiparticles, photons, plasmons, or further phase motion.

In a chain or array, the analogy becomes richer. Let Φ_j denote node fluxes and n_j their conjugate Cooper-pair numbers:

$$[\Phi_j, n_k] = i\hbar\delta_{jk}.$$

A generic circuit Hamiltonian has the form

$$H = \frac{1}{2}(\mathbf{n} - \mathbf{n}_g)^T C^{-1}(\mathbf{n} - \mathbf{n}_g) - \sum_{\ell} E_{J\ell}\cos(\Delta_\ell \boldsymbol{\phi} - A_\ell) + U_{\text{drive}}(\boldsymbol{\phi}).$$

The charging term supplies the kinetic metric, the Josephson terms supply the landscape, and the drive supplies the tilt. Discharge is then relaxation through a network of metastable sectors.

4. The Flap Viewpoint

For a single phase-slip event, the bounce-and-cut language is often adequate. One finds a Euclidean bounce, cuts it at a surface where the pseudo-momentum matches real-time data, and continues into Lorentzian evolution.

But sequential discharge is different. After the first event, the system is not back in the original background. It contains real-time products. The circuit has emitted a voltage pulse, redistributed energy among modes, and changed its local or global flux configuration. A second tunneling event occurs in this new dynamical environment.

This motivates a more general picture:

$$\boxed{\text{real-time evolution} + \text{Euclidean flap} + \text{real-time evolution} + \text{Euclidean flap} + \cdots}$$

rather than a collection of isolated bounces.

In this formulation, the Euclidean segment is an insertion into a Lorentzian history. Its endpoints, duration, and shape are determined by a global stationarity problem involving both the

Euclidean flap and the surrounding real-time evolution. This is the natural framework when the products of one discharge event influence later discharge events.

A useful slogan is

Bounces describe tunneling events; flaps describe tunneling events in context.

For circuit discharge, this distinction is not cosmetic. The experimentally measured object is not only the tunneling exponent of one isolated event, but a stochastic time record of switching, relaxation, pulses, and subsequent events. The flap viewpoint keeps the causal Lorentzian environment in the calculation.

5. Sequential Discharge and Cascades

Suppose that the circuit has a ladder of metastable sectors labeled by an integer N. Let the stored energy be approximately

$$E_N = \frac{1}{2L}(\Phi_{\text{ext}} - N\Phi_0)^2.$$

A phase-slip event changes

$$N \to N + 1,$$

and releases

$$\Delta E_N = E_N - E_{N+1}.$$

The semiclassical transition rate has the form

$$\Gamma_N \sim A_N \exp[-S_N/\hbar],$$

where S_N grows as the drive is discharged and the available energy decreases.

Thus a circuit can realize a discrete relaxation cascade:

$$N_0 \to N_0 + 1 \to N_0 + 2 \to \cdots ,$$

with strongly nonuniform waiting times

$$\tau_N \sim \Gamma_N^{-1}.$$

As the system approaches the final low-energy sector, the action increases and the waiting times grow rapidly. This is the circuit analogue of slow or eternal discharge.

In more complex circuits, several different elementary transitions may compete:

$$N \to N + \mathbf{m},$$

where $\mathbf{m}$ is an integer vector describing a collective phase-slip move. The action then depends not only on the released energy, but also on the spatial structure and complexity of the move:

$$\Gamma_{\mathbf{m}} \sim A_{\mathbf{m}} \exp[-S_{\mathbf{m}}/\hbar].$$

This leads naturally to an instanton landscape: a discrete set of possible tunneling moves, each with its own action, phase, and real-time aftermath.

6. Observable Signatures

Circuit discharge is experimentally rich because it can be monitored in real time. Possible observables include:

switching-rate plateaus and crossovers,
temperature-independent quantum escape,
voltage-pulse statistics,
sequential waiting-time distributions,
microwave emission from real-time products,
offset-charge modulation of tunneling amplitudes,
interference between competing phase-slip paths.

The rate itself measures the bounce action. The response to weak probes can measure how long the trajectory lingers in different regions of configuration space. Offset charges and fluxes can modify relative phases between alternative tunneling histories. In this way, the discharge circuit is not merely an analogue simulator of Schwinger physics; it is a platform for instanton tomography.

7. Bottom Line

The discharge problem provides the first case study in engineered instanton physics.

A biased superconducting circuit stores energy in a global variable and releases it through discrete tunneling eve

The analogy to $1+1$-dimensional QED is direct:

QED_{1+1}	superconducting circuit
electric flux	node flux / phase winding
charged-pair creation	phase slip / Cooper-pair tunneling
field-energy release	bias-energy release
Schwinger bounce	phase-slip bounce
sequential discharge	switching cascade
flux sectors	winding / charge sectors

The deepest lesson is methodological. Discharge is not only a Euclidean tunneling problem. It is a mixed real- and imaginary-time history. The rare event is Euclidean, but the products are Lorentzian and causal. For single events, a bounce may suffice. For cascades, the flap philosophy is the natural language.

Circuit discharge turns Schwinger physics into a controllable, observable, and engineerable instanton process.

Case Study II: The Kronecker Clock in Superconducting Circuits

1. The Basic Idea

A conventional clock is a system whose state moves through a sequence of distinguishable configurations. The simplest idealization is a rotor:

$$\phi(t) = \omega t + \phi_0,$$

with the clock hand represented by the angle ϕ. But an angle by itself is periodic. It only gives time modulo

$$T = \frac{2\pi}{\omega}.$$

To recover long times, one normally counts windings:

$$\phi_{\mathrm{lift}}(t) = \phi_0 + \omega t \in \mathbb{R}.$$

In a superconducting circuit this lifted phase is not merely a mathematical convenience. It is physically accessible through phase slips, voltage pulses, and node flux variables. The Josephson relation gives

$$V = \frac{\hbar}{2e}\dot{\phi},$$

so that the integrated voltage records the lifted phase:

$$\Phi(t) = \int^t dt' \, V(t') = \frac{\hbar}{2e}\phi_{\mathrm{lift}}(t).$$

Thus a superconducting circuit naturally promotes the compact phase variable into a winding-sensitive clock variable.

The Kronecker clock adds a simple but powerful twist. Instead of using one periodic observable, use two:

$$\cos(\alpha t), \qquad \cos(\beta t),$$

with

$$\frac{\alpha}{\beta} \notin \mathbb{Q}.$$

The pair of phases

$$(\alpha t, \beta t) \mod 2\pi$$

moves quasi-periodically on a two-torus. Because the frequencies are incommensurate, the trajectory does not close. It samples the torus densely.

The fundamental clock statement is therefore

$$\boxed{t \longmapsto \left(e^{i\alpha t}, e^{i\beta t}\right)}$$

with α/β irrational. For an ideal noiseless clock, the pair of phases determines t uniquely up to the available resolution and time range.

This is the physical content of the Kronecker clock:

$$\boxed{\text{two incommensurate periodic observables can reconstruct an effectively nonperiodic time.}}$$

2. Rotor Hamiltonian and Time Reconstruction

Consider two independent superconducting rotors, or two washboard variables, with lifted phases

$$\phi_1(t) = \omega_1 t + \phi_{1,0}, \qquad \phi_2(t) = \omega_2 t + \phi_{2,0},$$

and

$$\frac{\omega_1}{\omega_2} \notin \mathbb{Q}.$$

The simplest effective Hamiltonian is

$$H = \frac{1}{2}E_{C1}(n_1 - n_{g1})^2 + \frac{1}{2}E_{C2}(n_2 - n_{g2})^2 - E_{J1}\cos\phi_1 - E_{J2}\cos\phi_2 + H_{\text{drive}}.$$

In a current- or voltage-biased regime, the phases drift:

$$\dot{\phi}_i = \omega_i.$$

The observables

$$X_1 = \cos\phi_1, \qquad X_2 = \cos\phi_2$$

are individually periodic and ambiguous, but jointly they define a point on the torus:

$$(X_1, X_2) = (\cos\omega_1 t, \cos\omega_2 t).$$

To reconstruct time, one solves the simultaneous congruences

$$\omega_1 t = \theta_1 \pmod{2\pi}, \qquad \omega_2 t = \theta_2 \pmod{2\pi}.$$

Equivalently,

$$t = \frac{\theta_1 + 2\pi m}{\omega_1} = \frac{\theta_2 + 2\pi n}{\omega_2}, \qquad m, n \in \mathbb{Z}.$$

Thus

$$\omega_2(\theta_1 + 2\pi m) = \omega_1(\theta_2 + 2\pi n).$$

For irrational ω_1/ω_2, exact equality selects at most one integer pair within any finite observation window. Approximate equality determines the time up to a resolution controlled by Diophantine approximation.

The reconstruction problem is therefore a physical version of Kronecker's theorem:

$$\boxed{\{\, m\omega_2 - n\omega_1 \,:\, m, n \in \mathbb{Z} \,\} \quad \text{comes arbitrarily close to any target phase mismatch.}}$$

The price of high accuracy is large integers. In clock language, resolving long times requires fine phase resolution.

3. Relation to Pauli's Objection

The Kronecker clock clarifies the traditional tension between clocks and the absence of an ideal time operator. Pauli's argument rules out a perfectly self-adjoint operator T satisfying

$$[H, T] = i\hbar$$

for a Hamiltonian bounded below, under suitable regularity assumptions. But physical clocks need not realize that ideal limit. They can work well over a large but finite range of times and energies.

The Kronecker clock is especially interesting because it is constructed from perfectly ordinary bounded observables:

$$\cos\phi_1, \qquad \cos\phi_2.$$

Neither is a global time operator. But together, with irrational frequency ratio, they encode enough information to reconstruct time over an enormous range:

$$t \sim \mathcal{R}(\cos\phi_1, \sin\phi_1, \cos\phi_2, \sin\phi_2),$$

where $\mathcal{R}$ is a reconstruction map that depends sensitively on the irrational ratio.

The price is fragility. If

$$\frac{\omega_1}{\omega_2}$$

has very good rational approximants, then distinct large times can produce almost indistinguishable points on the torus. Thus the clock's useful range and resolution are controlled by number theory.

A useful summary is

$$\boxed{\text{Pauli forbids an ideal global time operator; Kronecker clocks realize practical time reconstruction from incomme}}$$

4. Stochastic Ticks: Tunneling as Radioactive Clockwork

In a deterministic rotor clock, phase advances smoothly. But in superconducting circuits the advance of a lifted phase can occur through tunneling events. A washboard potential provides the simplest model:

$$U_i(\phi_i) = -E_{Ji}\cos\phi_i - F_i\phi_i,$$

where F_i is the bias force. The phase escapes from one well to the next by quantum tunneling:

$$\phi_i \to \phi_i + 2\pi.$$

Each event produces a voltage pulse:

$$\int dt\, V_i = \frac{\hbar}{2e}\Delta\phi_i = \frac{h}{2e}.$$

Thus the lifted phase is a counting variable:

$$\phi_i(t) = 2\pi N_i(t) + \varphi_i(t),$$

where $N_i(t)$ is the number of phase slips and φ_i is the residual phase inside a well.

If tunneling is rare, the clock becomes stochastic. The probability of a slip in channel i is

$$\Gamma_i \sim A_i e^{-S_i/\hbar}.$$

Thus

$$N_i(t)$$

is approximately a Poisson process:

$$P(N_i = N) = \frac{(\Gamma_i t)^N}{N!}e^{-\Gamma_i t},$$

with

$$\langle N_i(t)\rangle = \Gamma_i t, \qquad \mathrm{Var}(N_i) = \Gamma_i t.$$

The clock is then reminiscent of radioactive dating. Time is inferred not from a deterministic hand, but from a stochastic accumulation of rare events.

With two incommensurate tunneling channels, we obtain a stochastic Kronecker clock:

$$\boxed{t \quad \longleftrightarrow \quad (N_1(t), N_2(t))}$$

with rates or phase advances that are incommensurate. The deterministic torus flow is replaced by a noisy walk on a discrete torus.

The resulting clock has two complementary limitations:

$$\text{Poisson noise:} \qquad \frac{\Delta t}{t} \sim \frac{1}{\sqrt{\Gamma t}},$$

and

$$\text{Diophantine ambiguity:} \qquad m\omega_2 - n\omega_1 \approx 0.$$

The first is statistical. The second is arithmetic.

5. Circuit Realization

A minimal superconducting realization uses two weakly coupled current-biased junctions, or two effective phase-slip elements, with different effective masses or plasma frequencies. The Euclidean Lagrangian may be written

$$L_E = \sum_{i=1}^{2} \left[\frac{M_i}{2} \dot{\phi}_i^2 + E_{Ji}(1 - \cos\phi_i) - F_i \phi_i \right] + L_{\text{int}}(\phi_1, \phi_2).$$

The effective masses are determined by capacitances:

$$M_i = \left(\frac{\hbar}{2e}\right)^2 C_i.$$

The tunneling actions are approximately

$$S_i = \int_{\phi_{i,-}}^{\phi_{i,+}} d\phi \sqrt{2M_i \left[U_i(\phi) - E_i\right]}.$$

If

$$\frac{S_1}{S_2} \notin \mathbb{Q}$$

or if the corresponding rates satisfy

$$\frac{\log \Gamma_1}{\log \Gamma_2} \notin \mathbb{Q},$$

then the hierarchy of waiting times and counts has an incommensurate structure.

Alternatively, one can engineer deterministic incommensurate phase drift by applying two voltage biases:

$$\dot{\phi}_i = \frac{2e}{\hbar} V_i, \qquad \frac{V_1}{V_2} \notin \mathbb{Q}.$$

Then the measured pair

$$(\cos\phi_1, \cos\phi_2)$$

is the direct circuit analogue of a Kronecker clock.

The key experimental knobs are

$$\boxed{C_i, \quad E_{Ji}, \quad I_i, \quad V_i, \quad n_{gi}, \quad \Phi_{\text{ext}}.}$$

They tune the clock frequencies, tunneling rates, offset-charge phases, and interference patterns.

6. Kronecker Clock as a Probe of Global Quantum Theory

The Kronecker clock is not merely a timing device. It exposes global aspects of quantum mechanics.

First, it depends on lifted phases:

$$\phi_i \in \mathbb{R},$$

not merely compact phases

$$e^{i\phi_i} \in U(1).$$

Thus it probes winding.

Second, it depends on number-theoretic properties of frequency ratios:

$$\frac{\omega_1}{\omega_2}.$$

Two irrational numbers can behave very differently depending on their rational approximants. A badly approximable irrational gives robust time reconstruction, while a well approximable irrational gives long near-recurrences.

Third, offset charges attach phases to tunneling events:

$$\mathcal{A}_{\mathbf{m}} \sim A_{\mathbf{m}} e^{-S_{\mathbf{m}}/\hbar} e^{2\pi i\,\mathbf{m}\cdot\mathbf{n}_g}.$$

Thus the clock is sensitive not only to classical drift, but also to coherent quantum phases on tunneling histories.

This is the bridge from Kronecker clocks to instanton landscapes. A circuit with many possible tunneling moves has amplitudes labeled by integer vectors

$$\mathbf{m} \in \mathbb{Z}^d.$$

The total amplitude for a transition can take the form

$$\mathcal{A} = \sum_{\mathbf{m}} A_{\mathbf{m}} \exp[-S_{\mathbf{m}}/\hbar + 2\pi i\,\mathbf{m}\cdot\mathbf{n}_g]\,.$$

The phases

$$2\pi\,\mathbf{m}\cdot\mathbf{n}_g$$

sample a torus. If the components of $\mathbf{n}_g$ are rationally independent, the sampling is quasi-periodic and eventually dense.

7. From Kronecker Clocks to Kronecker Cascades

In the discharge problem, the system relaxes through a sequence of discrete tunneling events. In a many-island circuit, the possible moves are integer vectors:

$$\mathbf{N} \to \mathbf{N} + \mathbf{m}.$$

Each move changes the stored energy:

$$\Delta E_{\mathbf{m}} = E(\mathbf{N}) - E(\mathbf{N} + \mathbf{m}),$$

and has a rate

$$\Gamma_{\mathbf{m}} \sim A_{\mathbf{m}} e^{-S_{\mathbf{m}}/\hbar}.$$

The action generally grows as the system approaches a minimum. But if the charges or offset phases are incommensurate, the approach to the minimum can be controlled by Diophantine approximation.

The basic arithmetic problem is

$$\mathbf{m} \cdot \boldsymbol{\alpha} \approx g, \qquad \mathbf{m} \in \mathbb{Z}^d,$$

where g is the residual field or bias to be cancelled. Kronecker's theorem says that approximate cancellation is always possible when the relevant ratios are irrational, but it may require large integer vectors.

Large integer vectors correspond physically to complicated collective tunneling moves. Their actions are large:

$$S_{\mathbf{m}} \sim s_0 |\mathbf{m}|.$$

Thus the system relaxes through a cascade of increasingly difficult approximations:

$$g_0 \to g_1 \to g_2 \to \cdots \to 0,$$

with waiting times

$$\tau_k \sim \exp[S_{\mathbf{m}_k}/\hbar].$$

This is a Kronecker cascade:

$$\boxed{\text{relaxation controlled by a hierarchy of increasingly accurate integer approximations.}}$$

It is the dynamical counterpart of the Kronecker clock. The clock uses irrational motion to encode time; the cascade uses irrational structure to delay equilibration.

8. Observable Signatures

A superconducting Kronecker clock or cascade could be recognized through:

$$\boxed{\begin{array}{l} \text{quasi-periodic phase traces,} \\ \text{long near-recurrences,} \\ \text{stochastic tick statistics,} \\ \text{strong sensitivity to irrational ratios,} \\ \text{hierarchical waiting times,} \\ \text{offset-charge modulation of transition amplitudes,} \\ \text{rare collective events involving large integer moves.} \end{array}}$$

In the deterministic regime, the signature is torus winding:

$$t \mapsto (\phi_1(t), \phi_2(t)) \bmod 2\pi.$$

In the stochastic regime, the signature is a noisy integer walk:

$$t \mapsto (N_1(t), N_2(t)).$$

In the landscape regime, the signature is a sequence of relaxation bottlenecks:

$$\tau_1 \ll \tau_2 \ll \tau_3 \ll \cdots.$$

9. Bottom Line

The Kronecker clock is a compact case study in global quantum mechanics. It brings together:

$$\boxed{\text{winding, irrationality, tunneling, stochasticity, and reconstruction of time.}}$$

In superconducting circuits, these ingredients are not abstract. The phases are measurable, the winding sectors are physical, and the tunneling events produce real voltage pulses.

The essential message is

$$\boxed{\text{incommensurate circuit phases can turn compact quantum variables into an effectively unbounded clock.}}$$

When tunneling supplies the ticks, the clock becomes stochastic:

$$\boxed{\text{the Kronecker clock becomes a radioactive clock.}}$$

And when many tunneling moves compete, the same arithmetic structure becomes a principle of relaxation:

$$\boxed{\text{Kronecker clocks become Kronecker cascades.}}$$

Case Study III: Three-Island Chain and Instanton Bifurcation

1. Why This Example Matters

The simplest instanton is a one-coordinate tunneling event. A particle crosses a barrier, or a single superconducting phase slips by 2π. Such examples are invaluable, but they hide an important possibility: a tunneling event can have internal structure.

The three-island chain is a minimal setting where this becomes explicit. The collective tunneling coordinate describes an end-to-end transfer, while a transverse coordinate describes how the middle island participates during the event. As a control parameter is varied, the symmetric collective instanton can lose transverse stability and split into two daughter instantons.

The conceptual lesson is

$$\boxed{\text{an instanton can bifurcate, acquiring internal geometry.}}$$

This is a controlled toy model for structured tunneling. It also provides a natural arena for interference between semiclassical histories, because the daughter instantons are distinct tunneling paths connecting the same initial and final sectors.

2. Minimal Three-Island Model

Consider three superconducting islands with phases

$$\theta_1, \theta_2, \theta_3,$$

subject to the global constraint

$$\theta_1 + \theta_2 + \theta_3 = 0.$$

Take the Josephson energy to be that of an open chain:

$$U_J = -J\left[\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3)\right].$$

The Euclidean kinetic term is

$$T_E = \frac{E_1}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_3^2\right) + \frac{E_2}{2}\dot{\theta}_2^2.$$

The parameter E_2/E_1 controls how heavy the middle island is relative to the two outer islands.

Introduce collective and internal coordinates

$$\phi \equiv \theta_1 - \theta_3, \qquad \chi \equiv \theta_1 - 2\theta_2 + \theta_3.$$

Using the constraint, the inverse transformation is

$$\theta_1 = \frac{\phi}{2} + \frac{\chi}{6}, \qquad \theta_2 = -\frac{\chi}{3}, \qquad \theta_3 = -\frac{\phi}{2} + \frac{\chi}{6}.$$

The junction phase differences are especially simple:

$$\theta_{12} \equiv \theta_1 - \theta_2 = \frac{\phi + \chi}{2}, \qquad \theta_{23} \equiv \theta_2 - \theta_3 = \frac{\phi - \chi}{2}.$$

Thus ϕ advances both junction phases together, while χ makes one junction lead and the other lag.

The reduced Euclidean Lagrangian is

$$L_E = \frac{E_1}{4}\dot{\phi}^2 + \frac{E_1 + 2E_2}{36}\dot{\chi}^2 + 2J\left(1 - \cos\frac{\phi}{2}\cos\frac{\chi}{2}\right).$$

Equivalently,

$$L_E = \frac{M_\phi}{2}\dot{\phi}^2 + \frac{M_\chi}{2}\dot{\chi}^2 + V(\phi, \chi),$$

with

$$M_\phi = \frac{E_1}{2}, \qquad M_\chi = \frac{E_1 + 2E_2}{18},$$

and

$$V(\phi, \chi) = 2J\left(1 - \cos\frac{\phi}{2}\cos\frac{\chi}{2}\right).$$

3. The Symmetric Collective Instanton

The symmetric tunneling trajectory lies on

$$\chi = 0.$$

Along this diagonal,

$$V_{\text{diag}}(\phi) = 2J\left(1 - \cos\frac{\phi}{2}\right).$$

It is convenient to define

$$\psi \equiv \frac{\phi}{2}.$$

Then

$$L_E(\chi = 0) = E_1\dot{\psi}^2 + 2J(1 - \cos\psi).$$

The Euclidean equation of motion is

$$E_1\ddot{\psi} = J\sin\psi.$$

The instanton solution is

$$\psi_{\rm inst}(\tau) = 4 \arctan e^{\omega\tau}, \qquad \omega = \sqrt{\frac{J}{E_1}}.$$

Equivalently,

$$\phi_{\rm inst}(\tau) = 8 \arctan e^{\omega\tau}.$$

This symmetric instanton satisfies

$$\theta_{12}(\tau) = \theta_{23}(\tau) = \frac{\phi_{\rm inst}(\tau)}{2}.$$

Thus the two junctions slip in perfect synchrony. The middle island remains unpolarized:

$$\chi(\tau) = 0, \qquad \theta_2(\tau) = 0.$$

4. Transverse Fluctuations

To test whether the symmetric instanton is stable, expand the potential for small χ:

$$V(\phi,\chi) = 2J\left(1 - \cos\frac{\phi}{2}\right) + \frac{J}{4}\cos\frac{\phi}{2}\,\chi^2 - \frac{J}{192}\cos\frac{\phi}{2}\,\chi^4 + \cdots .$$

The transverse curvature is therefore

$$K(\phi) = \left.\frac{\partial^2 V}{\partial\chi^2}\right|_{\chi=0} = \frac{J}{2}\cos\frac{\phi}{2}.$$

On the instanton,

$$\cos\frac{\phi_{\rm inst}}{2} = \cos\psi_{\rm inst} = 1 - 2\,\mathrm{sech}^2(\omega\tau).$$

So the transverse fluctuation operator is

$$\mathcal{O}_\perp = -M_\chi \frac{d^2}{d\tau^2} + \frac{J}{2} - J\,\mathrm{sech}^2(\omega\tau).$$

This is a Pöschl–Teller spectral problem.

The lowest transverse eigenvalue determines the fate of the symmetric instanton:

$$\mathcal{O}_\perp \eta_n = \lambda_n \eta_n.$$

If

$$\lambda_0 > 0,$$

the symmetric instanton is stable. If

$$\lambda_0 < 0,$$

it has become unstable against developing internal structure.

Rescale

$$x = \omega\tau, \qquad \omega^2 = \frac{J}{E_1}.$$

Then

$$\mathcal{O}_\perp = J\left[-\alpha\frac{d^2}{dx^2} + \frac{1}{2} - \mathrm{sech}^2 x\right],$$

where

$$\alpha \equiv \frac{M_\chi}{E_1} = \frac{E_1 + 2E_2}{18E_1}.$$

The critical condition is

$$\alpha = \frac{1}{2}.$$

Thus, with the present conventions,

$$\boxed{E_{2,c} = 4E_1.}$$

At criticality, the soft mode is

$$\eta_0(\tau) \propto \operatorname{sech}(\omega\tau).$$

The most invariant way to state the result is

$$\boxed{\lambda_0(E_2) = 0 \quad \Longleftrightarrow \quad \text{instanton bifurcation.}}$$

5. Normal Form Near the Bifurcation

Near the critical point, write the transverse deformation as

$$\chi(\tau) = a\,\eta_0(\tau) + \cdots, \qquad \eta_0(\tau) = \operatorname{sech}(\omega\tau),$$

where a is a soft collective coordinate. Projecting the action onto this mode gives a Landau normal form

$$S[a] = S_0 + \frac{1}{2}\lambda a^2 + \frac{1}{4}g a^4 + \cdots, \qquad g > 0.$$

Here

$$\lambda = \lambda_0(E_2),$$

with

$$\lambda(E_2) \simeq \frac{J}{27E_1}(E_2 - E_{2,c})$$

in the present normalization.

For

$$E_2 > E_{2,c},$$

we have $\lambda > 0$, and the stable saddle is the symmetric instanton

$$a = 0.$$

For

$$E_2 < E_{2,c},$$

we have $\lambda < 0$, and two daughter instantons appear:

$$a_\pm = \pm\sqrt{-\frac{\lambda}{g}}.$$

Their actions are lowered relative to the symmetric saddle:

$$S_\pm = S_0 - \frac{\lambda^2}{4g}.$$

Thus

$$\boxed{S_0 - S_\pm \propto (E_{2,c} - E_2)^2.}$$

This is a pitchfork bifurcation in instanton space. The symmetric tunneling path becomes unstable, and the system chooses one of two symmetry-related internally structured paths.

6. Explicit Daughter Instantons

To leading order near criticality,

$$\phi(\tau) = \phi_{\mathrm{inst}}(\tau) + O(a^2),$$

and

$$\chi_{\pm}(\tau) = \pm a_0 \,\mathrm{sech}(\omega\tau) + O(a_0^3),$$

where

$$a_0 = \sqrt{-\frac{\lambda}{g}}.$$

In terms of the original island phases,

$$\theta_1(\tau) = \frac{1}{2}\phi_{\mathrm{inst}}(\tau) \pm \frac{a_0}{6}\,\mathrm{sech}(\omega\tau),$$

$$\theta_2(\tau) = \mp\frac{a_0}{3}\,\mathrm{sech}(\omega\tau),$$

$$\theta_3(\tau) = -\frac{1}{2}\phi_{\mathrm{inst}}(\tau) \pm \frac{a_0}{6}\,\mathrm{sech}(\omega\tau).$$

In terms of junction phase differences,

$$\theta_{12}(\tau) = \frac{1}{2}\phi_{\mathrm{inst}}(\tau) \pm \frac{a_0}{2}\,\mathrm{sech}(\omega\tau),$$

$$\theta_{23}(\tau) = \frac{1}{2}\phi_{\mathrm{inst}}(\tau) \mp \frac{a_0}{2}\,\mathrm{sech}(\omega\tau).$$

Thus the daughter instantons are distinguished by which junction leads during the tunneling event:

$$\boxed{\theta_{12} > \theta_{23} \quad \text{or} \quad \theta_{23} > \theta_{12}.}$$

The middle island develops a transient polarization. One daughter has

$$\theta_2(\tau) < 0$$

near the center of the event, while the other has

$$\theta_2(\tau) > 0.$$

The physical interpretation is therefore simple:

$$\boxed{\text{collective tunneling splits into two internal orderings.}}$$

7. Tilt, Bias, and Imperfect Pitchforks

In an ideal symmetric model, the two daughter instantons are related by

$$\chi \to -\chi$$

and have equal actions. In a real circuit, small asymmetries, offset charges, or bias terms may break this symmetry. The soft-mode action then becomes

$$S[a] = S_0 + \delta a + \frac{1}{2}\lambda a^2 + \frac{1}{4}g a^4 + \cdots .$$

The parameter δ acts like a symmetry-breaking field. It selects one daughter branch over the other.

More generally, if the leading asymmetry is cubic, one may write

$$S[a] = S_0 + \frac{1}{2}\lambda a^2 + \frac{1}{6}\Gamma_3 a^3 + \frac{1}{4} g a^4 + \cdots .$$

Then the daughter action splitting scales as

$$\Delta S \equiv S_+ - S_- \sim \Gamma_3 \left(-\frac{\lambda}{g}\right)^{3/2} .$$

Thus

$$\boxed{\Delta S \propto |E_2 - E_{2,c}|^{3/2}}$$

near criticality, while the mean action lowering scales as

$$\boxed{S_0 - \frac{S_+ + S_-}{2} \propto |E_2 - E_{2,c}|^2 .}$$

This distinction is important. The bifurcation creates two channels, but a small bias can quickly turn the two-channel problem into branch selection.

8. Aharonov–Casher Phases and Interference

In superconducting circuits, tunneling amplitudes can carry offset-charge phases. For two daughter instantons, the transition amplitude has the form

$$\mathcal{A} = A_+ e^{-S_+/\hbar + i\gamma_+} + A_- e^{-S_-/\hbar + i\gamma_-} .$$

The relative phase

$$\Delta\gamma = \gamma_+ - \gamma_-$$

can be controlled by offset charges:

$$\Delta\gamma = 2\pi\, \mathbf{m} \cdot \mathbf{n}_g ,$$

where $\mathbf{m}$ is the effective charge-transport vector distinguishing the two histories.

In the near-symmetric regime,

$$S_+ \simeq S_-, \qquad A_+ \simeq A_-,$$

so

$$\mathcal{A} \simeq 2A e^{-S/\hbar} \cos\frac{\Delta\gamma}{2} .$$

The rate becomes

$$\Gamma \propto |\mathcal{A}|^2 \simeq 4A^2 e^{-2S/\hbar} \cos^2\frac{\Delta\gamma}{2} .$$

Thus the bifurcated instanton becomes an interferometer:

$$\boxed{\text{two daughter saddles can interfere.}}$$

This gives a direct route to instanton tomography. By tuning offset charges and measuring rates, one can infer the existence and relative phase of distinct tunneling histories.

9. Observable Signatures

The three-island bifurcation should leave several characteristic signatures:

$$\boxed{\begin{array}{l}\text{softening of a transverse mode near } E_{2,c},\\ \text{enhanced tunneling rate below the bifurcation,}\\ \text{appearance of two competing escape channels,}\\ \text{strong offset-charge modulation of the rate,}\\ \text{interference fringes between daughter instantons,}\\ \text{sensitivity to small symmetry-breaking perturbations,}\\ \text{possible decoherence between branch histories.}\end{array}}$$

The most direct diagnostic is the tunneling rate as a function of E_2/E_1 and offset charge. Near the bifurcation:

$$\Gamma \sim \exp\left[-\frac{2S_0}{\hbar} + \frac{2}{\hbar}\frac{\lambda^2}{4g}\right]$$

up to interference factors. Thus the rate should increase as the daughter branches lower the action, while also developing phase-sensitive modulation if both branches remain coherent.

10. Bottom Line

The three-island chain is the minimal controlled model of an instanton bifurcation.

$$\boxed{\text{A symmetric collective phase slip can become transversely unstable and split into two daughter instantons.}}$$

The daughter instantons have a transparent physical meaning:

$$\boxed{\text{one junction leads and the other lags, or vice versa.}}$$

In original island variables, the bifurcation is a transient polarization of the middle island. In semiclassical language, it is a pitchfork bifurcation of tunneling saddles. In circuit language, it is an engineered two-path instanton interferometer.

This case study supplies the bridge from ordinary phase-slip tunneling to instanton landscapes:

$$\boxed{\text{once instantons can bifurcate, tunneling becomes a problem in the geometry of saddle networks.}}$$

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Gloss: Interference and Aharonov–Casher Phases as a Universal Dial

1. From Multiple Saddles to Interference

The three-island bifurcation gives a concrete example of a more general phenomenon. Once a transition can proceed through more than one semiclassical history, the total tunneling amplitude is not a probability sum, but an amplitude sum:

$$\mathcal{A} = \sum_\alpha A_\alpha \exp\left[-\frac{S_\alpha}{\hbar} + i\gamma_\alpha\right].$$

Here α labels instanton histories: different spatial paths, different junctions, different daughter branches, or different collective tunneling moves.

The rate is

$$\Gamma \propto |\mathcal{A}|^2.$$

Thus the relative phases $\gamma_\alpha - \gamma_\beta$ can strongly enhance, suppress, or even nearly cancel a tunneling process.

For two comparable saddles,

$$\mathcal{A} = A_+ e^{-S_+/\hbar + i\gamma_+} + A_- e^{-S_-/\hbar + i\gamma_-}.$$

If

$$S_+ \simeq S_-, \qquad A_+ \simeq A_-,$$

then

$$\mathcal{A} \simeq 2A e^{-S/\hbar} e^{i(\gamma_+ + \gamma_-)/2} \cos\frac{\Delta\gamma}{2},$$

where

$$\Delta\gamma = \gamma_+ - \gamma_-.$$

The rate is therefore

$$\boxed{\Gamma \simeq 4A^2 e^{-2S/\hbar} \cos^2\frac{\Delta\gamma}{2}.}$$

This is the simplest instanton interferometer.

2. Offset Charge as a Phase Dial

In superconducting circuits, offset charges provide a nearly universal way to tune the phases of phase-slip amplitudes. The reason is that offset charges appear as total-derivative terms in the Euclidean action:

$$L_E = \frac{1}{2}\dot{\boldsymbol{\phi}}^T C\, \dot{\boldsymbol{\phi}} + U(\boldsymbol{\phi}) + i\hbar\, \mathbf{n}_g \cdot \dot{\boldsymbol{\phi}}.$$

For a tunneling history α, the topological term contributes

$$S_{\text{top}}^{(\alpha)} = i\hbar \int_\alpha d\tau\, \mathbf{n}_g \cdot \dot{\boldsymbol{\phi}} = i\hbar\, \mathbf{n}_g \cdot \Delta\boldsymbol{\phi}_\alpha.$$

If the tunneling move is described by an integer winding vector

$$\Delta\boldsymbol{\phi}_\alpha = 2\pi\, \mathbf{m}_\alpha, \qquad \mathbf{m}_\alpha \in \mathbb{Z}^N,$$

then its amplitude contains the phase

$$\boxed{\exp\left(i2\pi\, \mathbf{n}_g \cdot \mathbf{m}_\alpha\right).}$$

Thus

$$\gamma_\alpha = 2\pi\, \mathbf{n}_g \cdot \mathbf{m}_\alpha.$$

For two histories,

$$\Delta\gamma = 2\pi\, \mathbf{n}_g \cdot (\mathbf{m}_+ - \mathbf{m}_-).$$

Offset charge is therefore a phase dial:

$$\boxed{\mathbf{n}_g \quad \text{tunes} \quad \Delta\gamma.}$$

3. Aharonov–Casher Interpretation

This phase is the circuit version of the Aharonov–Casher effect. In the Aharonov–Bohm effect, a charge moving around magnetic flux acquires a phase. In the Aharonov–Casher effect, a flux or phase-slip object moving around charge acquires a phase.

The schematic duality is

Aharonov–Bohm	Aharonov–Casher
charge encircles flux	flux/phase slip encircles charge
$q \oint A \cdot dx$	$2\pi n_g\, m$
magnetic flux tunes phase	offset charge tunes phase

In a Josephson circuit, a phase slip changes the superconducting phase by 2π. If two phase-slip paths enclose different offset charges, they differ by an Aharonov–Casher phase. This makes tunneling amplitudes sensitive to gate-controlled charge offsets.

For two paths,

$$\mathcal{A} = A_1 e^{-S_1/\hbar} + A_2 e^{-S_2/\hbar} e^{i2\pi n_g}.$$

In the symmetric case,

$$A_1 = A_2 = A, \qquad S_1 = S_2 = S,$$

so

$$\mathcal{A} = 2A e^{-S/\hbar} e^{i\pi n_g} \cos(\pi n_g),$$

and

$$\boxed{\Gamma \propto \cos^2(\pi n_g).}$$

At

$$n_g = \frac{1}{2},$$

the leading tunneling amplitude is suppressed by destructive interference.

4. Connection to the Three-Island Bifurcation

In the three-island model, the bifurcation produces two daughter instantons:

$$\chi_+(\tau) \quad \text{and} \quad \chi_-(\tau).$$

They connect the same initial and final sectors but differ in their internal ordering:

$$\theta_{12} \text{ leads} \qquad \text{or} \qquad \theta_{23} \text{ leads}.$$

Semiclassically, the transition amplitude is

$$\mathcal{A} = A_+ e^{-S_+/\hbar + i\gamma_+} + A_- e^{-S_-/\hbar + i\gamma_-}.$$

If the two branches transport charge around different offset-charge environments, then

$$\Delta\gamma = 2\pi\, \mathbf{n}_g \cdot (\mathbf{m}_+ - \mathbf{m}_-).$$

Near the bifurcation,

$$S_+ \simeq S_-,$$

so interference is strong. The tunneling rate becomes a direct probe of the relative branch phase:

$$\Gamma(\mathbf{n}_g) \simeq \Gamma_0 \cos^2[\pi\, \mathbf{n}_g \cdot (\mathbf{m}_+ - \mathbf{m}_-)] \,.$$

This is the central experimental opportunity:

$$\boxed{\text{offset charge converts daughter instantons into a tunable interferometer.}}$$

The existence of charge-dependent fringes would be strong evidence that both daughter histories contribute coherently. Conversely, disappearance of the fringes would diagnose decoherence or strong action imbalance between the two branches.

5. Universal Role in Instanton Landscapes

The same mechanism applies far beyond the three-island model. In any circuit with many possible phase-slip histories, the transition amplitude can be written

$$\mathcal{A} = \sum_{\mathbf{m}} A_{\mathbf{m}} \exp\left[-\frac{S_{\mathbf{m}}}{\hbar} + i2\pi\, \mathbf{n}_g \cdot \mathbf{m}\right].$$

The integer vector $\mathbf{m}$ labels the net winding or charge-transport pattern of the instanton.

Thus offset charges act as a universal set of knobs on the instanton landscape:

$$\boxed{\mathbf{n}_g \quad \text{landscapes phases, while} \quad C, E_J, \Phi_{\text{ext}} \quad \text{landscape actions.}}$$

This distinction is useful:

circuit knob	instanton effect
C_{ij}	changes kinetic metric and actions
$E_{J\ell}$	changes potential barriers and paths
Φ_{ext}	changes frustration and path degeneracy
$n_{g,i}$	changes interference phases
I, V	tilt the landscape

In this sense, offset charge is the most direct "phase dial" for instanton tomography.

6. Bottom Line

The Aharonov–Casher phase is a universal dial for coherent phase-slip physics.

$$\boxed{\text{Multiple instantons give amplitudes to sum; offset charges tune their relative phases.}}$$

For two comparable saddles,

$$\boxed{\Gamma(n_g) \propto \cos^2(\pi n_g)}$$

in the simplest symmetric case.

For structured instantons, such as the daughter branches of the three-island bifurcation, the same principle turns internal instanton geometry into an experimentally visible interference pattern.

$$\boxed{\text{Aharonov–Casher interference makes instanton structure empirical.}}$$

Case Study IV: Instanton Landscapes, Landscaping, and Kronecker Cascades

1. From Individual Instantons to Instanton Landscapes

The previous case studies focused on individual tunneling events: a phase slip discharges a biased circuit; two incommensurate phases make a Kronecker clock; a three-island chain produces a bifurcating instanton; offset charges tune interference between alternative histories.

The natural next step is to stop thinking of instantons one at a time. In a many-island circuit, there is not just one tunneling path. There is a network of possible phase-slip moves, each with its own action, phase, and real-time aftermath. This network is the instanton landscape.

A convenient way to describe it is to label metastable sectors by an integer vector

$$\mathbf{N} \in \mathbb{Z}^d.$$

A tunneling event is a move

$$\mathbf{N} \longrightarrow \mathbf{N} + \mathbf{m}, \qquad \mathbf{m} \in \mathbb{Z}^d.$$

The vector $\mathbf{m}$ records which phases wind, which charges move, or which flux sectors change.

The semiclassical amplitude for such a move has the schematic form

$$\mathcal{A}_{\mathbf{m}} = A_{\mathbf{m}} \exp\left[-\frac{S_{\mathbf{m}}}{\hbar} + i2\pi\, \mathbf{n}_g \cdot \mathbf{m}\right].$$

The action $S_{\mathbf{m}}$ is controlled by the circuit landscape:

$$C_{ij}, \quad E_{J\ell}, \quad \Phi_{\text{ext}}, \quad I, \quad V,$$

while the phase is controlled by offset charges:

$$\mathbf{n}_g.$$

Thus a superconducting circuit provides two complementary forms of control:

$$\boxed{\begin{array}{ll} \text{landscaping actions:} & C_{ij}, E_{J\ell}, \Phi_{\text{ext}}, I, V, \\ \text{landscaping phases:} & \mathbf{n}_g. \end{array}}$$

This is the basic meaning of instanton landscaping:

$$\boxed{\text{designing not just a potential, but a network of rare tunneling moves.}}$$

2. The Master Hamiltonian as a Landscaping Machine

For a superconducting circuit with node phases $\boldsymbol{\phi}$ and conjugate Cooper-pair numbers $\mathbf{n}$, a useful master Hamiltonian is

$$H = \frac{1}{2}(\mathbf{n} - \mathbf{n}_g)^T C^{-1} (\mathbf{n} - \mathbf{n}_g) - \sum_{\ell} E_{J\ell} \cos(\Delta_\ell \boldsymbol{\phi} - A_\ell) + U_{\text{drive}}(\boldsymbol{\phi}).$$

Here

term	role
C^{-1}	kinetic metric / charging cost
$E_{J\ell}\cos(\Delta_\ell\phi - A_\ell)$	periodic terrain
$A_\ell, \Phi_{\rm ext}$	frustration / magnetic holonomy
$\mathbf{n}_g$	Aharonov–Casher phases
$U_{\rm drive}$	tilt / discharge drive

The Euclidean Lagrangian has the corresponding form

$$L_E = \frac{1}{2}\dot{\boldsymbol{\phi}}^T C\,\dot{\boldsymbol{\phi}} + U(\boldsymbol{\phi}) + i\hbar\,\mathbf{n}_g \cdot \dot{\boldsymbol{\phi}}.$$

The final term is a total derivative:

$$i\hbar \int d\tau\,\mathbf{n}_g \cdot \dot{\boldsymbol{\phi}} = i\hbar\,\mathbf{n}_g \cdot \Delta\boldsymbol{\phi}.$$

For an instanton move

$$\Delta\boldsymbol{\phi} = 2\pi\mathbf{m},$$

this gives the phase

$$e^{i2\pi\mathbf{n}_g\cdot\mathbf{m}}.$$

This decomposition is powerful. The same circuit graph supplies:

$$\boxed{\text{geometry} + \text{metric} + \text{topological phases} + \text{drive.}}$$

Thus one can deliberately shape:

$$\boxed{\text{which instantons exist, which dominate, and how they interfere.}}$$

3. Effective Graph of Tunneling Moves

At low energy, a circuit with many metastable sectors can often be reduced to an effective graph. Vertices are sectors:

$$\mathbf{N} \in \mathbb{Z}^d.$$

Edges are allowed tunneling moves:

$$\mathbf{N} \to \mathbf{N} + \mathbf{m}.$$

Each edge carries a complex amplitude:

$$t_{\mathbf{m}} = A_{\mathbf{m}} e^{-S_{\mathbf{m}}/\hbar} e^{i2\pi\mathbf{n}_g\cdot\mathbf{m}}.$$

The effective Hamiltonian on the sector graph has the tight-binding form

$$H_{\rm eff} = \sum_{\mathbf{N}} E(\mathbf{N})\,|\mathbf{N}\rangle\langle\mathbf{N}| + \sum_{\mathbf{N},\mathbf{m}} \left[t_{\mathbf{m}}|\mathbf{N}+\mathbf{m}\rangle\langle\mathbf{N}| + t^*_{\mathbf{m}}|\mathbf{N}\rangle\langle\mathbf{N}+\mathbf{m}|\right].$$

This Hamiltonian is not merely a computational device. It is the low-energy description of the instanton landscape.

The landscape has two layers:

energetic landscape	$E(\mathbf{N})$
instanton landscape	$t_{\mathbf{m}}$

The first tells us which sectors are high or low in energy. The second tells us how hard it is to move between them, and with what quantum phase.

4. Discharge as Motion on the Landscape

Suppose the circuit stores energy in a residual bias or flux g. A move $\mathbf{m}$ changes it by

$$g \to g - \mathbf{m} \cdot \boldsymbol{\alpha},$$

where the vector $\boldsymbol{\alpha}$ encodes the amount by which different elementary moves cancel the residual field or bias.

The energy before and after the move may be approximated by

$$E(g) = \frac{K}{2} g^2,$$

so that

$$\Delta E_{\mathbf{m}} = \frac{K}{2} \left[g^2 - (g - \mathbf{m} \cdot \boldsymbol{\alpha})^2 \right].$$

A move is energetically favorable if

$$\Delta E_{\mathbf{m}} > 0.$$

The tunneling rate has the semiclassical form

$$\Gamma_{\mathbf{m}} \sim A_{\mathbf{m}} \exp[-S_{\mathbf{m}}/\hbar].$$

A crude but useful estimate for the action of a collective move is

$$S_{\mathbf{m}} \sim s_0 \, |\mathbf{m}|,$$

or more generally

$$S_{\mathbf{m}} \sim \sqrt{\frac{M_{\mathbf{m}}^3}{\Delta E_{\mathbf{m}}}} \, F_{\mathbf{m}},$$

where $M_{\mathbf{m}}$ is an effective mass and $F_{\mathbf{m}}$ a barrier scale associated with the move. Large or collective moves are exponentially costly.

Thus discharge is not simply downhill motion. It is constrained by the discrete set of available integer moves:

$$\mathbf{m} \in \mathbb{Z}^d.$$

5. Kronecker Cascades

The Kronecker problem appears when the components of $\boldsymbol{\alpha}$ are rationally independent. Then no finite set of elementary discharges exactly cancels a generic residual bias, but integer combinations can approximate it arbitrarily well.

The arithmetic problem is

$$\mathbf{m} \cdot \boldsymbol{\alpha} \approx g, \qquad \mathbf{m} \in \mathbb{Z}^d.$$

Kronecker's theorem implies that, for rationally independent components of $\boldsymbol{\alpha}$, such approximations exist to arbitrary accuracy. But the required vector $\mathbf{m}$ may be large.

In physics language:

$$\boxed{\text{good cancellation requires complicated collective tunneling moves.}}$$

The system therefore relaxes through a sequence

$$g_0 \to g_1 \to g_2 \to \cdots,$$

where

$$g_{k+1} = g_k - \mathbf{m}_k \cdot \boldsymbol{\alpha},$$

and each step uses a better integer approximation:

$$|\mathbf{m}_{k+1} \cdot \boldsymbol{\alpha} - g_k| < |\mathbf{m}_k \cdot \boldsymbol{\alpha} - g_{k-1}|.$$

But better approximations generally require larger $|\mathbf{m}_k|$, and hence larger actions:

$$S_{\mathbf{m}_{k+1}} > S_{\mathbf{m}_k}.$$

The waiting times grow rapidly:

$$\tau_k \sim \Gamma_{\mathbf{m}_k}^{-1} \sim \exp[S_{\mathbf{m}_k}/\hbar].$$

This is a Kronecker cascade:

$$\boxed{\text{a relaxation process organized by increasingly accurate Diophantine approximations.}}$$

It is the circuit counterpart of slow or eternal discharge. The system keeps finding ways to improve the residual bias, but the improvements require increasingly rare tunneling events.

6. Continued Fractions and the One-Dimensional Case

For two incommensurate discharge quanta α and β, the relevant approximation problem is

$$k\alpha + \ell\beta \approx g, \qquad k, \ell \in \mathbb{Z}.$$

Close to equilibrium, successive improvements often require

$$k\alpha + \ell\beta \approx 0.$$

Equivalently,

$$\frac{\alpha}{\beta} \approx -\frac{\ell}{k}.$$

Thus the best late-stage relaxation moves are governed by rational approximants to

$$\frac{\alpha}{\beta}.$$

These are supplied by continued fractions:

$$\frac{\alpha}{\beta} = [a_0; a_1, a_2, a_3, \ldots].$$

The convergents

$$\frac{p_n}{q_n}$$

give especially good approximations:

$$\left| \frac{\alpha}{\beta} - \frac{p_n}{q_n} \right| \lesssim \frac{1}{q_n^2}.$$

The corresponding collective moves have integer sizes of order

$$|\mathbf{m}_n| \sim q_n.$$

Therefore their actions scale roughly as

$$S_n \sim s_0 q_n,$$

and their waiting times as

$$\tau_n \sim \exp(s_0 q_n/\hbar).$$

The number-theoretic quality of the irrational matters. A badly approximable irrational has moderate and regular convergents. A well approximable irrational can produce rare, enormous jumps in the denominators q_n, leading to extremely long bottlenecks.

Thus

$$\boxed{\text{different irrational ratios imply different relaxation histories.}}$$

7. Landscaping the Cascade

In a circuit, the ingredients of the cascade are not fixed by nature. They can be engineered.

The discharge quanta $\boldsymbol{\alpha}$ are set by circuit topology, capacitances, and how different phase-slip moves change the stored energy. The actions $S_{\mathbf{m}}$ are set by junction energies, capacitances, and barriers. The phases of the tunneling amplitudes are set by offset charges.

Thus one can design:

$$\boxed{\begin{array}{l}\text{which integer moves are allowed,}\\ \text{which moves have small action,}\\ \text{which moves interfere destructively,}\\ \text{which residual biases are long-lived,}\\ \text{which bottlenecks dominate relaxation.}\end{array}}$$

This is instanton landscaping in its strongest form:

$$\boxed{\text{engineering the arithmetic and geometry of rare-event dynamics.}}$$

Some design principles are:

Goal	Circuit strategy
fast initial discharge	low-action elementary moves
long-lived residual bias	incommensurate discharge quanta
protected sector	destructive AC interference
large hierarchy of timescales	badly or selectively approximable ratios
rare collective event	engineered low-action large $\mathbf{m}$

The remarkable point is that arithmetic becomes dynamical. Continued fractions, integer lattices, and rational independence are not decorative; they organize the actual sequence of rare quantum events.

8. Relation to Instanton Tomography

An instanton landscape can be probed by watching how transition rates respond to controlled deformations. Varying E_J, capacitances, fluxes, or biases changes the actions:

$$S_{\mathbf{m}} \to S_{\mathbf{m}} + \delta S_{\mathbf{m}}.$$

Varying offset charges changes the phases:

$$2\pi \mathbf{n}_g \cdot \mathbf{m} \to 2\pi(\mathbf{n}_g + \delta \mathbf{n}_g) \cdot \mathbf{m}.$$

Thus the measured transition rates

$$\Gamma(\mathbf{n}_g, \Phi_{\text{ext}}, I, V)$$

can reveal which instanton moves contribute.

In a two-path problem, interference fringes reveal a relative winding vector. In a landscape, the Fourier content of the rate as a function of offset charge can reveal the integer vectors

$$\mathbf{m}$$

of the dominant instantons:

$$\Gamma(\mathbf{n}_g) = \sum_{\mathbf{r}} \Gamma_{\mathbf{r}} e^{i2\pi \mathbf{r}\cdot \mathbf{n}_g}.$$

The harmonics $\mathbf{r}$ are fingerprints of tunneling moves and their interference.

Thus instanton tomography becomes a kind of spectroscopy on the integer lattice:

$$\boxed{\text{measure phase dependence to infer the topology of hidden tunneling histories.}}$$

9. Observable Signatures of Kronecker Cascades

A Kronecker cascade should have a distinctive experimental phenomenology:

$$\boxed{\begin{array}{l}\text{step-like relaxation of a residual bias or flux,}\\ \text{waiting times growing by large factors,}\\ \text{late-stage events involving collective moves,}\\ \text{sensitivity to rational approximants of control ratios,}\\ \text{strong sample-to-sample dependence for different irrational choices,}\\ \text{offset-charge modulation of bottleneck rates,}\\ \text{long-lived metastable residual fields without exact conservation.}\end{array}}$$

One would expect early relaxation to be relatively generic:

$$\text{small } |\mathbf{m}|, \quad \text{moderate action.}$$

Late relaxation should be arithmetic:

$$\text{large } |\mathbf{m}|, \quad \text{good Diophantine approximation,} \quad \text{huge action.}$$

The resulting time trace may look glassy:

$$g(t) = g_0 \quad \longrightarrow \quad g_1 \quad \longrightarrow \quad g_2 \quad \longrightarrow \quad \cdots$$

with waiting times

$$\tau_0 \ll \tau_1 \ll \tau_2 \ll \cdots.$$

But unlike ordinary glassiness, the hierarchy is not accidental. It is organized by discrete topology and number theory.

10. Bottom Line

The landscape viewpoint unifies the earlier case studies.

case study	landscape lesson
circuit discharge	rare moves relax stored energy
Kronecker clock	irrational winding organizes time
three-island bifurcation	one move can split into branches
AC interference	offset charge tunes phases of moves
Kronecker cascade	integer approximations organize relaxation

The key conceptual shift is

$$\boxed{\text{from instantons as isolated saddles to instantons as a designed network of rare moves.}}$$

Superconducting circuits make this shift concrete. They allow us to design the metric, terrain, topology, phases, and drive of the instanton landscape.

The final slogan is

$$\boxed{\text{Instanton landscaping turns arithmetic into dynamics.}}$$

And the strongest form of the message is

Kronecker cascades are discrete, number-theoretic relaxation histories generated by engineered instanton landsca