Exercise 2.7

Verify that

$$\mathbf{l}^2 = -r^2\nabla^2 + \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}.$$

$$\mathbf{l} = \mathbf{r}\times\mathbf{p} = -i\hbar\mathbf{r}\times\nabla = -i\hbar\left(\hat{x}\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) + \hat{y}\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) + \hat{z}\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}\right)\right)$$

With operators $\mathbf{a}, \mathbf{b}, \mathbf{c}$ that **commute** we have the triple product relations given in the exercise, but here $\mathbf{r}$, and ∇ do not commute. Taking care of the order of the operators we can write:

$$\mathbf{l}^2 = -\hbar^2\left(\mathbf{r}\times\nabla\right)\cdot\left(\mathbf{r}\times\nabla\right) = \mathbf{r}\cdot\left(\nabla\times\left(\mathbf{r}\times\nabla\right)\right) = \mathbf{r}\cdot\left(\frac{\partial}{\partial x}\mathbf{r}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\mathbf{r}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\mathbf{r}\frac{\partial}{\partial z} - \left(\nabla\cdot\mathbf{r}\right)\nabla\right)$$

Here we can use the general rule that

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a} + \left[\mathbf{a},\mathbf{b}\right].$$

and thus here

$$\frac{\partial}{\partial x}\mathbf{r}\frac{\partial}{\partial x} = \mathbf{r}\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x},$$

which gives

$$\mathbf{l}^2 = -\hbar^2\left(\mathbf{r}^2\nabla^2 + \mathbf{r}\cdot\nabla - \left(x\nabla\cdot\mathbf{r}\frac{\partial}{\partial x} + y\nabla\cdot\mathbf{r}\frac{\partial}{\partial y} + z\nabla\cdot\mathbf{r}\frac{\partial}{\partial z}\right)\right)$$

which we were asked to show as a first step.

We can now use the commutator relation again:

$$\nabla\cdot\mathbf{r} = \mathbf{r}\cdot\nabla + \left[\nabla,\mathbf{r}\right] = \mathbf{r}\cdot\nabla + 3$$

which gives

$$\mathbf{l}^2 = -\hbar^2\left(\mathbf{r}^2\nabla^2 - 2\mathbf{r}\cdot\nabla - \left(x\mathbf{r}\cdot\nabla\frac{\partial}{\partial x} + y\nabla\cdot\mathbf{r}\frac{\partial}{\partial y} + z\nabla\cdot\mathbf{r}\frac{\partial}{\partial z}\right)\right).$$

Since $r = \sqrt{x^2+y^2+z^2}$ and thus

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} = \frac{x}{r}\frac{\partial}{\partial r}$$

we have

$$\mathbf{r}\cdot\nabla = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} = \frac{1}{r}\left(x^2+y^2+z^2\right)\frac{\partial}{\partial r} = r\frac{\partial}{\partial r}$$

giving

$$\mathbf{l}^2 = -\hbar^2\left(\mathbf{r}^2\nabla^2 - 2r\frac{\partial}{\partial r} - \left(xr\frac{\partial}{\partial r}\frac{x}{r}\frac{\partial}{\partial r} + yr\frac{\partial}{\partial r}\frac{y}{r}\frac{\partial}{\partial r} + zr\frac{\partial}{\partial r}\frac{z}{r}\frac{\partial}{\partial r}\right)\right).$$

The ratios $x/r = \cos\phi\sin\theta$, $y/r = \sin\phi\sin\theta$, are r-independent and commute with $\partial/\partial r$ and thus we have finally

$$\mathbf{l}^2 = -\hbar^2\left(\mathbf{r}^2\nabla^2 - 2r\frac{\partial}{\partial r} - r^2\frac{\partial^2}{\partial r^2}\right) = -\hbar^2\left(\mathbf{r}^2\nabla^2 - \frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\right).$$