



Marco Tarzia



Anomalous corrections to the IPR in the localized phase of the Anderson model on the Bethe lattice

In collaboration with

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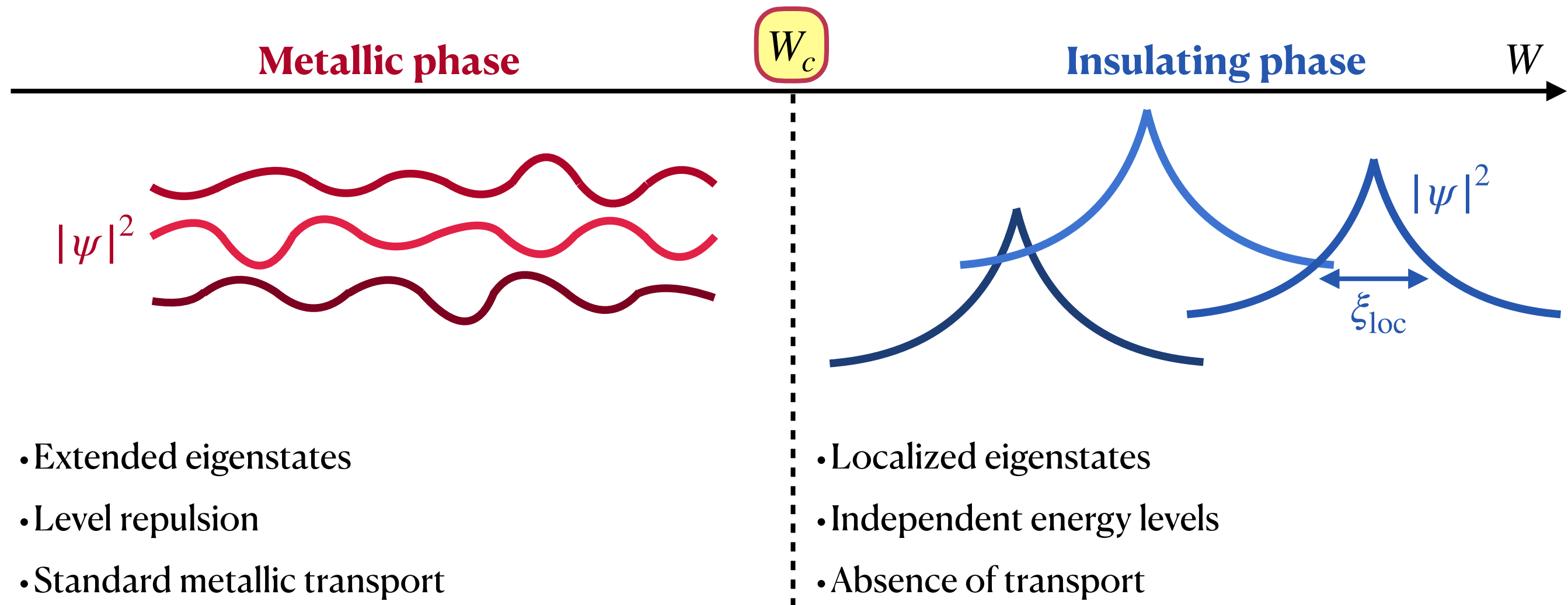
Quantum thermalization breakdown:

Localization and multifractality

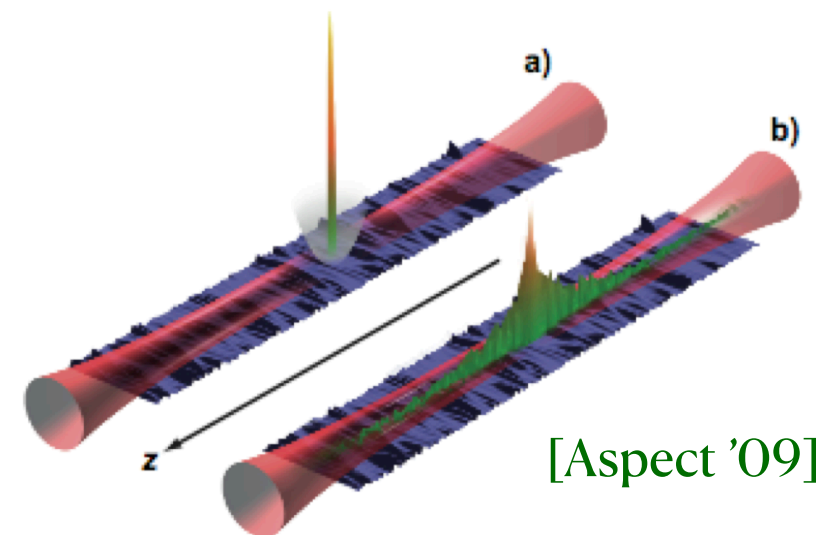
Nordita, Stockholm, 12th – 16th October 2026

Anderson localization

Localization transition of non-interacting quantum particles in a disordered potential [Anderson '58]



- The localisation transition exists in any dimension $d > 2$
- Experimental observation (e.g., in cold atom gases)
- One of the most spectacular phenomena of condensed matter physics with crucial implications in transport, random matrix theory, quantum chaos, ...

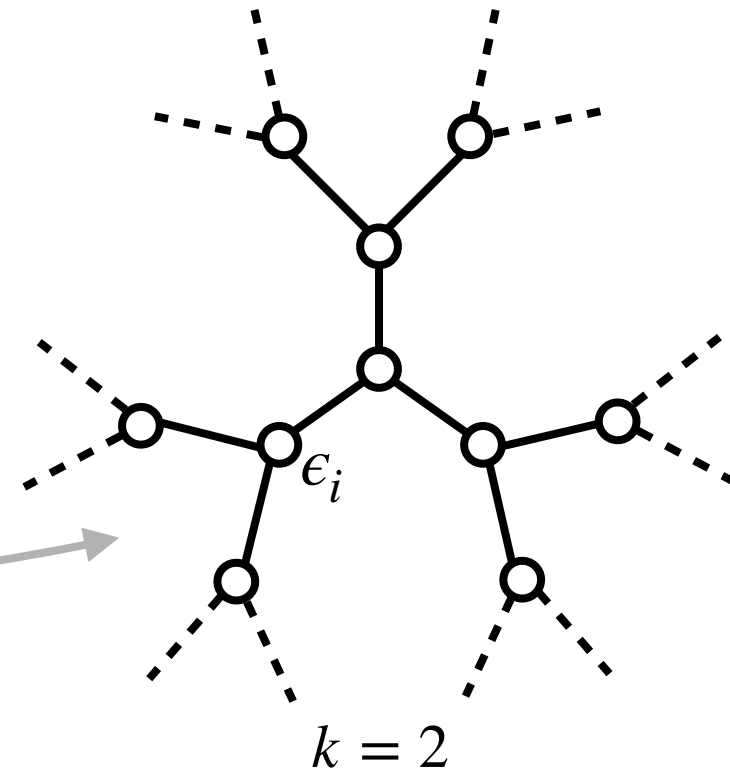


[Aspect '09]

Anderson localization on the Bethe lattice

$$H = - \sum_{\langle i,j \rangle} \left(c_i^\dagger c_j + c_j^\dagger c_i \right) + \sum_{i=1}^N \epsilon_i \hat{n}_i \quad \epsilon_i \in [-W/2, W/2]$$

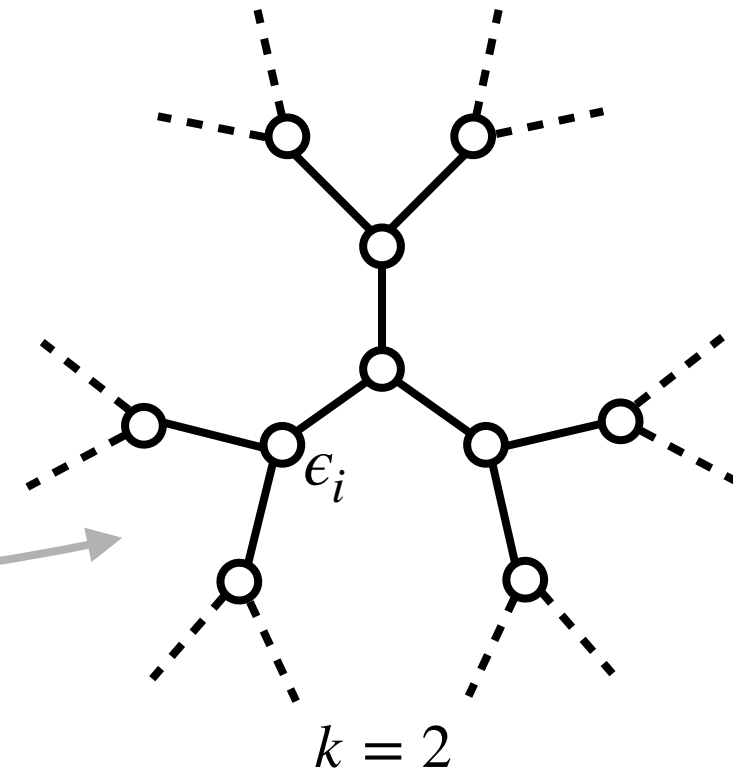
Bethe lattice $\longrightarrow$ an infinite loop-less hierarchical lattice
with fixed branching ratio k



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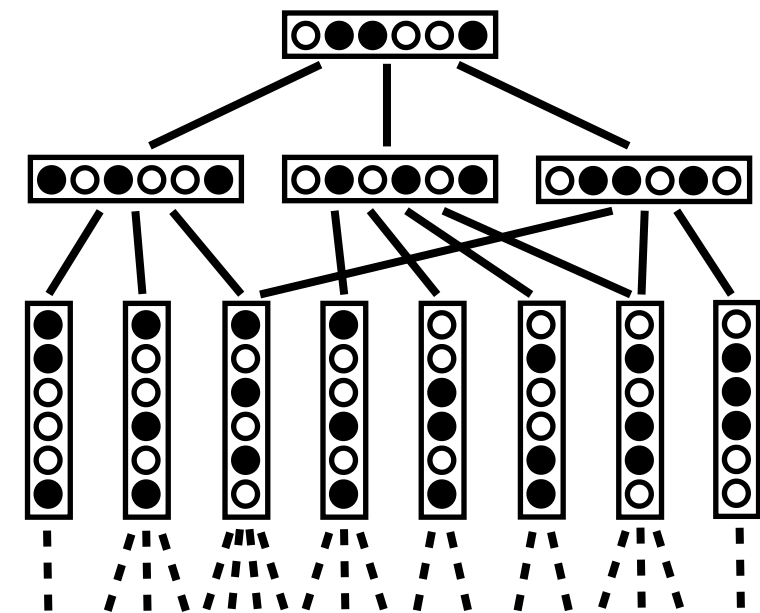
Bethe lattice $\longrightarrow$ an infinite loop-less hierarchical lattice
with fixed branching ratio k



- **Analytical solution** yielding the **transition point** and the **critical behavior**

[Abou-Chacra, Thouless, Anderson '73; Zirnbauer '96; Verbaarschot '88;
Mirlin & Fyodorov '91, '92, '94; Tikhonov & Mirlin '19; Rizzo & Tarzia '24]

- Pictorial representation of **high-dimensional Hilbert space graphs**
of **quantum interacting systems** [Altshuler & al '97]



- The upper critical dimension of Anderson localization is $d_u = \infty$ [Castellani & al '86; Tarquini & al '17]
The Bethe lattice solution corresponds to a **singular limit** [Mirlin & Fyodorov '94; Baroni & al '24]
Natural starting point to perform $1/d$ expansion around the mean-field solution

The inverse participation ratio

Order parameter of the localized phase $\longrightarrow$

$$I_2 = \left\langle \sum_{i=1}^N |\psi_\alpha(i)|^4 \delta(E_\alpha) \right\rangle$$

- Delocalized phase $\longrightarrow |\psi_\alpha(i)|^2 = O(1/N) \quad \forall i \quad \implies I_2 \propto 1/N$
- Localized phase $\longrightarrow |\psi_\alpha(i)|^2 = O(1) \text{ on } O(1) \text{ nodes} \quad \implies I_2 = O(1)$

Inverse volume
occupied by the
eigenstates

The inverse participation ratio

Order parameter of the localized phase $\longrightarrow$

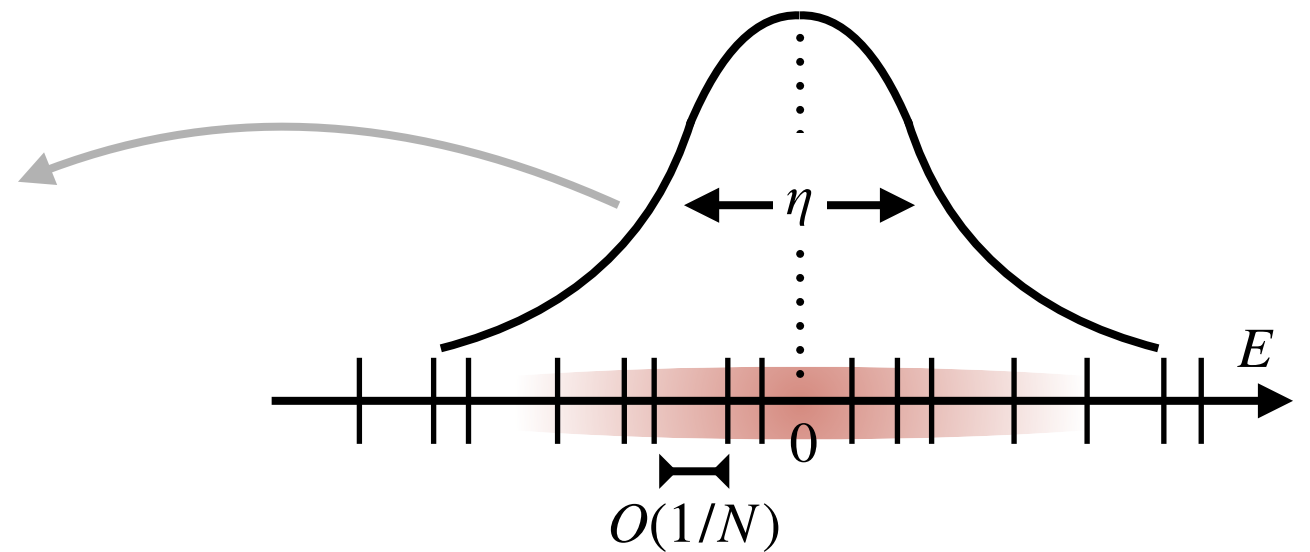
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• Localized phase $\longrightarrow |\psi_\alpha(i)|^2 = O(1)$ on $O(1)$ nodes $\implies I_2 = O(1)$

Inverse volume
occupied by the
eigenstates

Spectral broadening $\delta(E_\alpha) \longrightarrow \frac{1}{\pi} \frac{\eta}{E_\alpha^2 + \eta^2}$



Resolvent matrix $\longrightarrow G(\eta) = [H - i\eta]^{-1}$

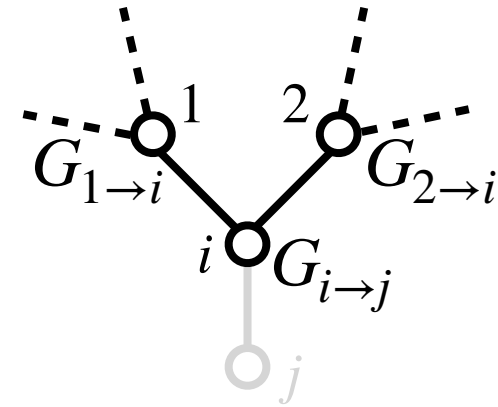
Spectral representation of the IPR $\longrightarrow I_2^\infty = \lim_{\eta \rightarrow 0^+} \lim_{N \rightarrow \infty} \frac{\langle \sum_{i=1}^N \eta |G_{ii}(\eta)|^2 \rangle}{\underbrace{\sum_{i=1}^N \langle \text{Im} G_{ii}(\eta) \rangle}_{I_2(\eta)}}$

Exact solution on the infinite Bethe lattice

Recursive equations for the “cavity” Green’s functions (**exact for $N \rightarrow \infty$**)

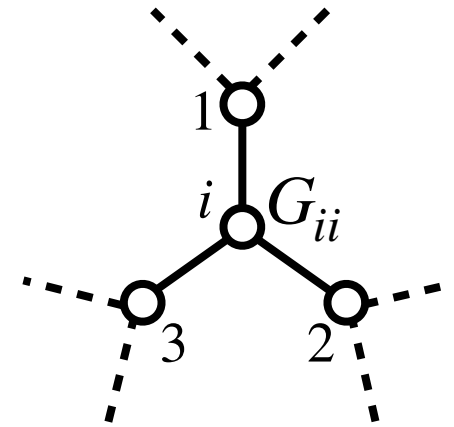
[Abou-Chacra, Thouless, Anderson '73]

$$G_{i \rightarrow j}^{-1} = \epsilon_i - i\eta - \sum_{m \in \partial i \setminus j} G_{m \rightarrow i}$$



Local Green’s function on node i of the Bethe lattice

$$G_{ii}^{-1} = M_{ii} = m_{ii} - i\eta \hat{m}_{ii} = \epsilon_i - i\eta - \sum_{m \in \partial i} G_{m \rightarrow i}$$

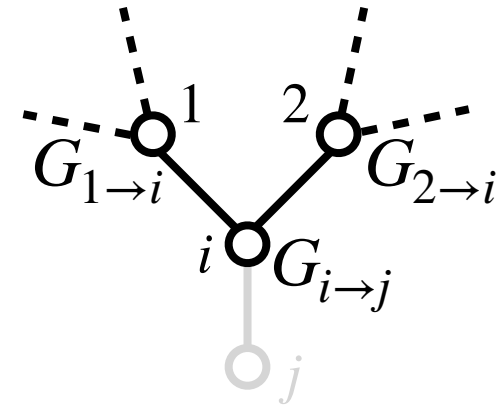


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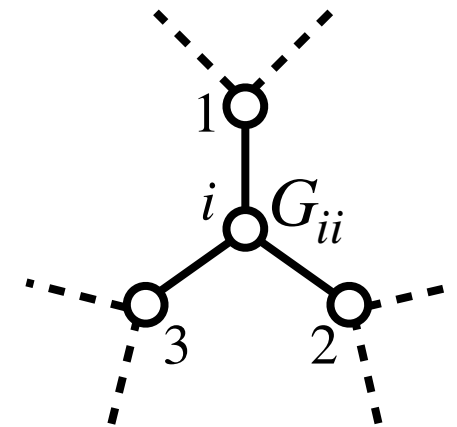
[Abou-Chacra, Thouless, Anderson ’73]

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Local Green’s function on node i of the Bethe lattice

$$G_{ii}^{-1} = M_{ii} = m_{ii} - i\eta \hat{m}_{ii} = \epsilon_i - i\eta - \sum_{m \in \partial i} G_{m \rightarrow i}$$



Joint probability distribution $Q_\eta(m, \hat{m})$

$$I_2(\eta) = \int_0^\infty d\hat{m} \bar{Q}_\eta(\hat{m}) \hat{m}^{-1}$$

$$\bar{Q}_\eta(\hat{m}) = \frac{Q_\eta(0, \hat{m})}{\int_0^\infty d\hat{m} Q_\eta(0, \hat{m})}$$

Directly related to the distributions of the $|\psi_\alpha(i)|^2$ [Mirlin & Fyodorov ’94]

Efficient numerical solution of the cavity equations via **population dynamics** with arbitrary numerical precision [Rizzo & Tarzia ’24]

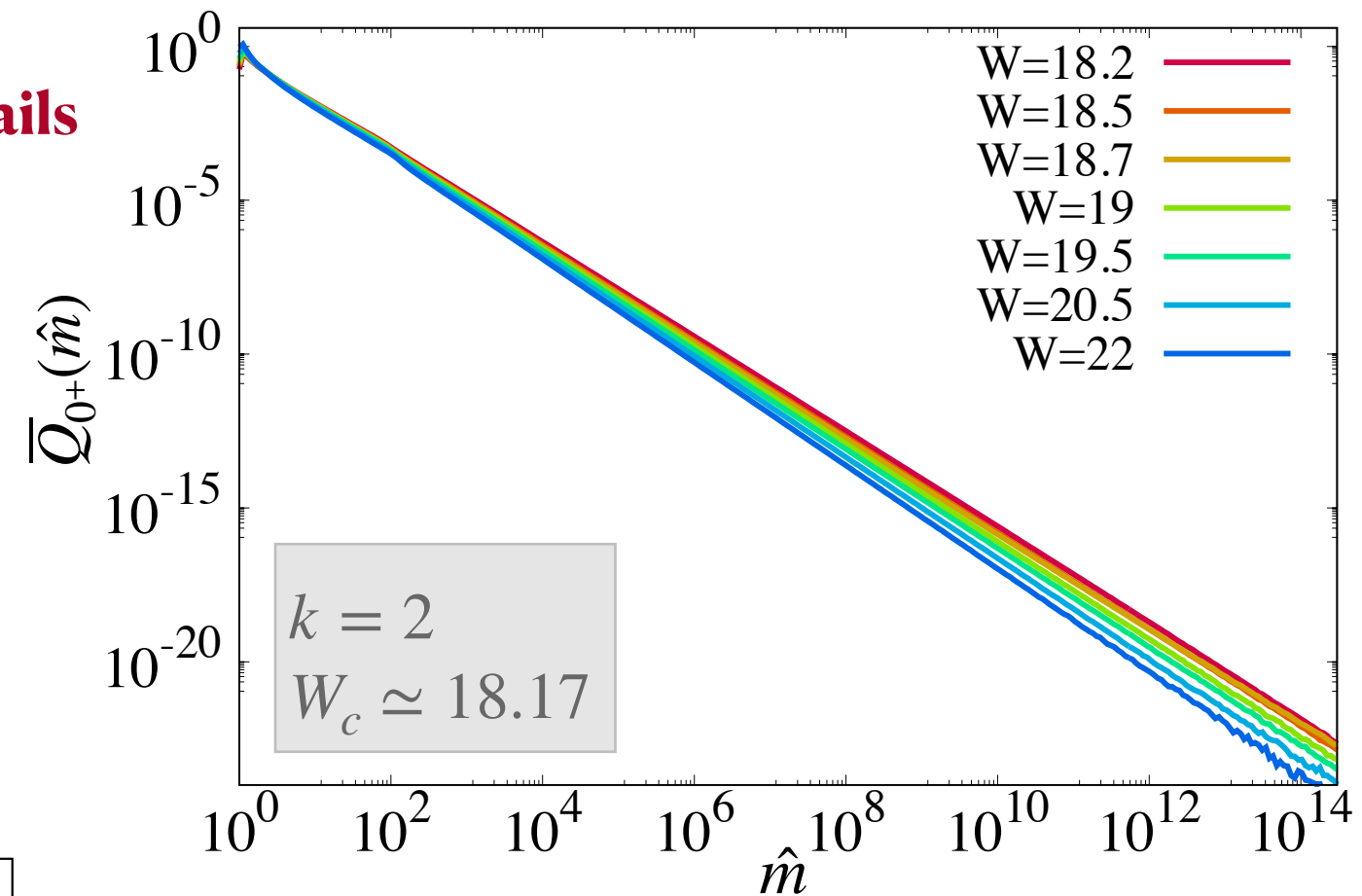
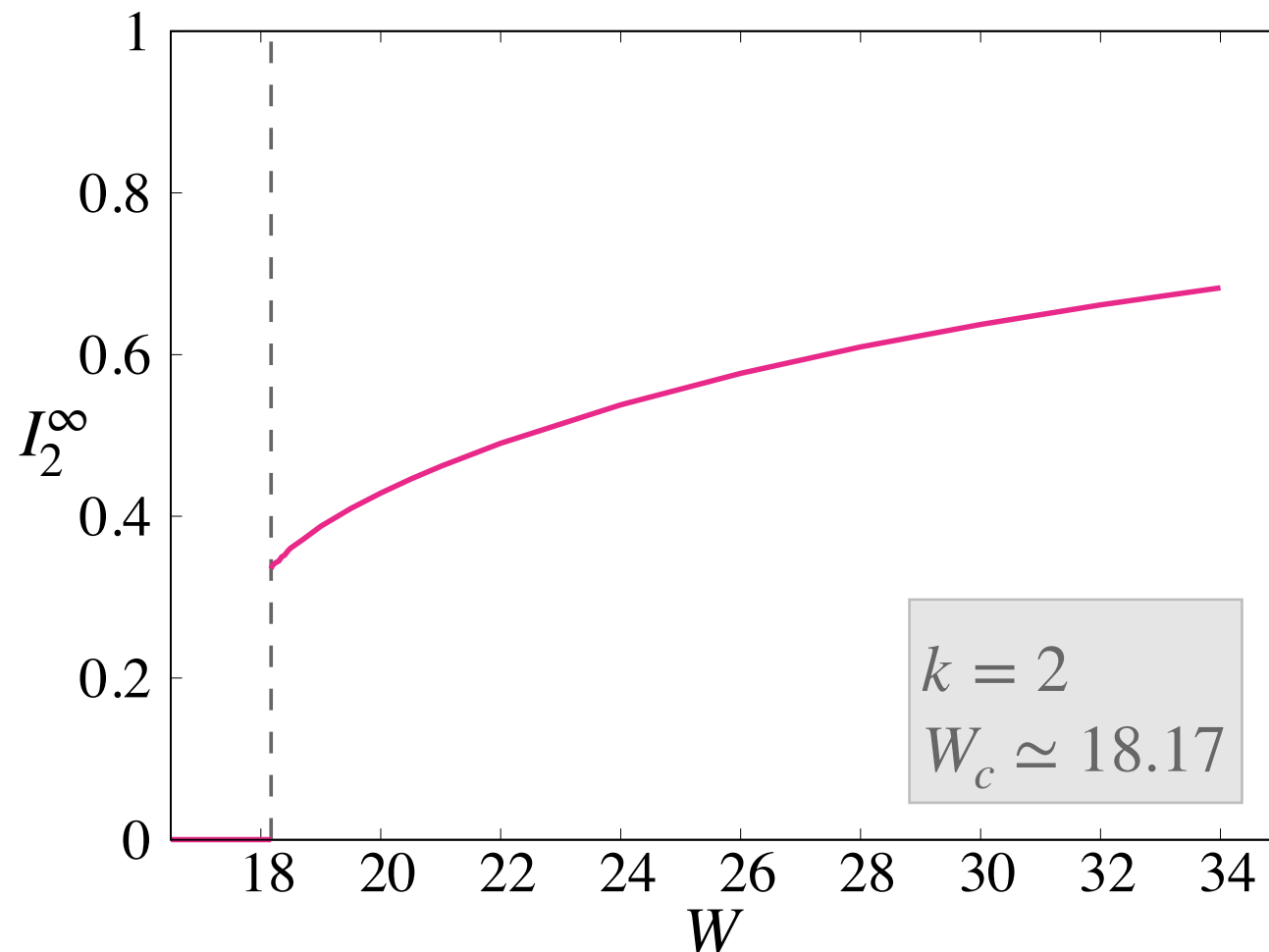
Exact solution on the infinite Bethe lattice ($W > W_c, \eta \rightarrow 0^+$)

In the $\eta \rightarrow 0^+$ limit $\bar{Q}_{0+}(\hat{m})$ develops **power law tails**

$$\bar{Q}_{0+}(\hat{m}) \simeq \text{cst } \hat{m}^{-1-\beta} \quad \hat{m} \gg 1$$

$$\beta \simeq 1/2 + c_\beta \sqrt{W - W_c}$$

[Abou-Chacra, Thouless, Anderson '73; Zirnbauer '96;
Verbaarschot '88; Mirlin & Fyodorov '91, '92, '94;
Tikhonov & Mirlin '19; Rizzo & Tarzia '24]



$$I_2^\infty(W) \simeq I_2^\infty(W_c) + a\sqrt{W - W_c}$$

[Rizzo & Tarzia '24]

Corrections to the asymptotic results

Intense debate on how to perform finite-size scaling analysis of the IPR

[García Mata & al '17, '22, '26; Sierant & al '23; Vanoni & al '23; Chen & al '24]

Precise numerical analysis of the corrections thanks to our very accurate determination of I_2^∞

Two types of corrections to I_2^∞ :

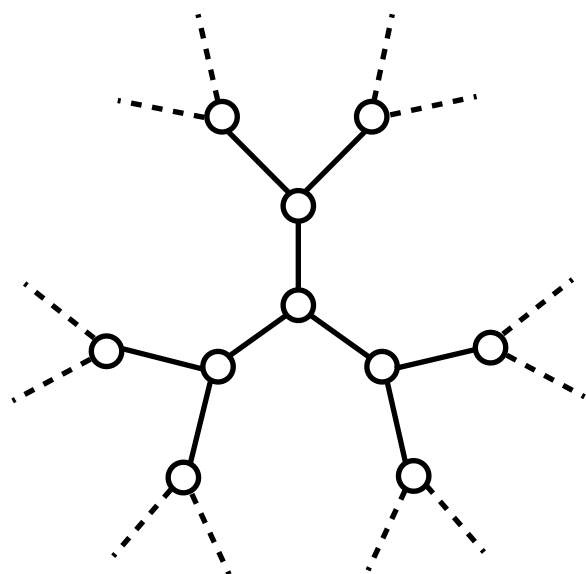
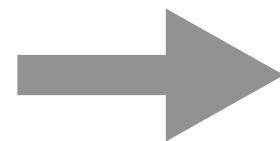
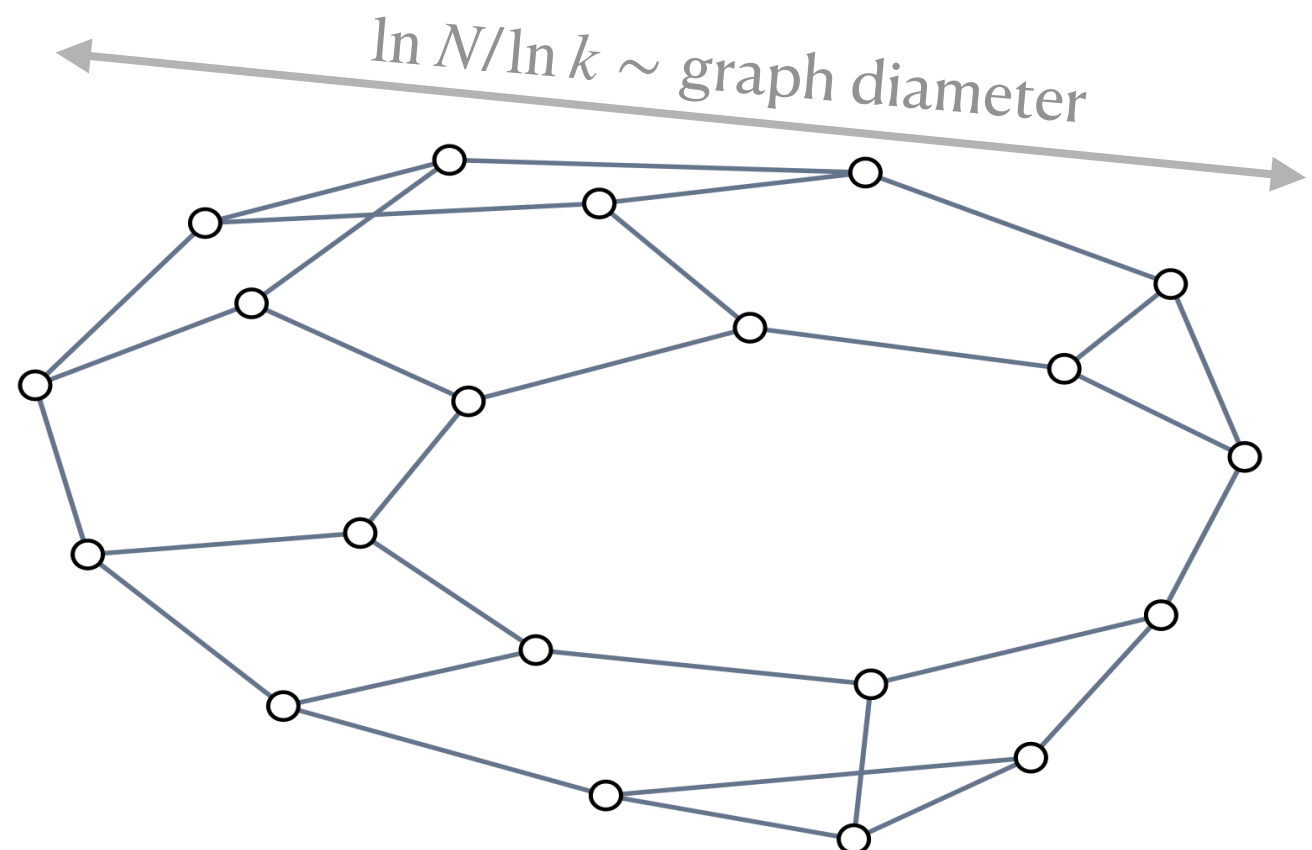
• **Finite- η corrections** $\longrightarrow I_2(N \rightarrow \infty, \eta) - I_2^\infty$

• **Finite- N corrections** $\longrightarrow I_2(N, \eta \rightarrow 0) - I_2^\infty$

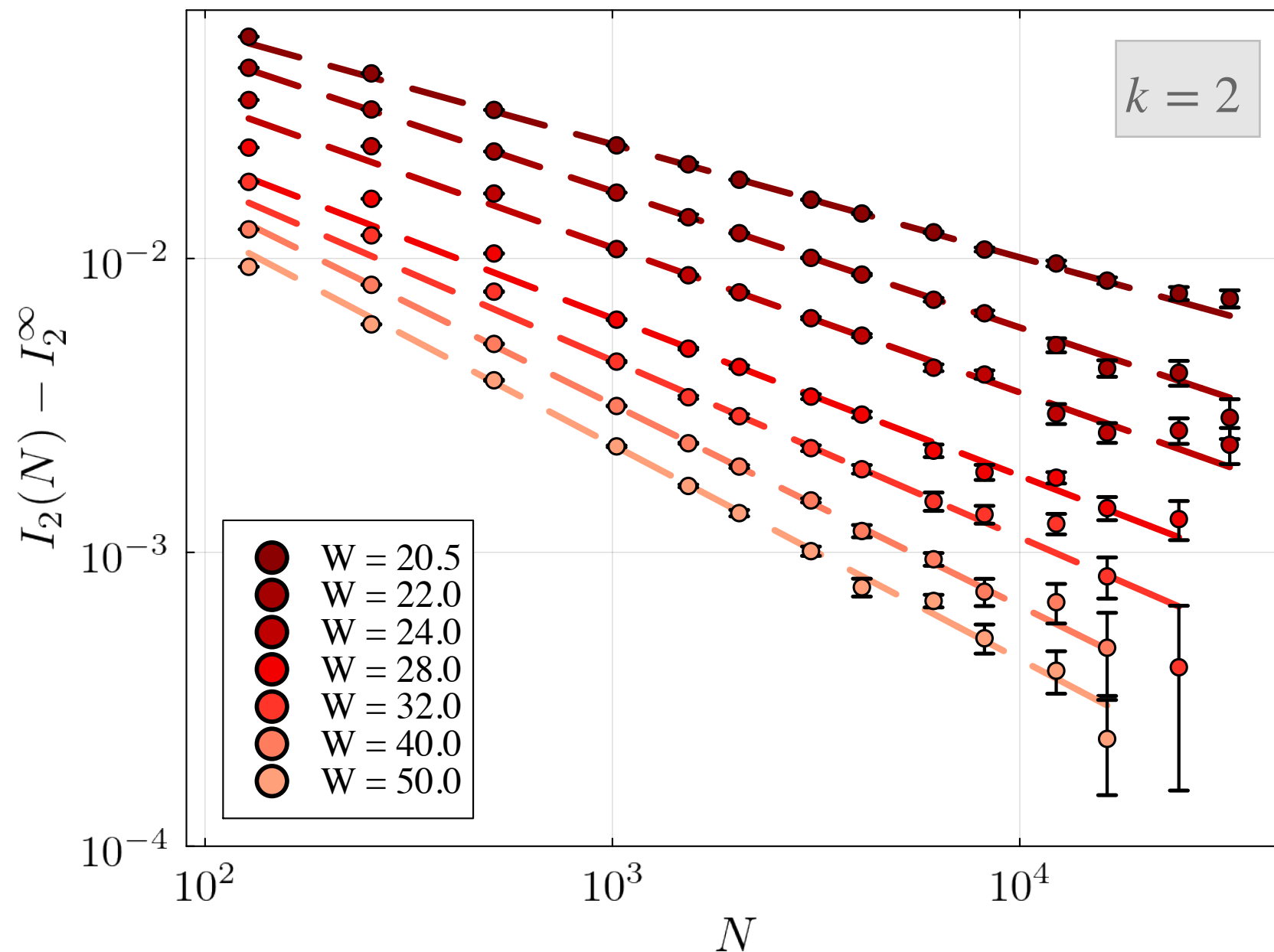
Finite-size version of the Bethe lattice $\longrightarrow$ **Random-Regular Graph**

- Locally tree-like (short loops are rare)

- Typical loop size $\sim \ln N / \ln k$



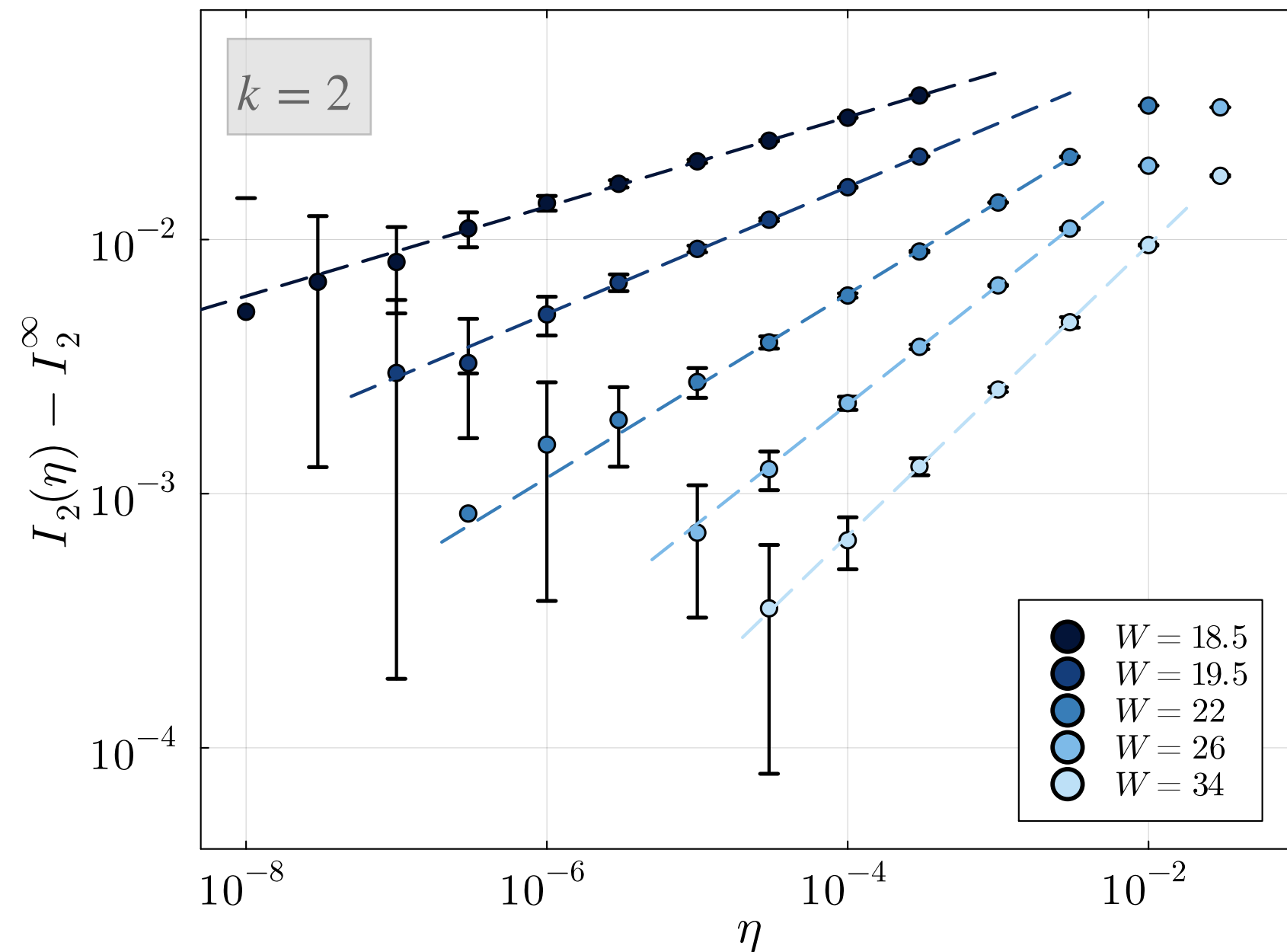
Finite- N corrections to the IPR



$$I_2(N, \eta = 0) \simeq I_2^\infty + \frac{c_N}{N^{\kappa_N(W)}}$$

- Exact diagonalizations on RRGs of N nodes, nearest eigenstates to $E_\alpha \simeq 0$
- $\kappa_N \in [0,1]$ varies continuously with the disorder ($\kappa_N \nearrow$ for $W \nearrow$)
- c_N of $O(1)$ varies smoothly with W

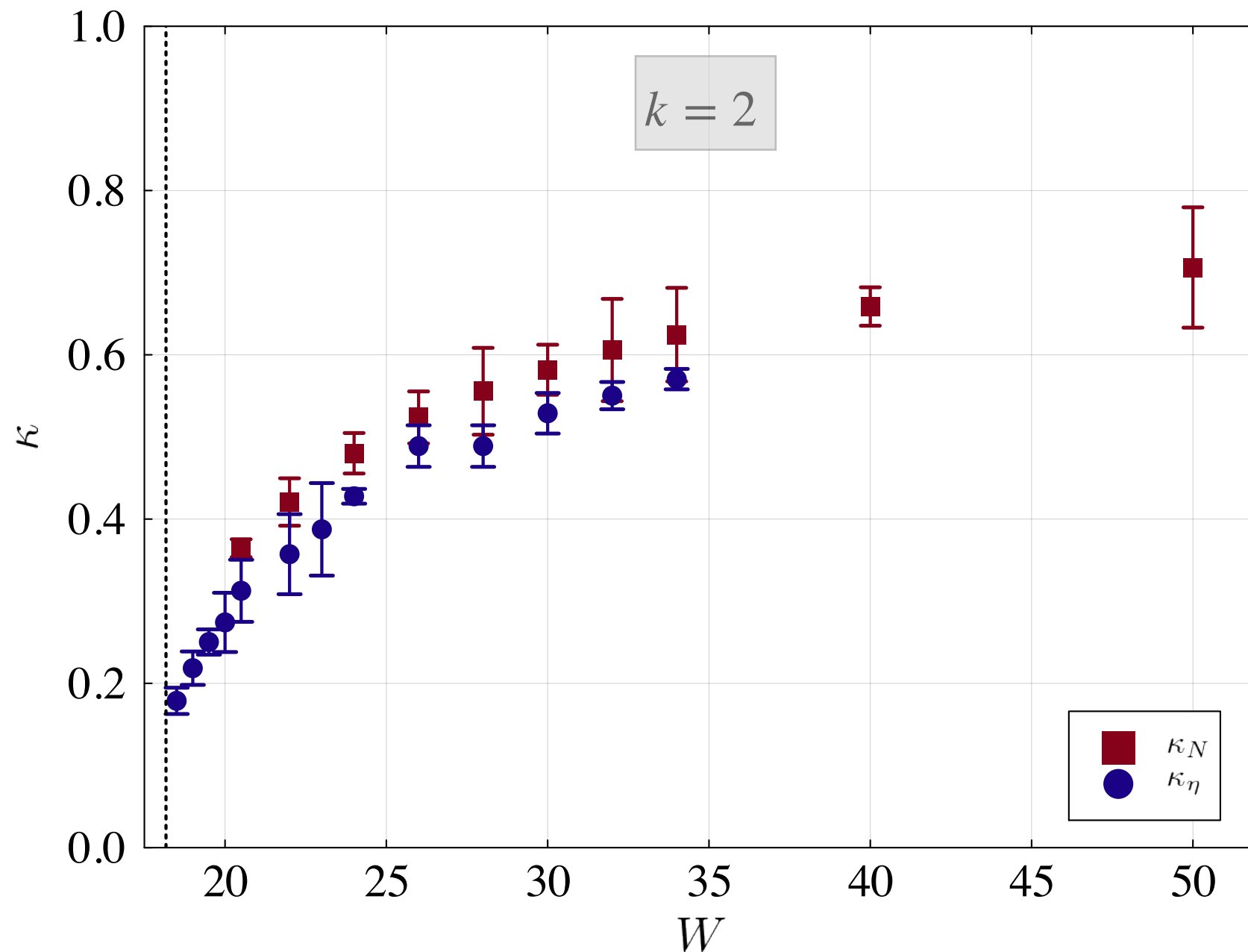
Finite- η corrections to the IPR



$$I_2(N \rightarrow \infty, \eta) \simeq I_2^\infty + c_\eta \eta^{\kappa_\eta(W)}$$

- Computed from the population dynamics solution of the cavity equations ($N \rightarrow \infty$)
- $\kappa_\eta \in [0, 1]$ varies continuously with the disorder ($\kappa_\eta \nearrow$ for $W \nearrow$)
- c_η of $O(1)$ varies smoothly with W

$$\kappa_N(W) = \kappa_\eta(W)$$



- The two exponents coincide within error bars across the whole localized phase
 ➔ **Finite- N and finite- η come a priori from different mechanisms : Same physical origin?**
- Only known example of order-parameter finite-size corrections governed by a continuously varying exponent even very far from criticality (similar to spin wave regime or Griffiths phases)
 ➔ **Physical origin of the anomalous corrections?**

Same exponent: Phenomenological argument

Spectral representation of the IPR at finite η :

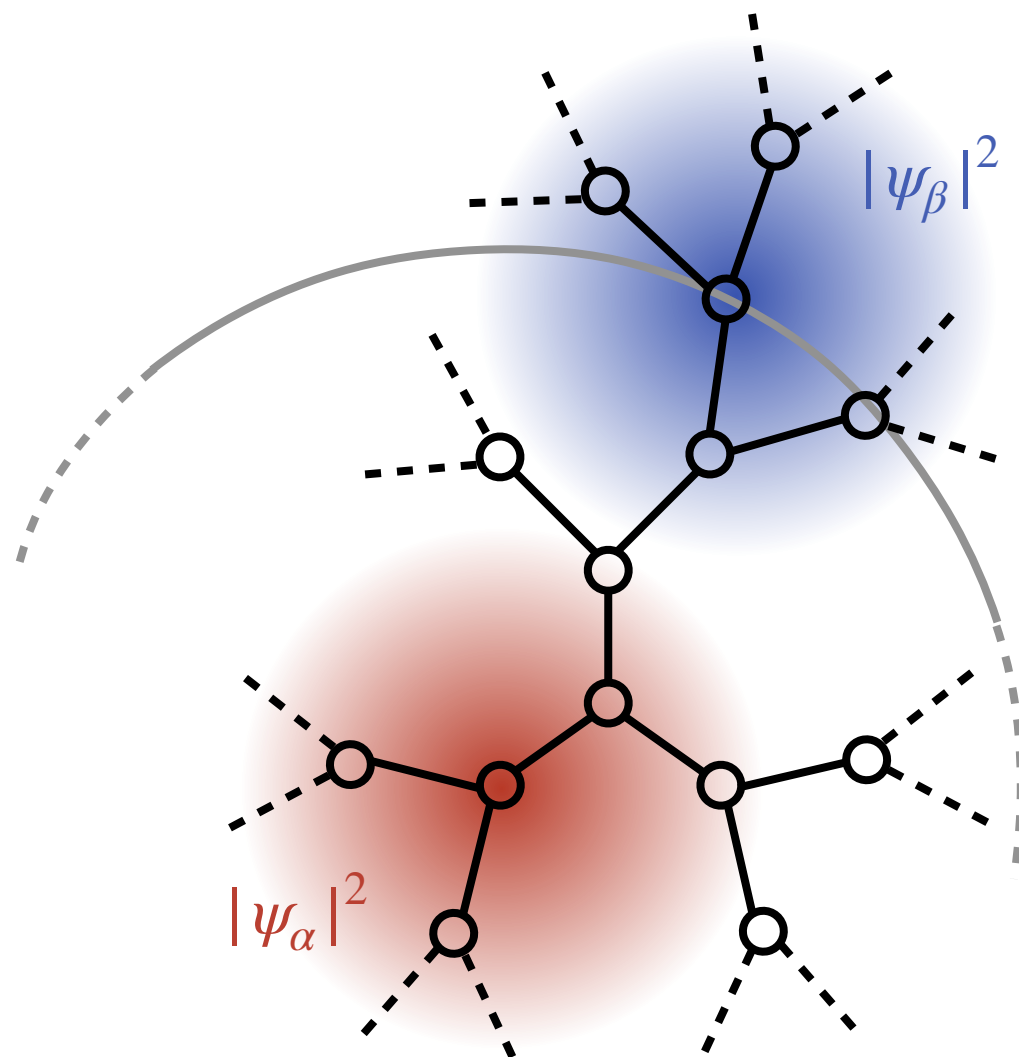
$$\frac{\eta}{N} \sum_i |G_{ii}|^2 = \underbrace{\frac{1}{N} \sum_{\alpha} \left(\sum_i |\psi_{\alpha}(i)|^4 \right) \frac{\eta}{E_{\alpha}^2 + \eta^2}}_{\text{Average of the IPR of } O(N\eta) \text{ eigenstates in an energy window of width } \eta \text{ around } E = 0} + \underbrace{\frac{1}{N} \sum_{\alpha \neq \beta} \left(\sum_i |\psi_{\alpha}(i)|^2 |\psi_{\beta}(i)|^2 \right) \frac{\eta (E_{\alpha} E_{\beta} + \eta^2)}{(E_{\alpha}^2 + \eta^2)(E_{\beta}^2 + \eta^2)}}_{\text{Overlap between pairs of eigenstates close in energy (localized in random positions on the graph)}}$$

Average of the IPR of $O(N\eta)$ eigenstates in an energy window of width η around $E = 0$

Overlap between pairs of eigenstates close in energy (localized in random positions on the graph)

ηk^d eigenstates in the energy window η within distance d from the localization center of ψ_{α}

$$\eta k^{d_{\text{typ}}} = 1 \implies d_{\text{typ}} = -\ln \eta / \ln k$$



$d_{\text{typ}} \sim$ effective graph diameter

Same exponent: Phenomenological argument

Spectral representation of the IPR at finite η :

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Turning on $\eta \iff$ effective graph diameter $d_{\text{typ}} = -\ln \eta / \ln k$ $\left\{ \begin{array}{l} \text{Diameter of a RRG of } N \text{ nodes} \longrightarrow \ln N / \ln k \end{array} \right\} \implies$ The two scales coincide when $N\eta = O(1)$

Single scaling form

$$I_2(N, \eta) \simeq I_2^{\infty} + N^{-\kappa} \Phi(N\eta)$$

$$\begin{cases} \Phi(x) \rightarrow c_N & \text{for } x \rightarrow 0 \\ \Phi(x) \rightarrow c_{\eta} x^{\kappa} & \text{for } x \rightarrow \infty \end{cases}$$

An exact derivation of the exponent κ

“Toy” non-linear σ model on the hyperbolic superplane [Zirnbauer '90, '91; Gruzberg & Mirlin '96; Dupre '96]

Same exotic criticality as the Anderson model on the Bethe lattice

$$\bar{\psi} = (\sigma, \pi_1, \pi_2, \bar{\xi}, -\xi) \text{ of unit hyperbolic norm } \longrightarrow \|\psi\|^2 = \sigma^2 - \pi_1^2 - \pi_2^2 - 2\bar{\xi}\xi = 1$$

$$\mathcal{H}_{\text{hs}} = J \sum_{\langle ij \rangle} \bar{\psi}_i \cdot \psi_j + H \sum_i \sigma_i \qquad J \leftrightarrow 1/W \qquad H \leftrightarrow \eta$$

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Exact cavity equation for the order parameter $\sigma = e^t/H + He^{-t}\bar{\xi}\xi$ [Gruzberg & Mirlin '96]

$$P_H(t) = \int_{\ln H}^{\infty} dt' L_H(t, t') P_H^k(t')$$

Solution of the cavity equation for $H = 0$ in the paramagnetic/localized phase

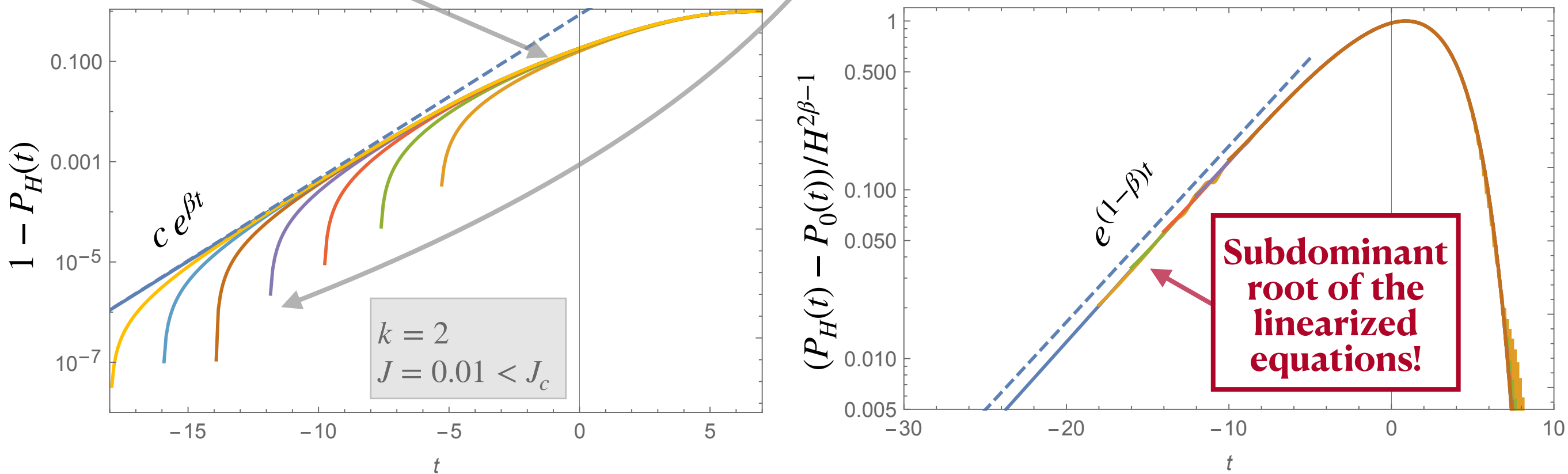
$$P_0(t) \simeq 1 - c e^{\beta t}$$

Same β as the tail exponent
of LDoS distribution

An exact derivation of the exponent κ

Solution of the cavity equations at finite H (for $k = 2$)

Matching the bulk regime at $t = O(1)$ with the boundary layer at $t = O(\ln H)$

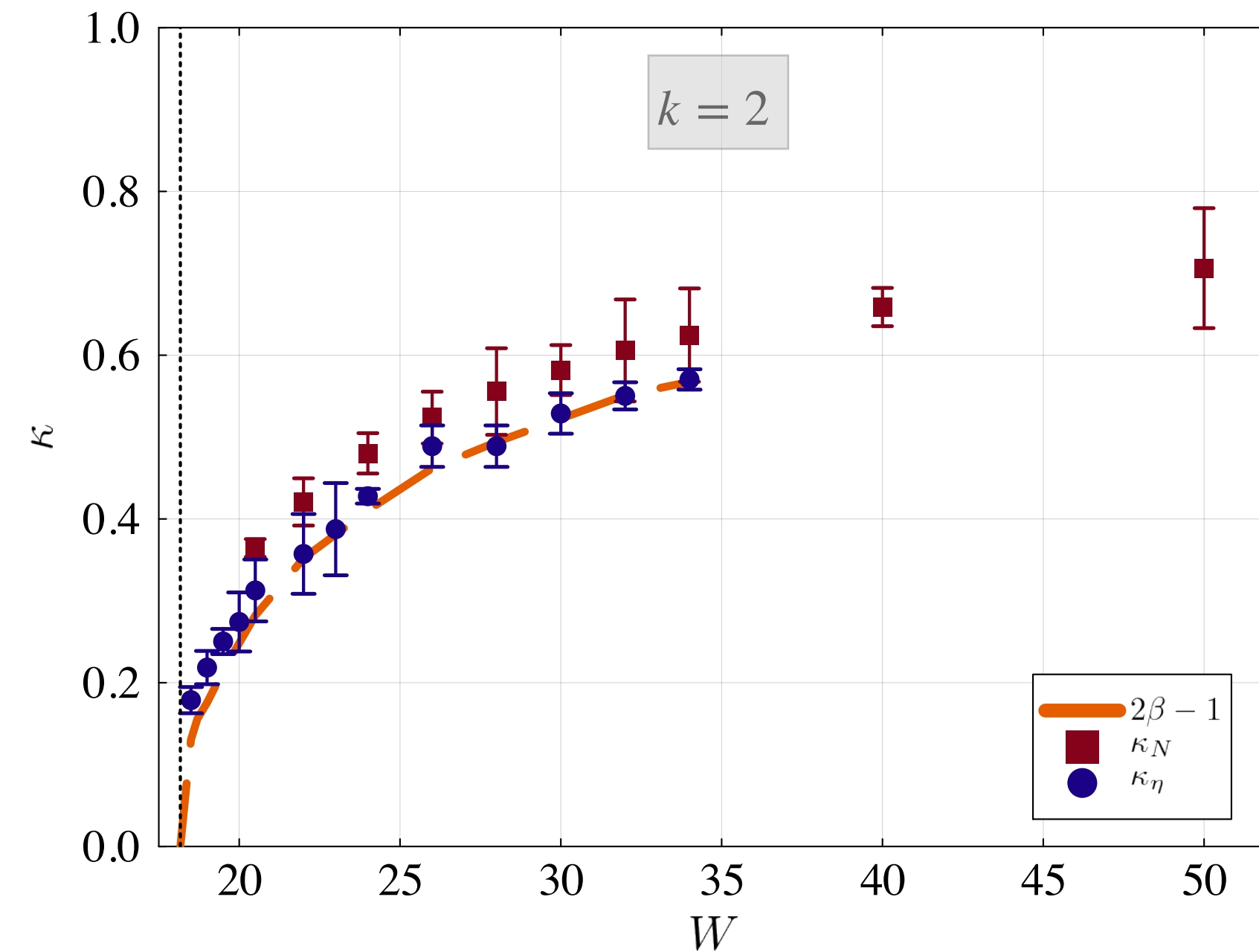


$$P_H(t) - P_0(t) = H^{2\beta-1} f_{\text{typ}}(t)$$

$$I_2 = \int_{\ln H}^{\infty} dt D_H(t) P_H^{k+1}(t) \implies I_2(H) - I_2(0) \propto H^{2\beta-1} \implies \kappa = 2\beta - 1$$

$$\beta \simeq 1/2 + c_\beta \sqrt{J_c - J} \implies \text{At the critical point } I_2(J_c, H) - I_2(J_c, 0) \propto 1/|\ln H|$$

An exact derivation of the exponent κ

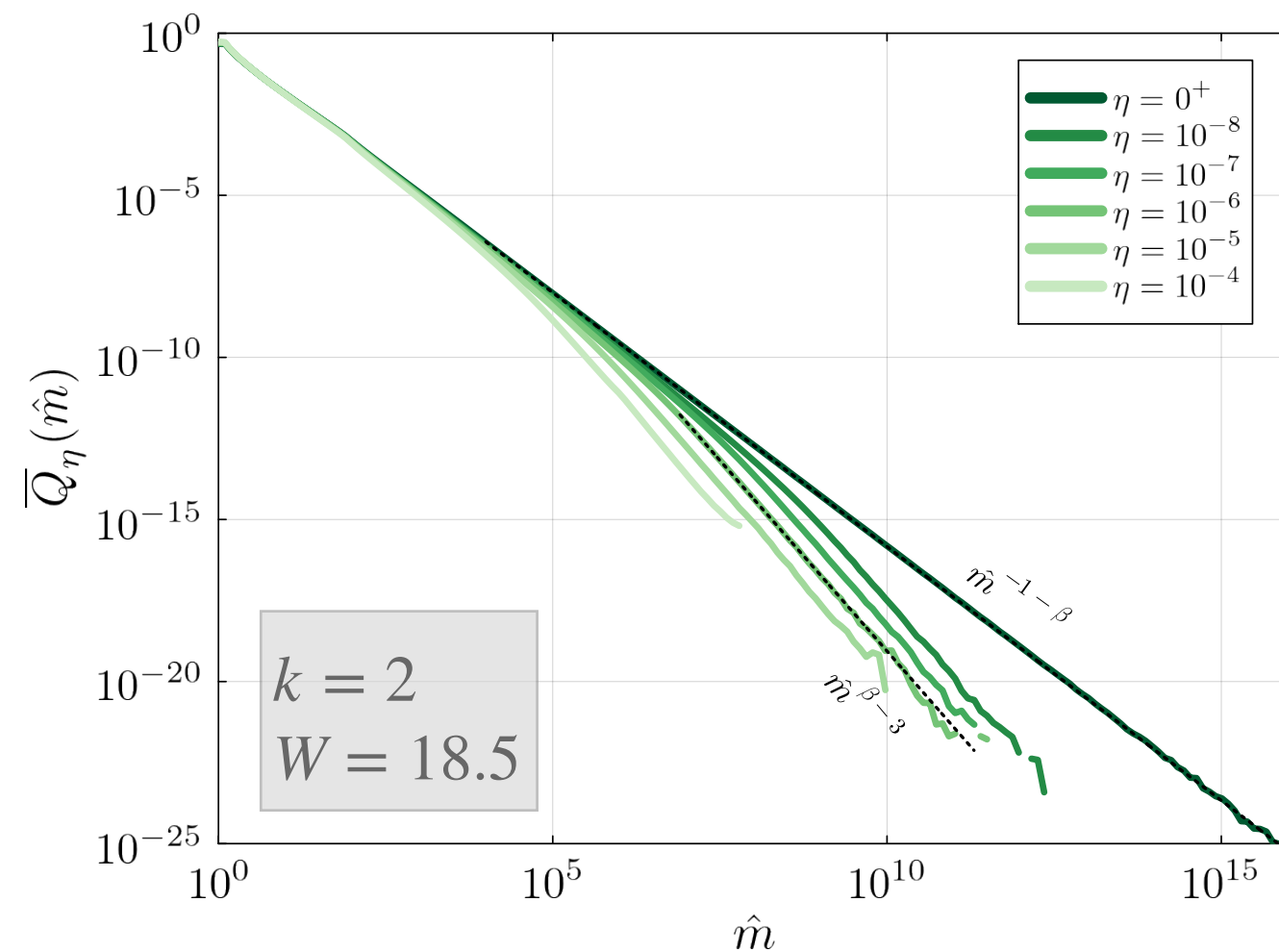


$$\kappa = 2\beta - 1$$

Excellent agreement with numerics !

Scaling of the full distributions

The analysis of the toy non-linear σ model suggests that the scaling holds beyond the average IPR and controls the full distributions



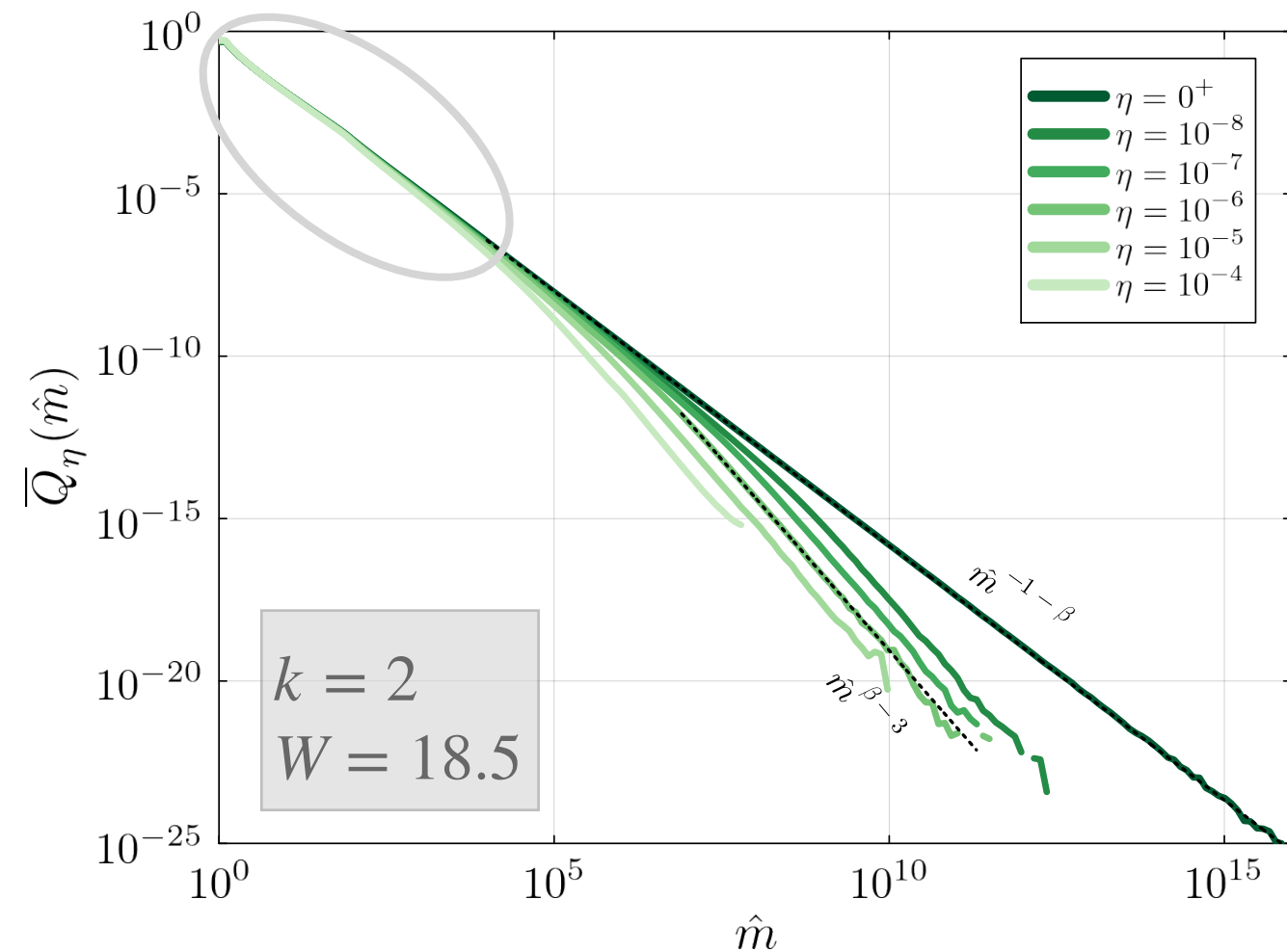
$$G_{ii}^{-1} = m_{ii} - \mathrm{i} \eta \hat{m}_{ii}$$

$$I_q = \left\langle \sum_{i=1}^N |\psi_\alpha(i)|^{2q} \delta(E_\alpha) \right\rangle$$

$$I_q(\eta) = \int d\hat{m} \hat{m}^{1-q} \overline{Q}_\eta(\hat{m})$$

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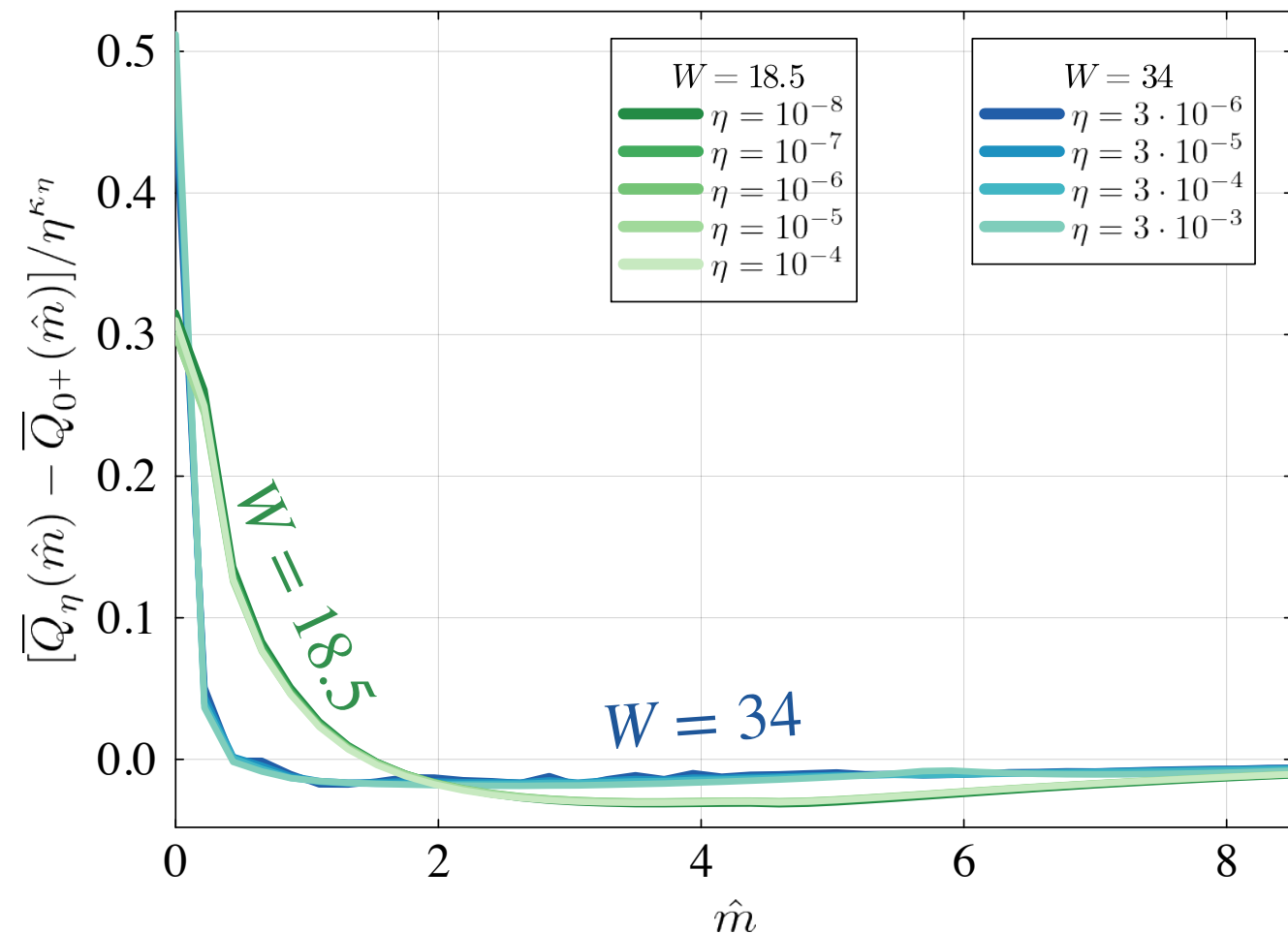


$$\bar{Q}_\eta(\hat{m}) - \bar{Q}_{0^+}(\hat{m}) = \eta^\kappa g_\eta(\hat{m})$$

$$G_{ii}^{-1} = m_{ii} - i \eta \hat{m}_{ii}$$

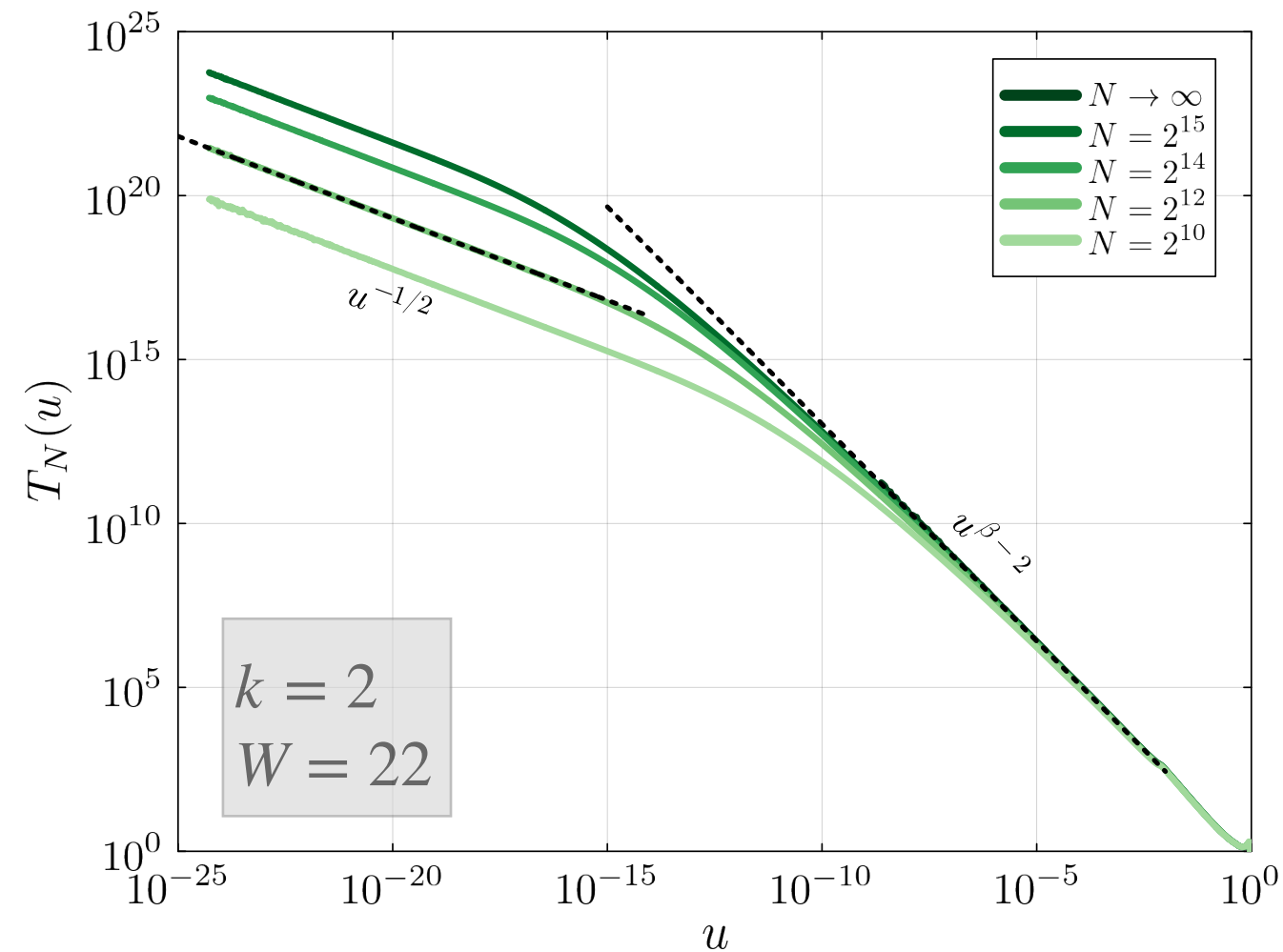
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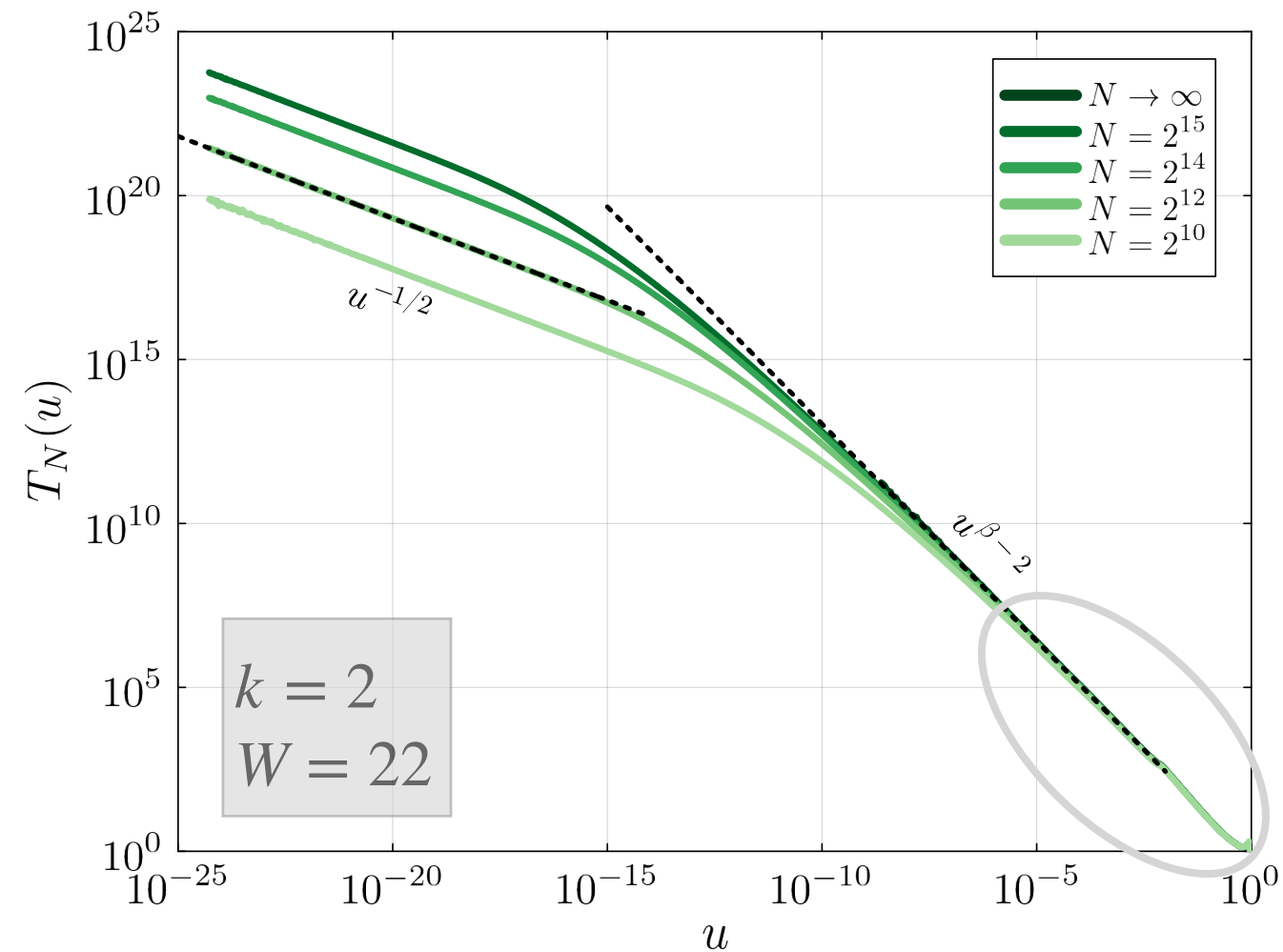
$$u = |\psi_\alpha(i)|^2$$

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$$I_q(N) = \int du u^q T_N(u)$$

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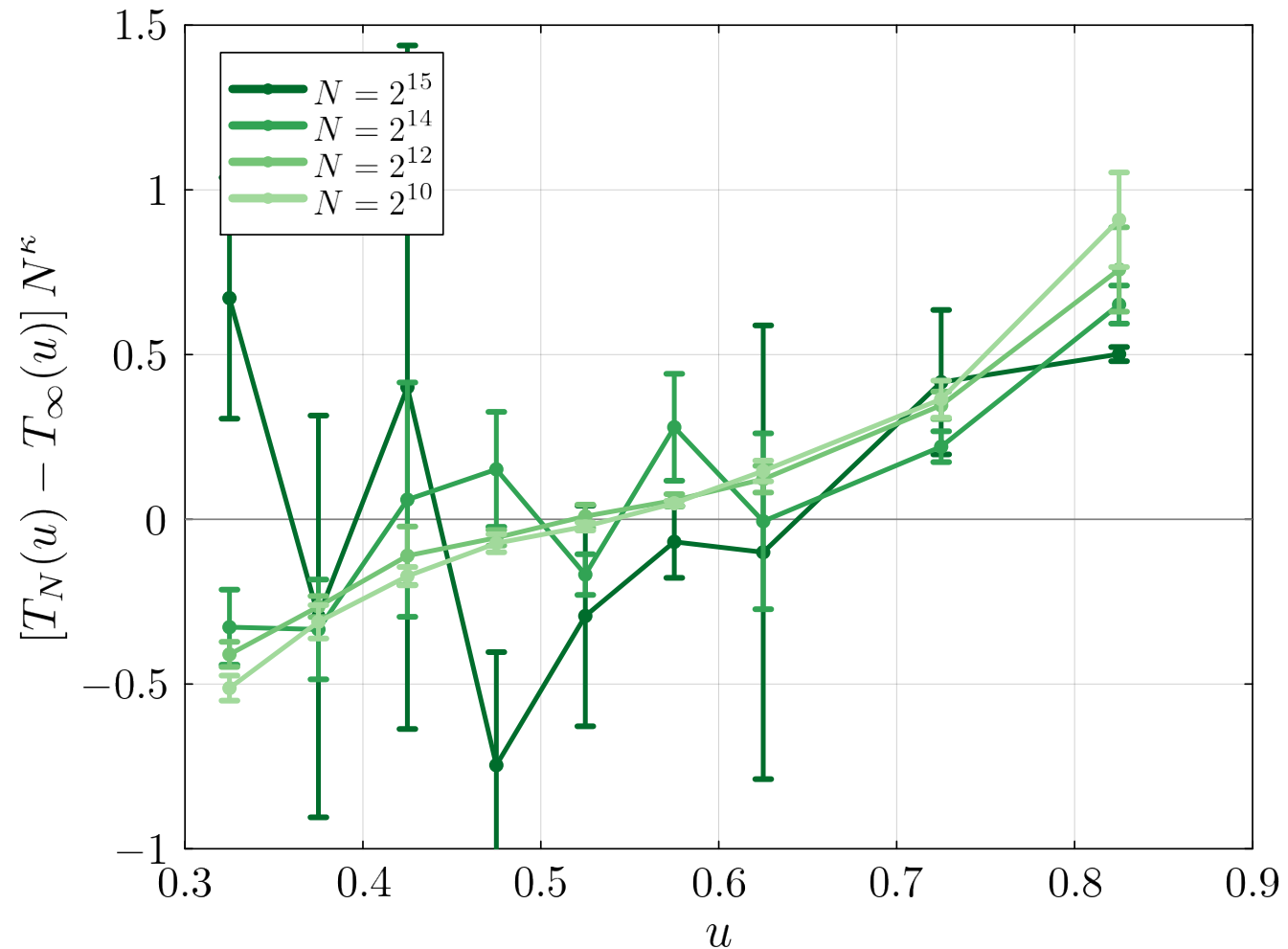


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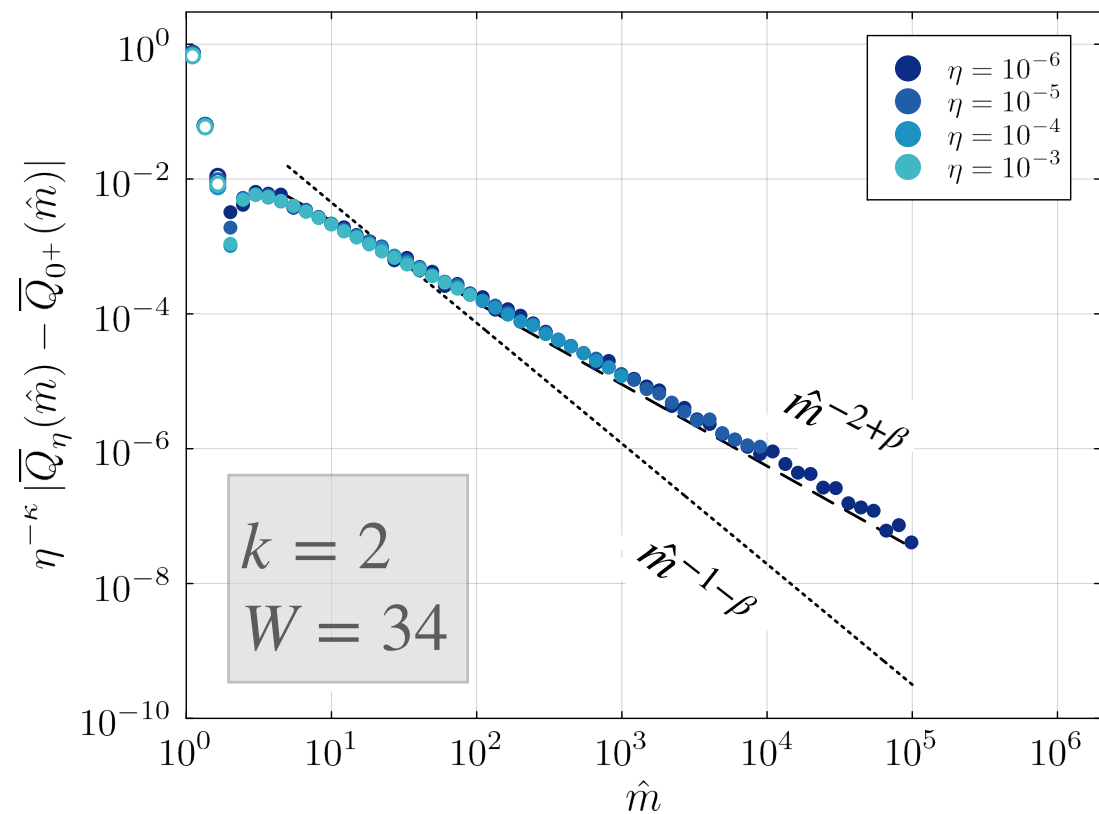
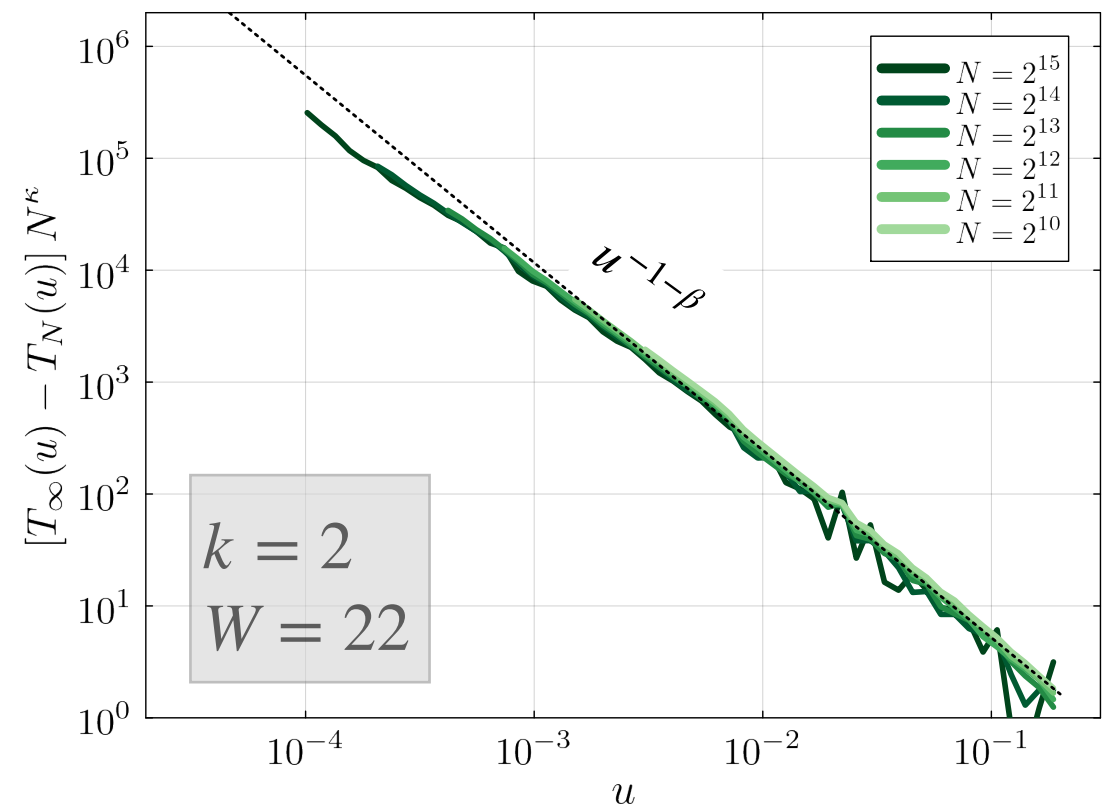
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$$T_N(u) - T_\infty(u) = N^{-\kappa} g_N(u)$$



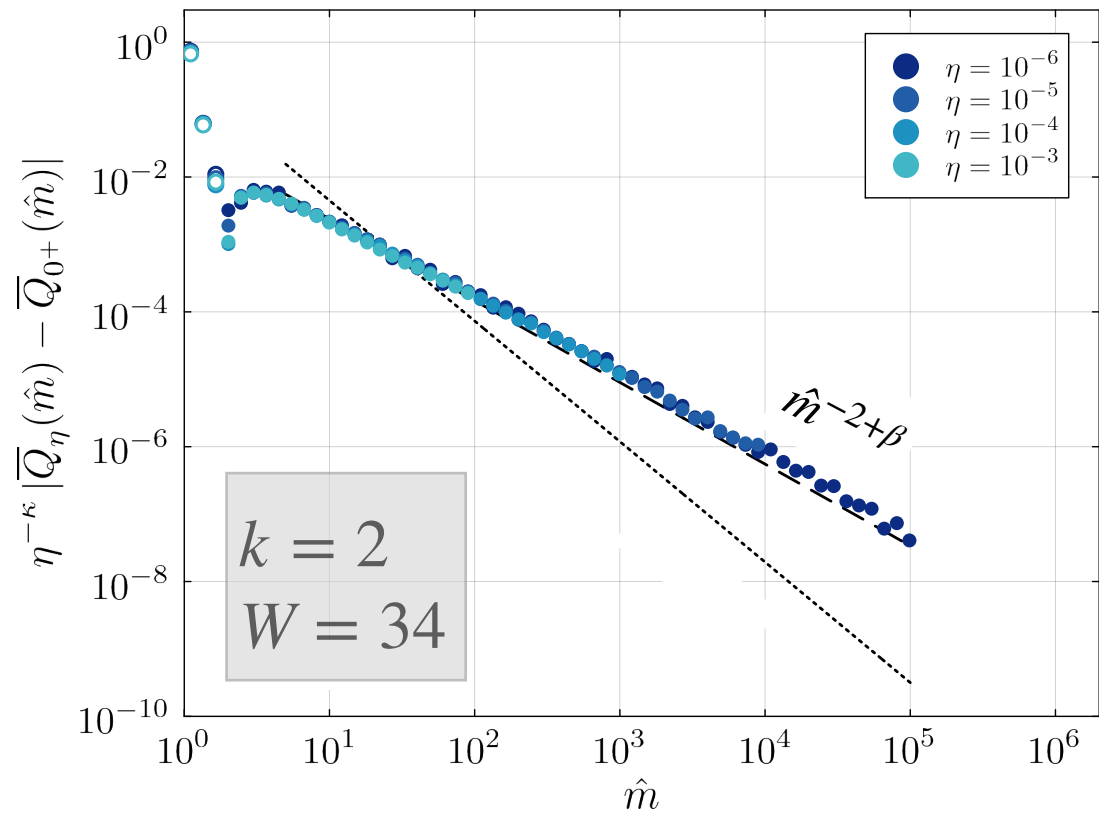
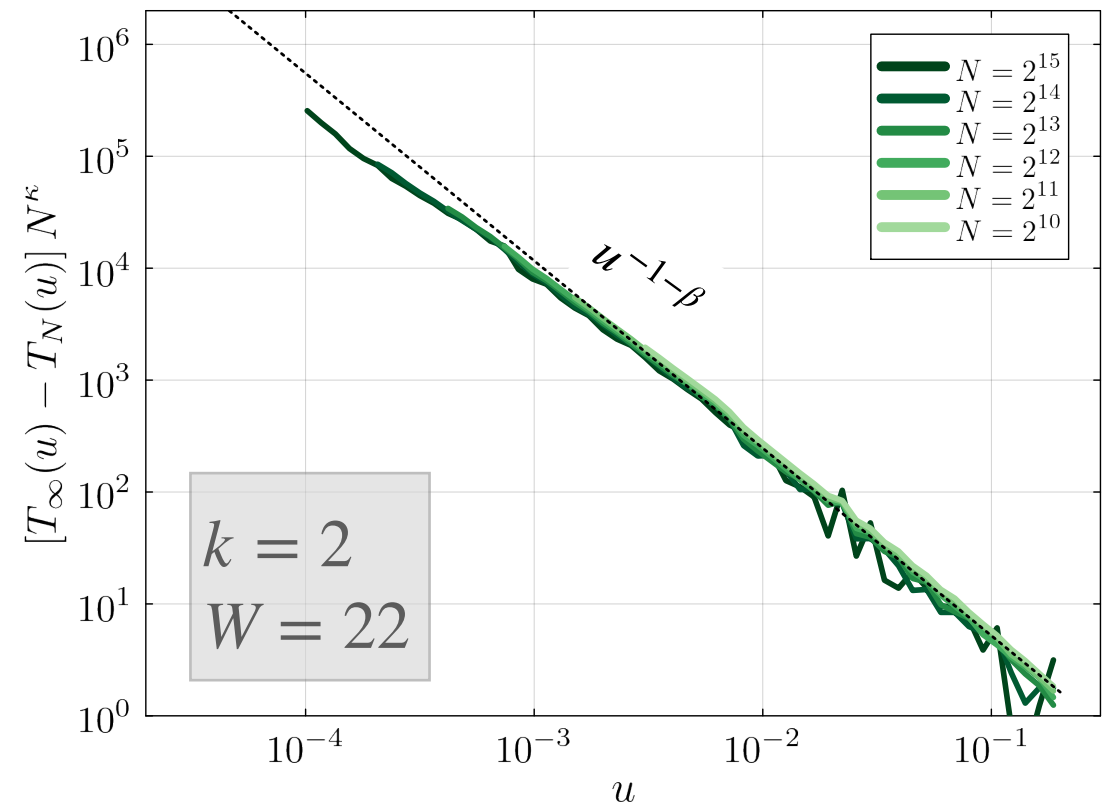
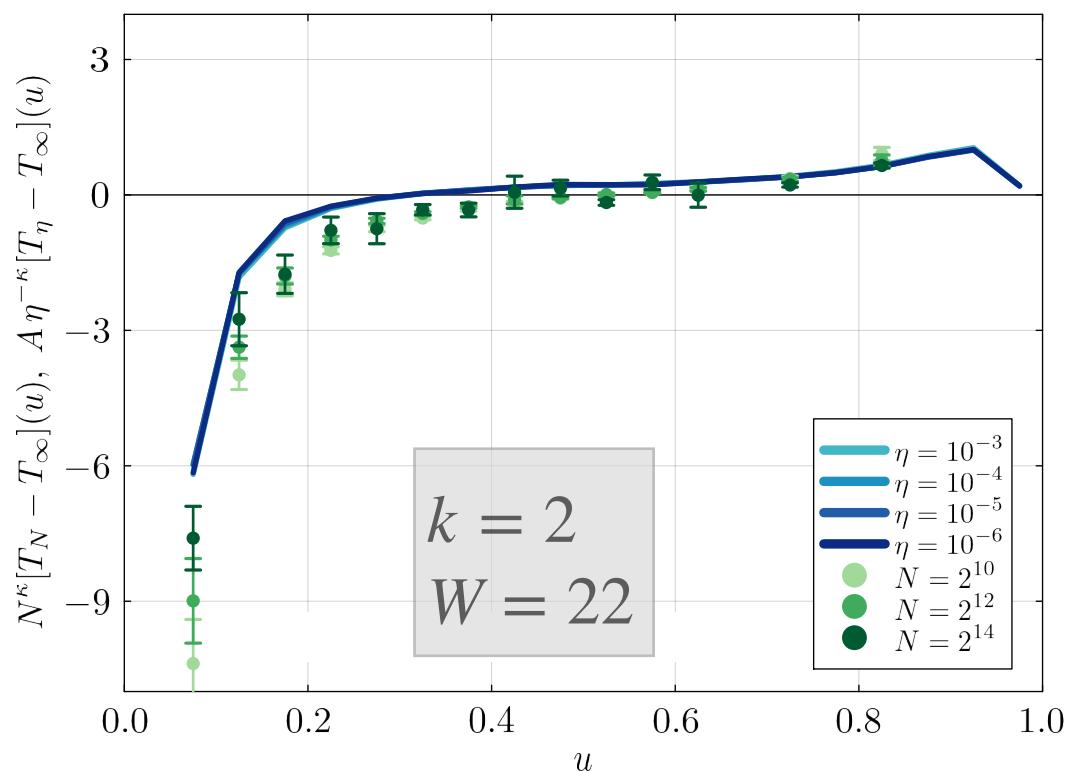
Scaling of the full distributions

The correction away from the bulk regime ($1 \ll \hat{m} \ll \eta^{-1}, N^{-1} \ll u \ll 1$) decays with the subdominant root with exponent $1 - \beta \longrightarrow$ **zero-mode of the linearized equation**



Scaling of the full distributions

The correction away from the bulk regime ($1 \ll \hat{m} \ll \eta^{-1}, N^{-1} \ll u \ll 1$) decays with the subdominant root with exponent $1 - \beta \longrightarrow$ **zero-mode of the linearized equation**



Same scaling function [Mirlin & Fyodorov '94]

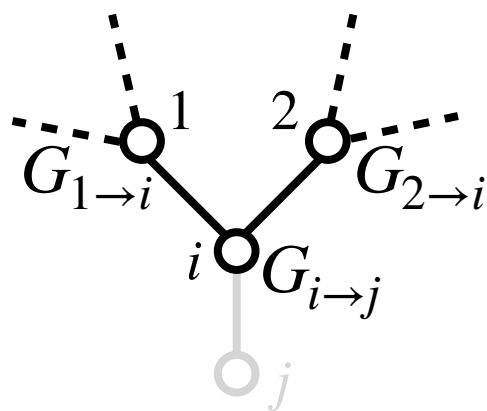
$$T_\infty(u) = u^{-3} \bar{Q}_{0+}(1/u)$$

$$g_N(u) \propto u^{-3} g_\eta(1/u)$$

Self-consistent 1-loop approximation

The corrections to the IPR asymptotic value are due to the presence of **loops** on finite RRGs

$$G_{i \rightarrow j}^{-1} = \epsilon_i - i\eta - \sum_{m \in \partial i \setminus j} G_{m \rightarrow i}$$



Self-consistent 1-loop approximation

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Generalized cavity equations, **exact on *any* lattice**, even in presence of loops
(Schur complement formula) [eg Prado Bandeira & al '26]

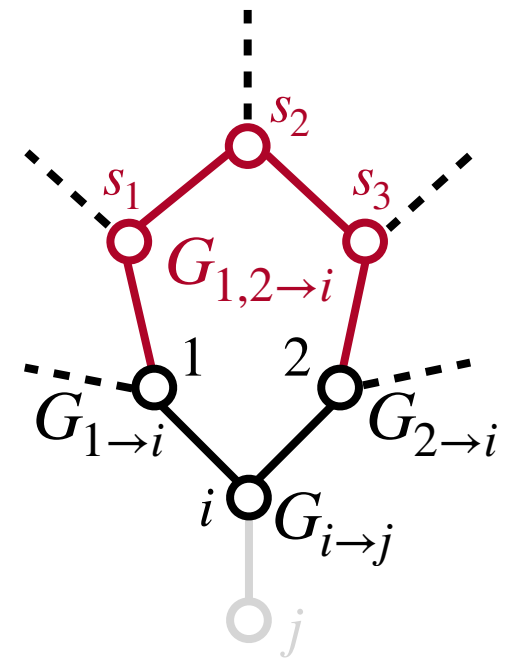
$$G_{i \rightarrow j}^{-1} = \epsilon_i - i\eta - \sum_{m \in \partial i \setminus j} G_{m \rightarrow i} - \sum_{m_1 \neq m_2 \in \partial i \setminus j} G_{m_1, m_2 \rightarrow i}$$



The off-diagonal terms vanish on the infinite Bethe lattice but cannot be neglected on finite RRGs

Self-consistent 1-loop approximation

$$G_{i \rightarrow j}^{-1} = \epsilon_i - i\eta - \sum_{m \in \partial i \setminus j} G_{m \rightarrow i} - \sum_{m_1 \neq m_2 \in \partial i \setminus j} G_{m_1, m_2 \rightarrow i}$$



- Assume that only the shortest loop contribute to the off-diagonal term

$$G_{1,2 \rightarrow i} \simeq G_{1 \rightarrow i} G_{s1 \rightarrow 1} G_{s2 \rightarrow s1} G_{s3 \rightarrow s2} G_{2 \rightarrow s3}$$

- On RRGs the number of paths of length ℓ between 1 and 2 (in absence of i) is a Poisson random variable of average $\langle \mathcal{N}(\ell) \rangle \propto k^\ell / N$ (for $\ell \ll \ln N / \ln k$)
[Wormald '99; Marinari & Monasson '04; Marinari & Semerjian '06]

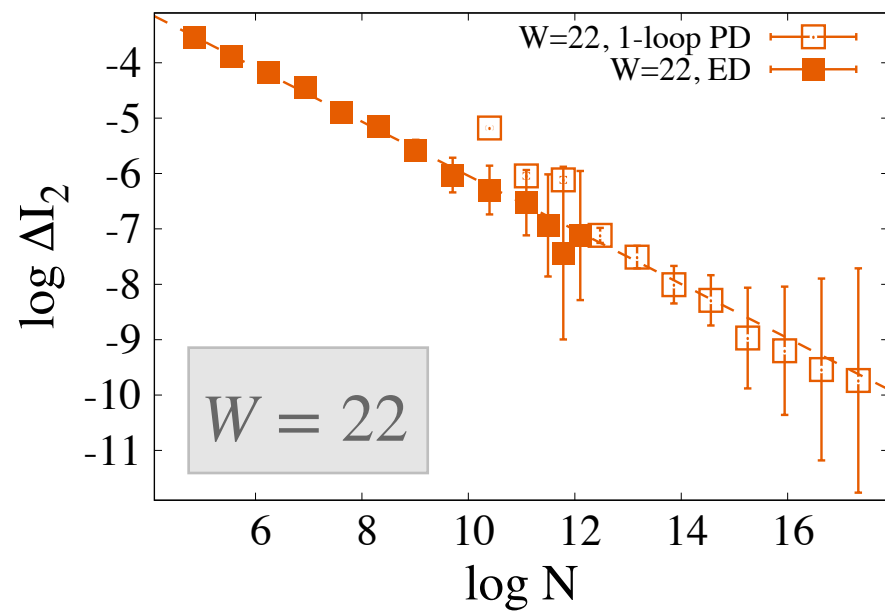
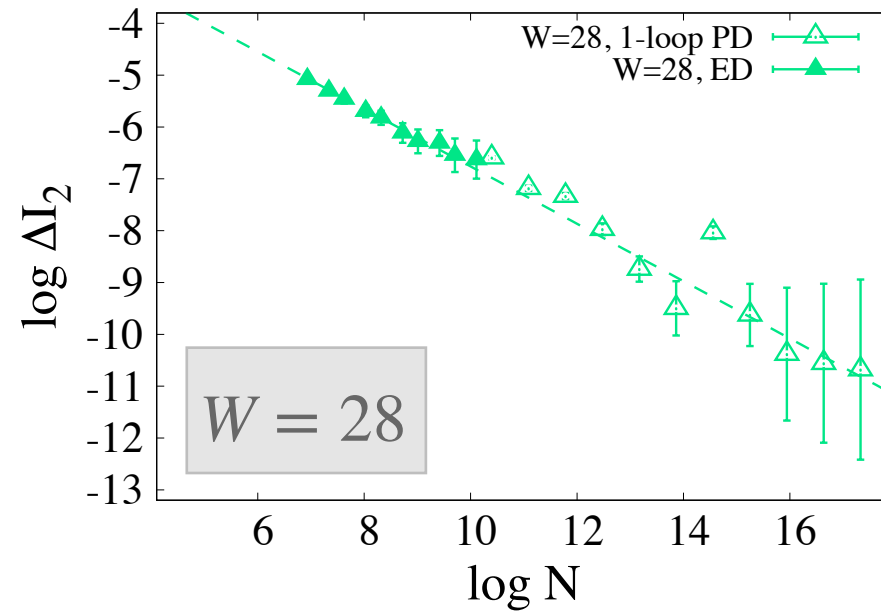
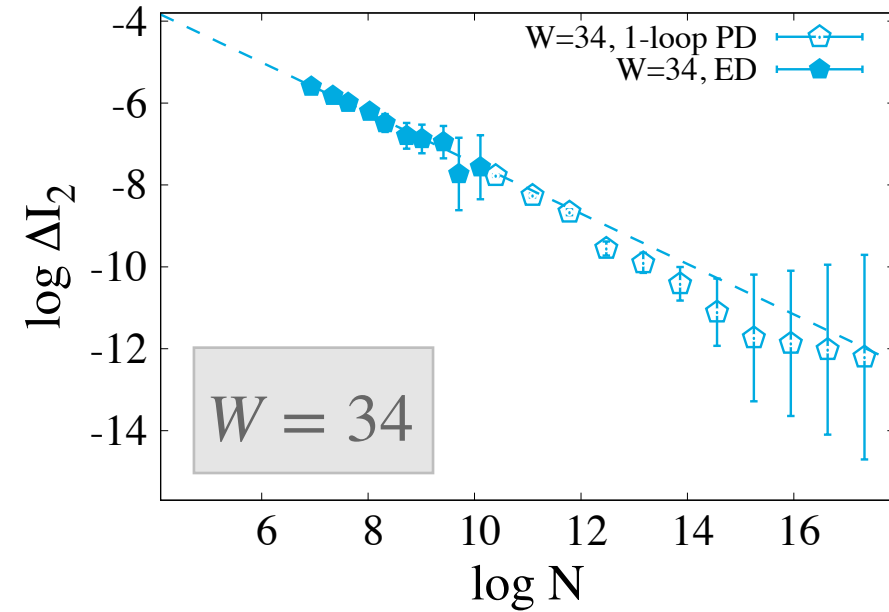
- Assume that the length ℓ of the shortest path between 1 and 2 is a i.i.d. random variable taken from the distribution

$$p(\ell) = C k^\ell \quad \text{for } \ell \in \{1, \dots, \ell_{\max}\}$$

$$\langle \mathcal{N}(\ell_{\max}) \rangle = \frac{k^{\ell_{\max}} + 2}{2N} \simeq 1 \iff \ln N \sim (\ell_{\max} + 1) \ln k$$

Self-consistent 1-loop approximation

Self-consistent 1-loop population dynamics algorithm



- Good quantitative agreement with exact diagonalizations
- Works well at strong disorder and for large N

Self-consistent 1-loop approximation

$$G_{i \rightarrow j}^{-1} = \epsilon_i - i\eta - \sum_{m \in \partial i \setminus j} G_{m \rightarrow i} - \sum_{m_1 \neq m_2 \in \partial i \setminus j} G_{m_1, m_2 \rightarrow i} = m_{i \rightarrow j} - i\eta \hat{m}_{i \rightarrow j}$$

$$\mathcal{G}_{\ell_{m_1, m_2}} = \prod_{a=0}^{\ell_{m_1, m_2}} g_a$$

$$G_{m_1, m_2 \rightarrow i} = \mathcal{G}_{\ell_{m_1, m_2}} \left(1 + i\eta \sum_{a=0}^{\ell_{m_1, m_2}} \frac{\hat{g}_a}{g_a} \right)$$

$$m_{i \rightarrow j} = \epsilon_i - \sum_{m \in \partial i \setminus j} g_{m \rightarrow i} - \sum_{m_1 \neq m_2 \in \partial i \setminus j} \mathcal{G}_{\ell_{m_1, m_2}}$$

$$\hat{m}_{i \rightarrow j} = 1 + \sum_{m \in \partial i \setminus j} \hat{g}_{m \rightarrow i} + \sum_{m_1 \neq m_2 \in \partial i \setminus j} \left(\mathcal{G}_{\ell_{m_1, m_2}} \sum_{a=0}^{\ell_{m_1, m_2}} \frac{\hat{g}_a}{g_a} \right)$$

Perturbation that the loops of a finite RRG produce on the Bethe lattice solution



Typical length of the path $\ell_{m_1, m_2} \sim \ell_{\max}$

Small and broadly distributed perturbation rare-event in the far tail of $\bar{Q}_{0+}(\hat{m})$ at a scale set by N

The response of the fixed point to any small perturbation is universal, controlled by the zero mode of the linearized equations

Finite- η and finite- N corrections are the response to the same zero mode, excited at the scale η^{-1} and N

Conclusions & Perspectives

Summary

- The corrections to the IPR in the localized phase of the Anderson model on the Bethe lattice display a highly unconventional singular behavior
- Continuously varying exponent $\kappa = 2\beta - 1$ throughout the entire localized phase
- Finite- N and finite- η corrections are governed by the same exponent
- The corrections are controlled by the heavy tail exponent β of the local density of states distribution through the zero mode solution of the linearized equations
- A self-consistent 1-loop treatment of the loops present on finite RRGs quantitatively reproduces the finite-size corrections

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- The corrections to the IPR in the localized phase of the Anderson model on the Bethe lattice display a highly unconventional singular behavior
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Perspectives / Open questions

- How to correctly perform a FSS analysis of the IPR (and of related observables)?
- Real-space origin of the anomalous exponent κ at the level of the eigenstates' geometric structure
[García Mata & al '22]
- Extending the analytical analysis from the toy non-linear σ model to the Anderson model on the Bethe lattice
- Adapt the loop expansion to more complex lattices (Hilbert space graphs of interacting quantum systems)