

Nonstabilizerness of ergodic and non-ergodic quantum dynamics

Piotr Sierant



**Barcelona
Supercomputing
Center**

Centro Nacional de Supercomputación

**Quantum thermalization breakdown:
localization and multifractality**

Oct 14th, 2026

Magic and quantum magic

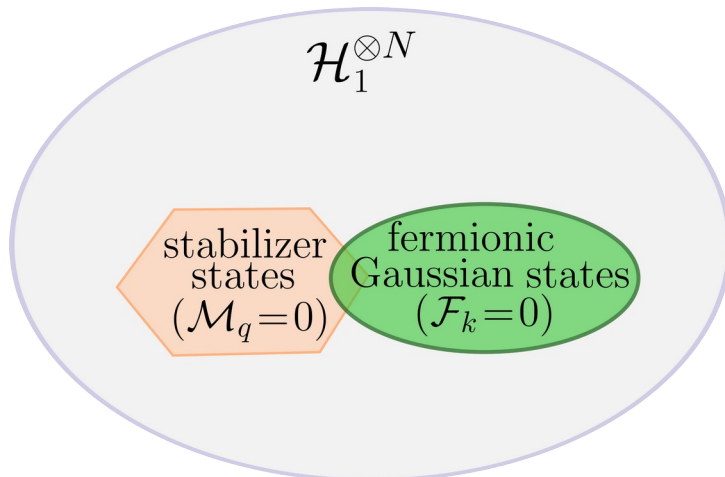
- Carlo Collodi, *The Adventures of Pinocchio* (1882) & *Pinocchio*, Walt Disney Productions (1940)



wooden puppet $\xrightarrow{\text{magic}}$ boy

- System of N qubits $\mathcal{H}_1^{\otimes N} \ni |\psi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |i_1 i_2 \dots i_N\rangle$

Classically **intractable**:



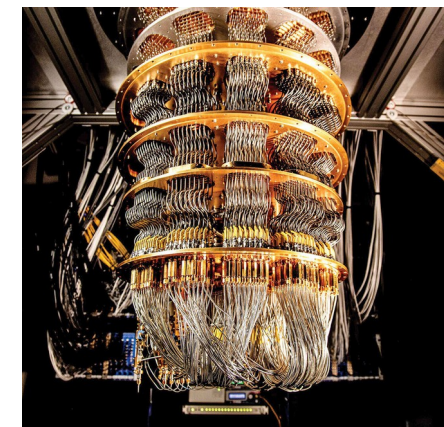
Classically **simulable**:

stabilizer states

or

fermionic Gaussian states

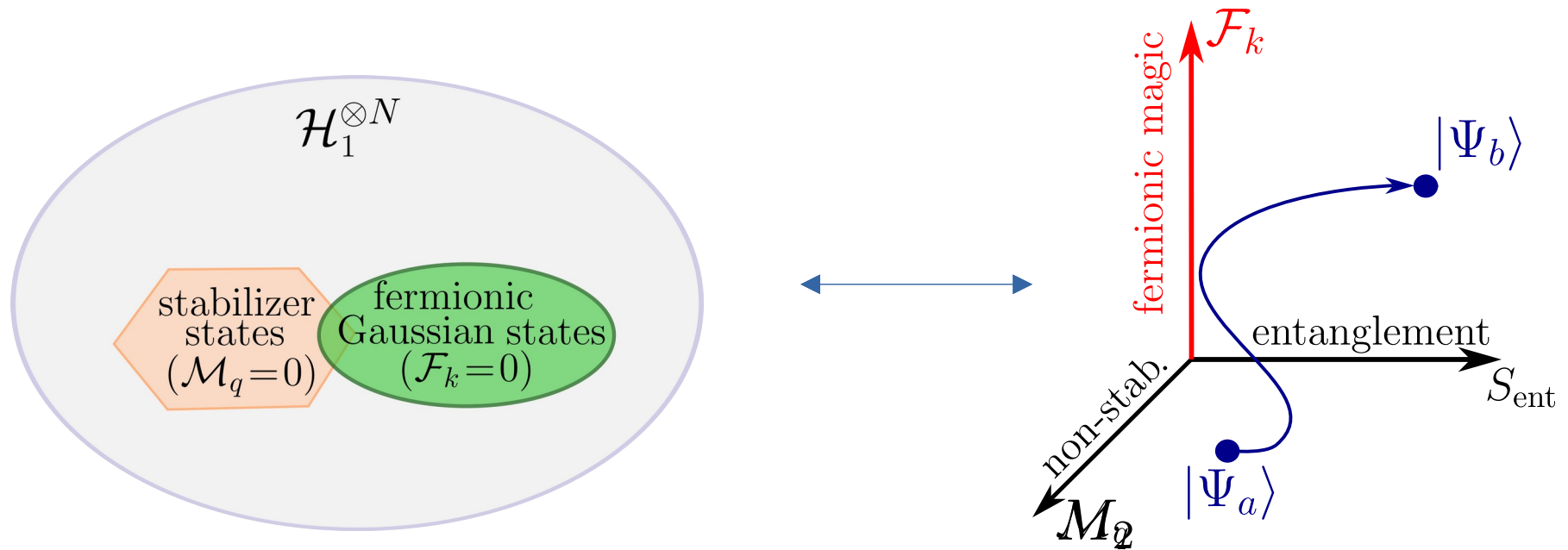
$\xrightarrow{\text{quantum magic}}$



Quantum Computer

S. Bravyi, A. Kitaev, Phys. Rev. A 71, 022316 (2005)
V. Veitch, et al., New J. Phys. 16 013009 (2014)

Magic resources in many-body systems



The idea: to quantify complexity of many-body states with respect to classically simulable ensembles of:

- stabilizer states $\longrightarrow$ nonstabilizerness, also “*magic*”
- fermionic Gaussian states $\longrightarrow$ fermionic non-Gaussianity, also “*fermionic magic*”

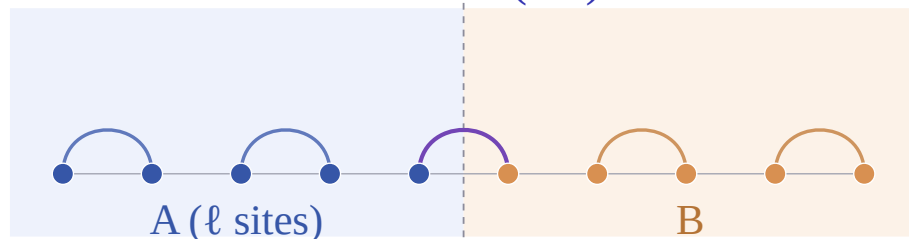
Quantum resources

- Quantum resource theories E. Chitambar, G. Gour, Rev. Mod. Phys. **91**, 025001 (2019)

Quantum resource	Entanglement	Nonstabilizerness	Fermionic non-Gaussianity
Free states	Separable states $\rho_{\text{sep}} = \sum_k p_k \rho_k^A \otimes \rho_k^B$		
Free operations	Local operations and classical communication (LOCC)		
Resourceful states	Entangled states $ \Phi^+\rangle = \frac{ 00\rangle + 11\rangle}{\sqrt{2}}$		

$$S_A = -\text{Tr}(\rho_A \log_2 \rho_A), \quad \rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$$

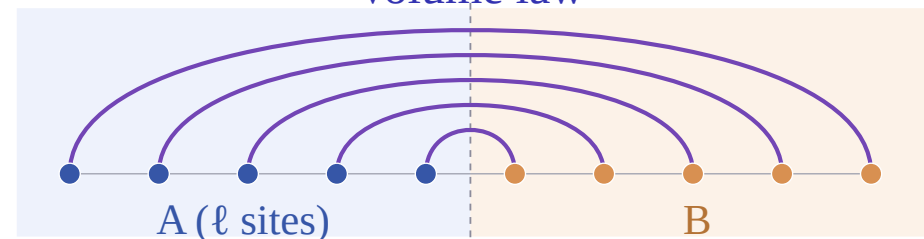
Area law (1D)



$$S_A = O(1)$$

J. Eisert, M. Cramer, M. B. Plenio, Rev. Mod. Phys. **82**, 277 (2010)
M. Žnidarič, T. Prosen, P. Prelovšek, Phys. Rev. B **77**, 064426 (2008)

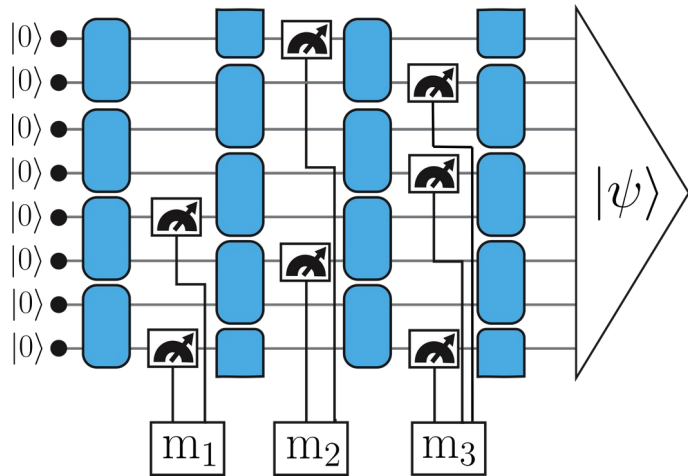
Volume law



$$S_A \propto \ell \quad (\ell \leq N/2)$$

E. Bianchi, L. Hackl, M. Kieburg, M. Rigol, L. Vidmar, PRX Quantum **3**, 030201 (2022)

Clifford circuits and non-stabilizerness



- Stabilizer state $P_j |\psi\rangle = |\psi\rangle$ where

$$P_j = e^{i\pi\phi_j} \prod_{k=1}^N X_k^{n_k^j} Z_k^{m_k^j}$$

is completely determined by the matrices (with entries=0,1)

$$M_X = [n_k^j] \quad M_Z = [m_k^j] \quad \Phi = [\phi_j]$$

- Clifford gates map Pauli string to Pauli string, simulation in time *polynomial* in N
[S. Aaronson, D. Gottesman, Phys. Rev. A **70**, 052328 \(2004\)](#)
[Stim: a fast stabilizer circuit simulator, C. Gidney, Quantum **5**, 497 \(2021\)](#)
- Given $|\psi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |i_1 i_2 \dots i_N\rangle$ how to check if it is a stabilizer state?
[H. Hamaguchi, K. Hamada, N. Yoshioka, Quantum **8**, 1461 \(2024\)](#) minimization procedure for $N \leq 9$
- Stabilizer Renyi entropy (SRE)
[L. Leone, S. Oliviero, A. Hamma, PRL **128**, 050402 \(2022\)](#)
[L. Leone, L. Bittel, PRA **110**, L040403 \(2024\)](#)
[PS, J. Valles-Muns, A. Garcia-Saez, Quantum **10**, 2059 \(2026\)](#)
 $\mathcal{P}_N$ - group of all 4^N Pauli strings

$$M_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$

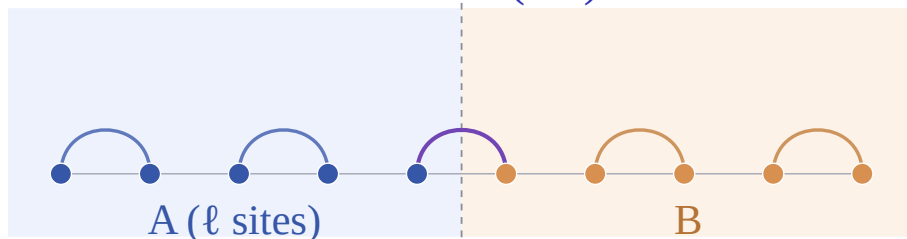
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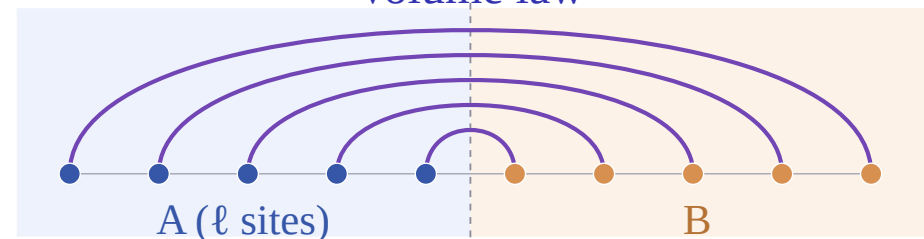
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Fermionic Gaussian states and unitaries

- System of N qubits, we define $2N$ Majorana fermions operators

$$\gamma_{2k-1} = \left(\prod_{m=1}^{k-1} Z_m \right) X_k, \quad \gamma_{2k} = \left(\prod_{m=1}^{k-1} Z_m \right) Y_k \quad \text{which anticommute} \quad \{\gamma_k, \gamma_l\} = 2\delta_{kl}\mathbb{I}$$

- fermionic Gaussian unitary $U_G = \exp \left[\frac{1}{4} \sum_{m,n=1}^{2N} H_{mn} \gamma_m \gamma_n \right]$ where $H_{mn} = -H_{nm}$

- fermionic Gaussian states $|\Psi_G\rangle = U_G |0\rangle^{\otimes N} \in \mathcal{H}_1^{\otimes N}$

- Under FGU Majorana fermions: $U_G^\dagger \gamma_m U_G = \sum_{n=1}^{2N} G_{mn} \gamma_n$ where $G = \exp(H)$

- Covariance Matrix, for any $|\Psi\rangle \in \mathcal{H}_1^{\otimes N}$ is a $2N \times 2N$ real antisymmetric matrix

$$M_{mn} = -\frac{i}{2} \langle \Psi | [\gamma_m, \gamma_n] | \Psi \rangle$$

- Under FGU $U_G: M \mapsto G^T M G$

- Wick's theorem:

[G. Wick, Phys. Rev. **80**, 268-272 \(1950\)](#)

$M = (M_{mn})$ of fermionic Gaussian state $|\Psi_G\rangle$

encodes all (higher point) correlation functions characterizing $|\Psi_G\rangle$

→ $|\Psi_G\rangle$ Simulation time polynomial in N

- Enable solution of 1D transverse field Ising model $H = -h_z \sum_{m=1}^N Z_m - J \sum_{m=1}^{N-1} X_m X_{m+1}$
[J. Surace, L. Tagliacozzo, SciPost Phys. Lect. Notes **54** \(2022\)](#)
[G. Mbeng, A. Russomanno, G. Santoro, SciPost Phys. Lect. Notes **82** \(2024\)](#)

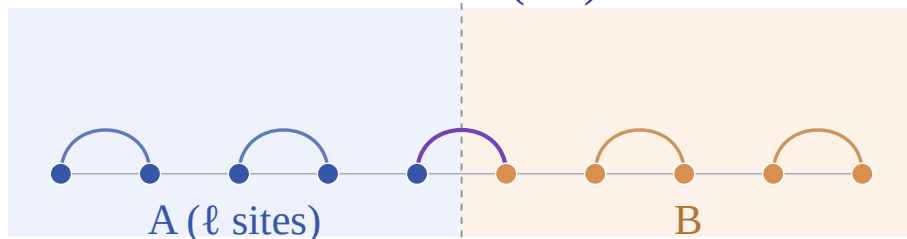
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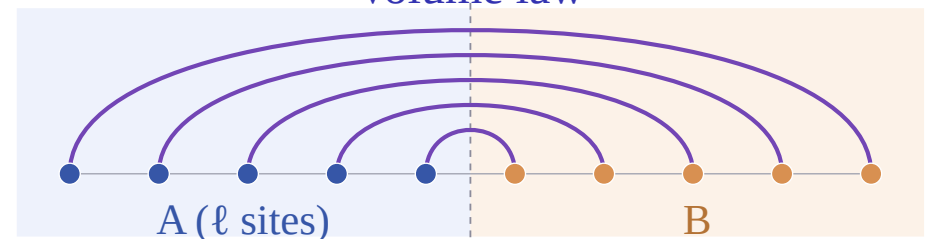
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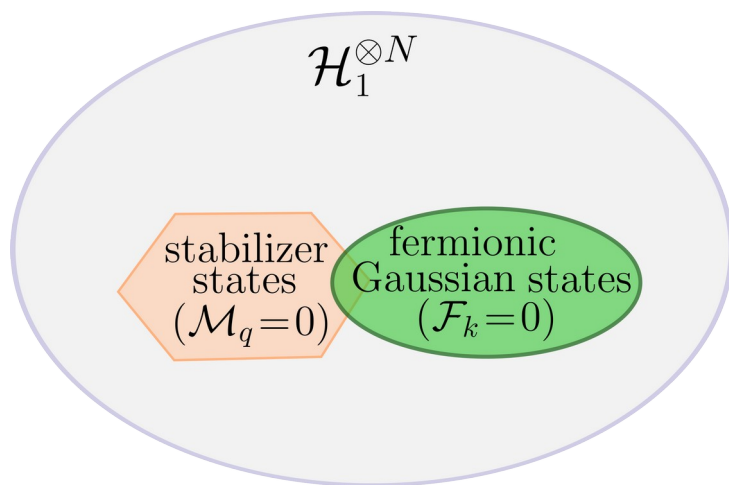


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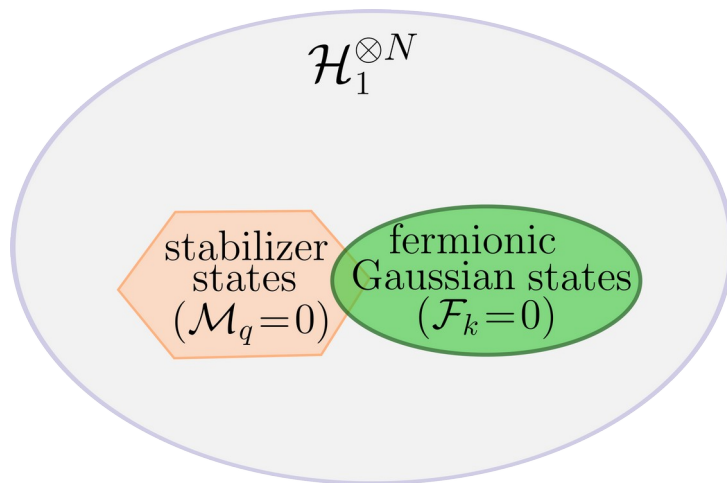
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Resourceful states	Entangled states $ \Phi^+\rangle = \frac{ 00\rangle + 11\rangle}{\sqrt{2}}$	Nonstabilizer states $T \psi_S\rangle \propto e^{i\frac{\pi}{8}Z} \psi_S\rangle$	Non-Gaussian states SWAP $ \psi_G\rangle$



Classically simulable:

stabilizer states

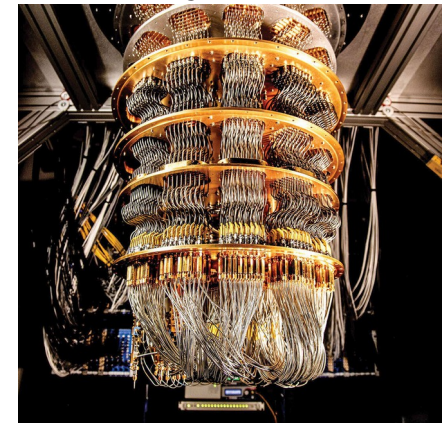
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fermionic Gaussian states

quantum magic



Classically intractable:



Measures of magic resources

- Quantum resource theories [E. Chitambar, G. Gour, Rev. Mod. Phys. **91**, 025001 \(2019\)](#)

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Resourceful states	$T \psi_S\rangle \propto e^{i\frac{\pi}{8}Z} \psi_S\rangle$	SWAP $ \psi_G\rangle$
Resource measures i) Faithful ii) Invariant iii) Additive	stabilizer fidelity $f_{\text{stab}}(\psi\rangle) = \max_{U_C} \langle\psi U_C 0\rangle^{\otimes N} ^2$	Gaussian fidelity $f_G(\psi\rangle) = \max_{U_G} \langle\psi U_G 0\rangle^{\otimes N} ^2$
	stabilizer Renyi entropy (SRE) L. Leone, S. Oliviero, A. Hamma, PRL 128, 050402 (2022) $M_2(\psi\rangle) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle\psi P \psi\rangle^4}{2^N} \right]$	Fermionic antiflatness (FAF) PS, P. Stornati, X. Turkeschi, PRX Quantum 7, 010302 (2026) $\mathcal{F}_k(\psi\rangle)$

Fermionic non-Gaussianity (or “fermionic magic”)

- How to quantify fermionic non-Gaussianity of $|\psi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |i_1 i_2 \dots i_N\rangle$?

- *Fermionic Gaussian rank*: minimal k such that $|\psi\rangle = \sum_{\alpha=1}^k b_{\alpha} |\Psi_{G, \alpha}\rangle$

M. Hebenstreit, R. Jozsa, B. Kraus, S. Strelchuk, M. Yoganathan, PRL **123**, 080503 (2019)

O. Reardon-Smith, M. Oszmaniec, and K. Korzekwa, Quantum **8**, 1549 (2024)

B. Dias and R. Koenig, Quantum **8**, 1350 (2024)

similarly: FLO extent

- Gaussian fidelity: $f_G(|\psi\rangle) = \max_{U_G} |\langle \psi | U_G | 0 \rangle^{\otimes N}|^2$
J. Cudby, S. Strelchuk, arXiv:2307.12654

- But we can avoid the optimization procedure:
PS, P. Stornati, X. Turkeshi, PRX Quantum **7**, 010302 (2026)

Fermionic antiflatness (FAF)

- For any $|\Psi\rangle \in \mathcal{H}_1^{\otimes N}$ and FGU $U_G = \exp \left[\frac{1}{4} \sum_{m,n=1}^{2N} H_{mn} \gamma_m \gamma_n \right]$
the covariance matrix $M = -\frac{i}{2} \langle \Psi | [\gamma_m, \gamma_n] | \Psi \rangle \mapsto G^T M G \implies \text{tr} (M^T M)$ is Gaussian invariant
- also: $|\Psi_G\rangle = U_G |0\rangle^{\otimes N} = \exp \left[\frac{1}{4} \sum_{m,n=1}^{2N} H_{mn} \gamma_m \gamma_n \right] |0\rangle^{\otimes N} \implies \text{tr} (M^T M) = 2N$ for FGS
- We define fermionic antiflatness (**FAF**) as $\mathcal{F}_k = N - \frac{1}{2} \text{tr} [(M^T M)^k]$
PS, P. Stornati, X. Turkeshi, PRX Quantum 7, 010302 (2026)
since: $M = Q^T \bigoplus_{m=1}^N \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix} Q$ $\mathcal{F}_k = N - \sum_{i=1}^N \lambda_i^{2k}$ measures “flatness” of $\{\lambda_i\}$
- FAF has the properties of fermionic non-Gaussianity measure:
 - faithfulness: $\mathcal{F}_k(|\psi\rangle) = 0 \iff |\psi\rangle \in \text{FGS}$ otherwise $\mathcal{F}_k(|\psi\rangle) > 0$
 - Gaussian invariance: $\mathcal{F}_k(U_G |\psi\rangle) = \mathcal{F}_k(|\psi\rangle)$
 - Additivity: $\mathcal{F}_k(|\psi\rangle \otimes |\phi\rangle) = \mathcal{F}_k(|\psi\rangle) + \mathcal{F}_k(|\phi\rangle)$ for states with fixed parity $\mathcal{P} = \prod_{k=1}^N Z_k$
- Characterization of fermionic non-Gaussianity measures that employ the covariance matrix: X. Turkeshi, PS, P. Tarabunga, arXiv:2608.10140

Quantum protocols for FAF measurement

- FAF is experimentally available via shadow tomography and related protocols:

K. Wan, W. J. Huggins, J. Lee, and R. Babbush, *Commun. Math. Phys.* **404**, 629 (2023)

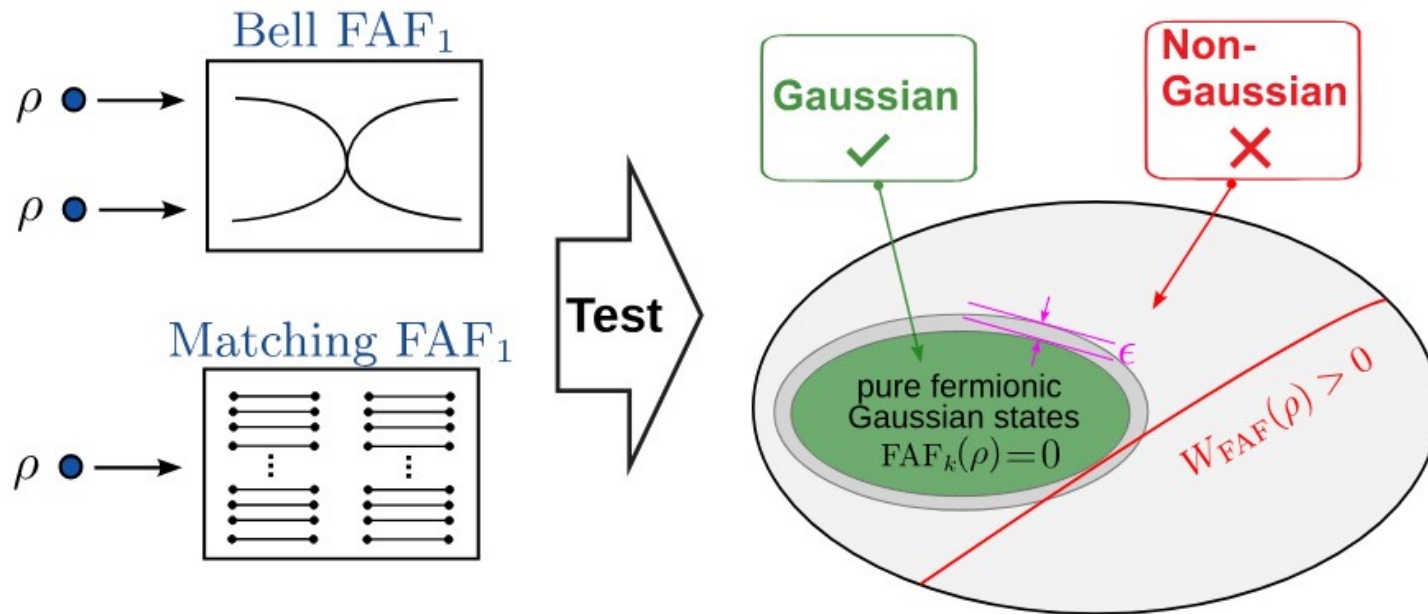
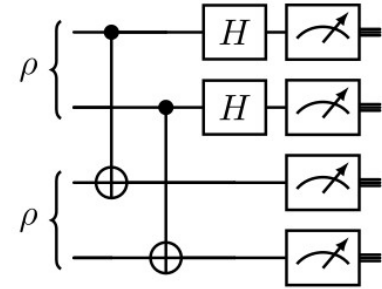
A. Zhao, N. C. Rubin, and A. Miyake, *Phys. Rev. Lett.* **127**, 110504 (2021)

J. Majsak, D. McNulty, M. Oszmaniec, *npj Quantum Info.* **11**, 61 (2025)

- Efficient protocols for FAF and Gaussianity testing

T. Haug, X. Turkeshi, PS, *arXiv:2605.26218*

$$\mathcal{F}_k = N - \frac{1}{2} \text{Tr} \left[|\psi\rangle\langle\psi|^{\otimes 2} \sum_{j \neq i} \gamma_i \gamma_j \otimes \gamma_i \gamma_j \right] = \frac{1}{2} \text{Tr} \left[\left(\sum_{a=1}^{2n} \gamma_a \otimes \gamma_a \right)^2 |\psi\rangle\langle\psi|^{\otimes 2} \right]$$

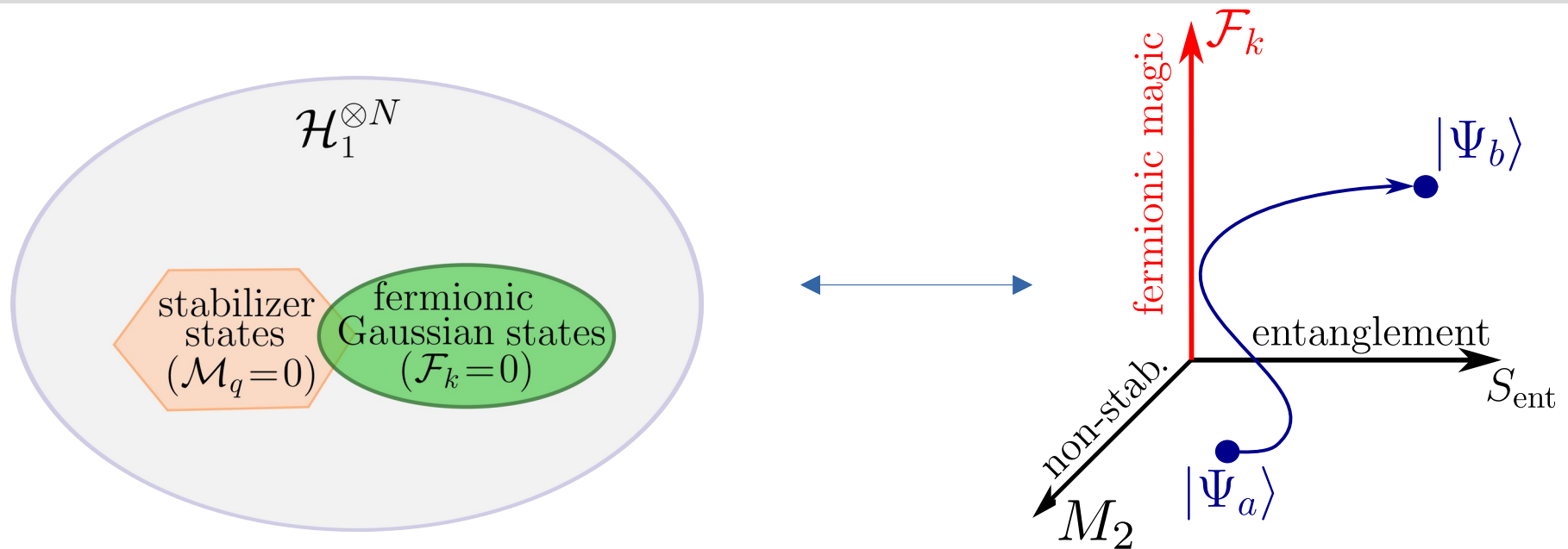


$$\epsilon_G(\psi) = D_{\text{tr}}(\psi, \phi_G) = \sqrt{1 - \max_{\phi_G \in \mathcal{G}} |\langle \phi_G | \psi \rangle|^2}$$

Lemma 1 (Distance-to-FAF bound). *For every pure state $|\psi\rangle$*

$$\frac{1}{4n} \text{FAF}_1(\psi) \leq \epsilon_G^2(\psi) \leq \frac{1}{2} \text{FAF}_1(\psi). \quad (5)$$

Magic resources in many-body systems



The idea: to quantify complexity of many-body states with respect to classically simulable ensembles of:

- stabilizer states $\longrightarrow$ nonstabilizerness, also “*magic*”

$$\text{SRE: } M_2(|\psi\rangle) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{|\langle \psi | P | \psi \rangle|^4}{2^N} \right]$$

- fermionic Gaussian states $\longrightarrow$ fermionic non-Gaussianity, also “*fermionic magic*”

$$\text{FAF: } \mathcal{F}_k(|\psi\rangle) \equiv N - \frac{1}{2} \text{tr} [(M^T M)^k]$$

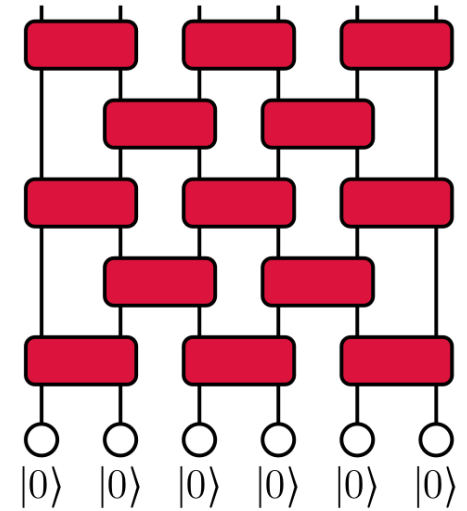
Random quantum circuit dynamics

- Random quantum circuits – minimal models of generic local unitary dynamics

M. Fisher, V. Khemani, A. Nahum, S. Vijay, *Ann. Rev. of Cond. Matt. Phys.* **14**, 335-379 (2022)

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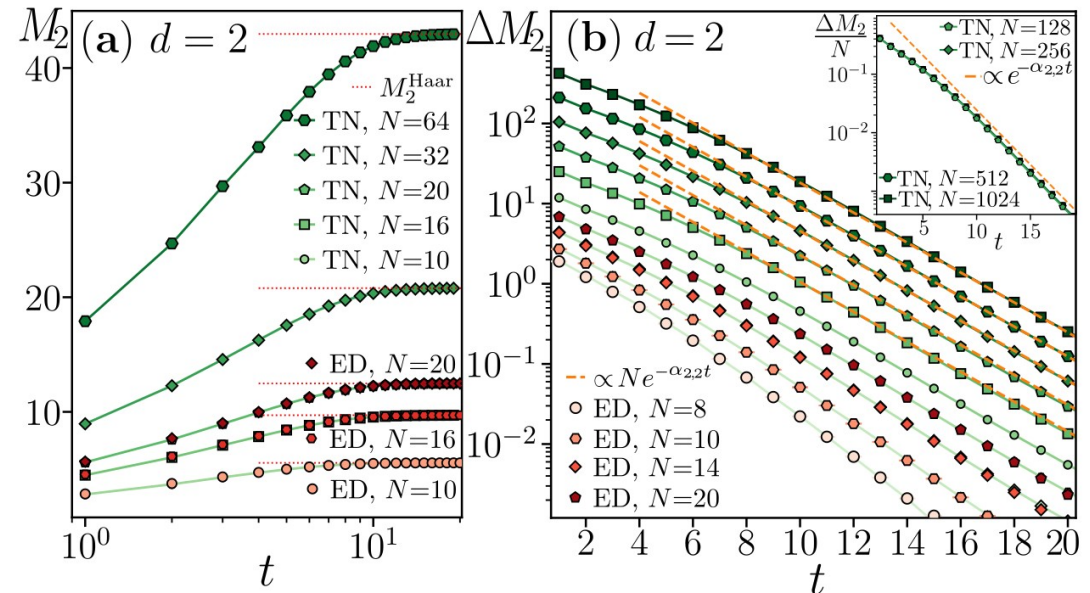
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- SRE under random circuit dynamics:

X. Turkeshi, E. Tirrito, PS, Nat. Comm. **16**, 2575 (2025)

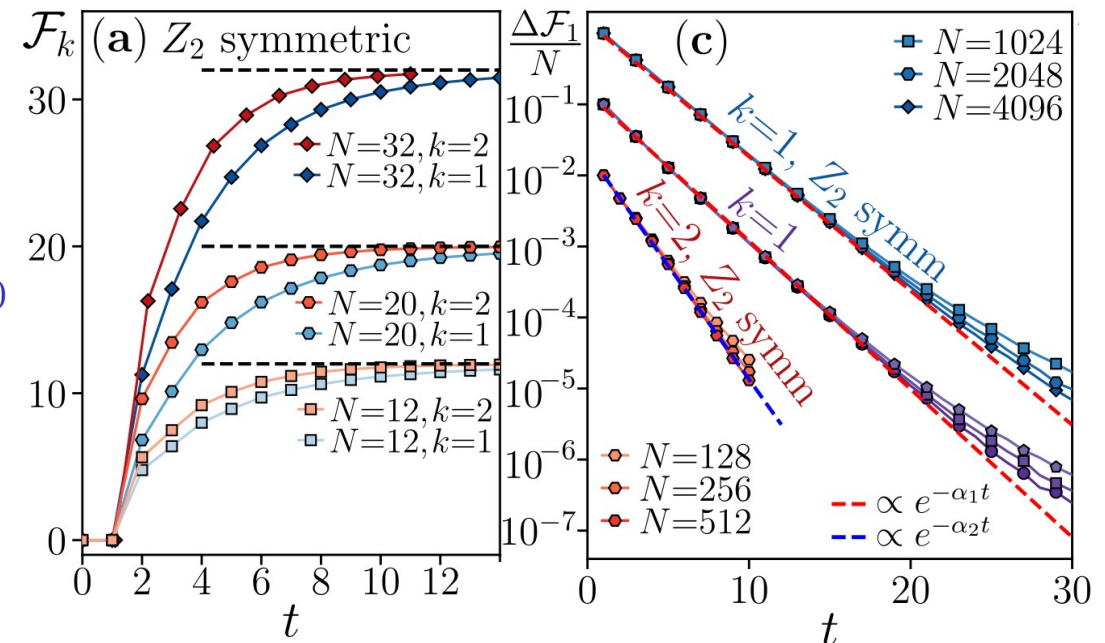
saturation at $t_{\text{sat}} \propto \log(N)$



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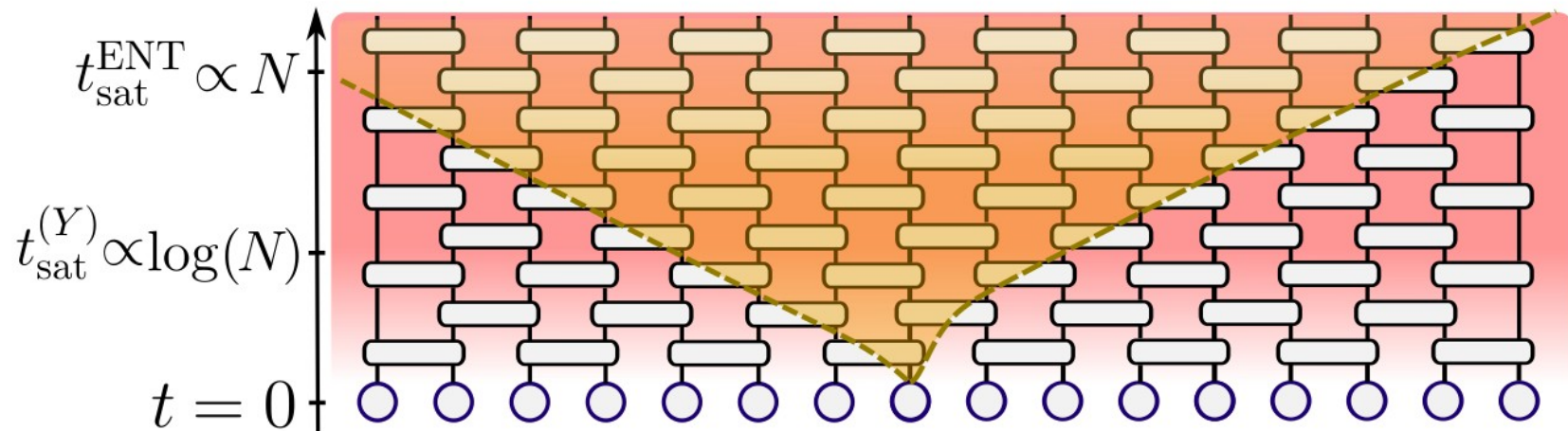
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SRE in mixed field Ising model

- Hamiltonian:
$$H = \sum_{i=1}^N gX_i + \sum_{i=1}^N h_i Z_i + \sum_{i=1}^{N-1} JZ_i Z_{i+1} \quad \mathcal{M}_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$

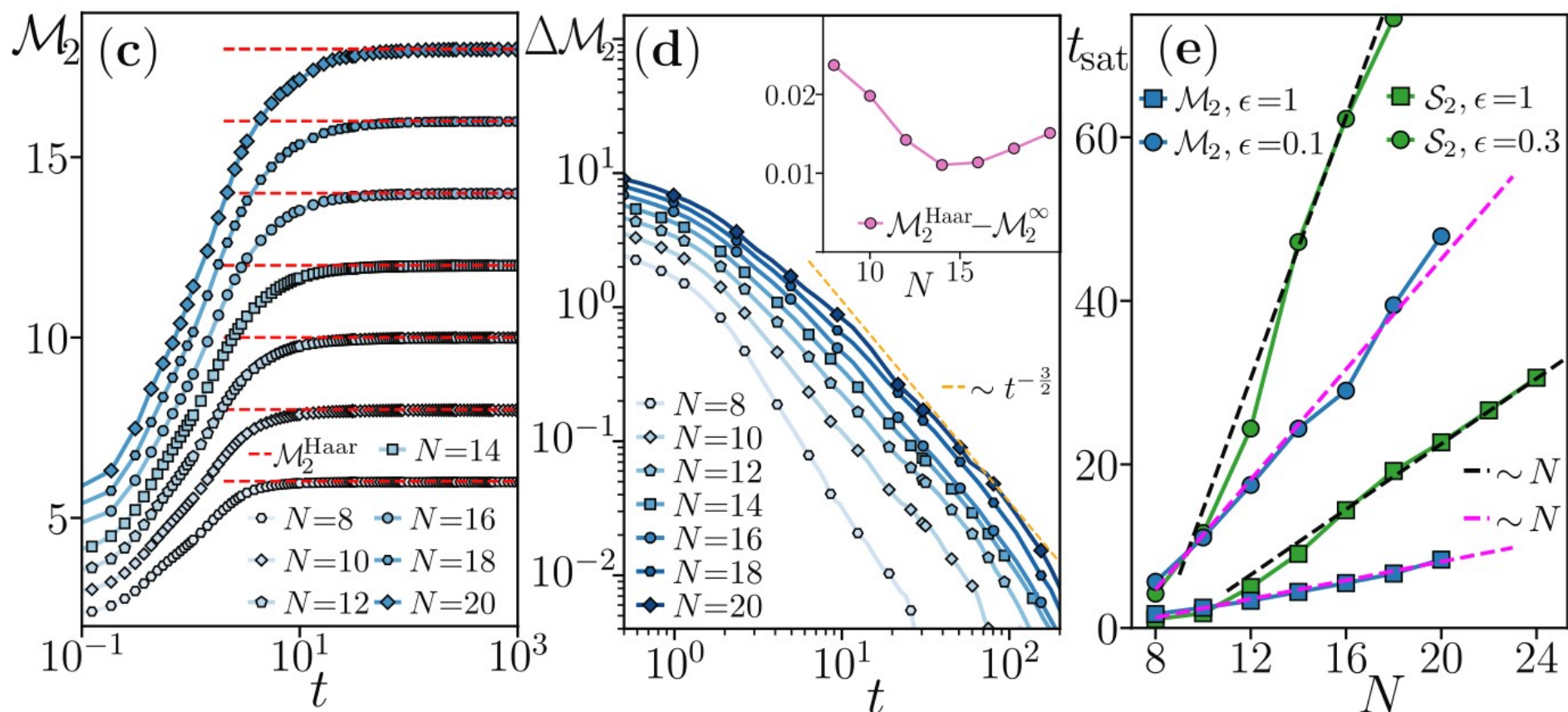
$$J=1, \quad g=(\sqrt{5}+5)/8, \quad h_i=(\sqrt{5}+1)/4$$

H. Kim, D. Huse, Phys. Rev. Lett. **111**, 127205 (2013)

M. Pandey, P. Claeys, D. Campbell, A. Polkovnikov, D. Sels, Phys. Rev. X **10**, 041017 (2020)

- Initial state: $|\psi_0\rangle = \prod_j U_j |0\rangle^{\otimes N}$ where $U_j \in SU(2)$ (Haar random)

$$\Delta \mathcal{M}_2(t) \equiv \mathcal{M}_2^\infty - \mathcal{M}_2(t)$$

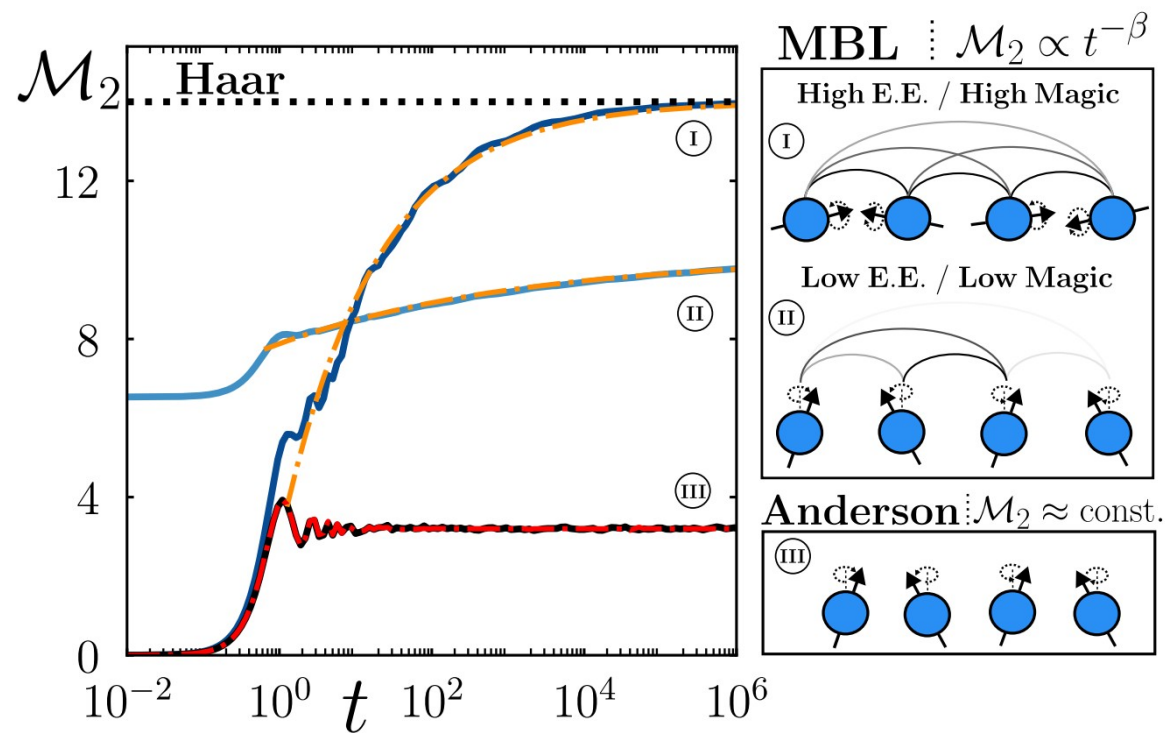
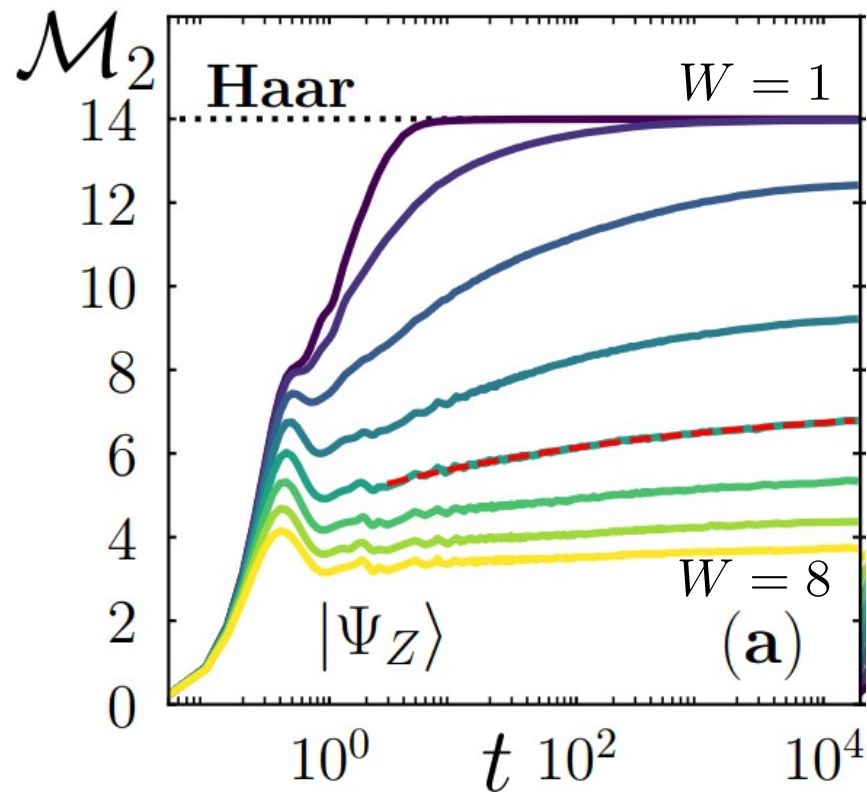


SRE in *disordered* mixed field Ising model

- Hamiltonian:
$$H_{\text{dis}} = \sum_{i=1}^{L-1} J_i Z_i Z_{i+1} + \sum_{i=1}^L h_i Z_i + \sum_{i=1}^L X_i$$

$$J_i \in [0.8, 1.2]$$

$$h_i \in [-W, W]$$
- SRE time evolution:



SRE in *disordered* mixed field Ising model

- Hamiltonian:
$$H_{\text{dis}} = \sum_{i=1}^{L-1} J_i Z_i Z_{i+1} + \sum_{i=1}^L h_i Z_i + \sum_{i=1}^L X_i$$

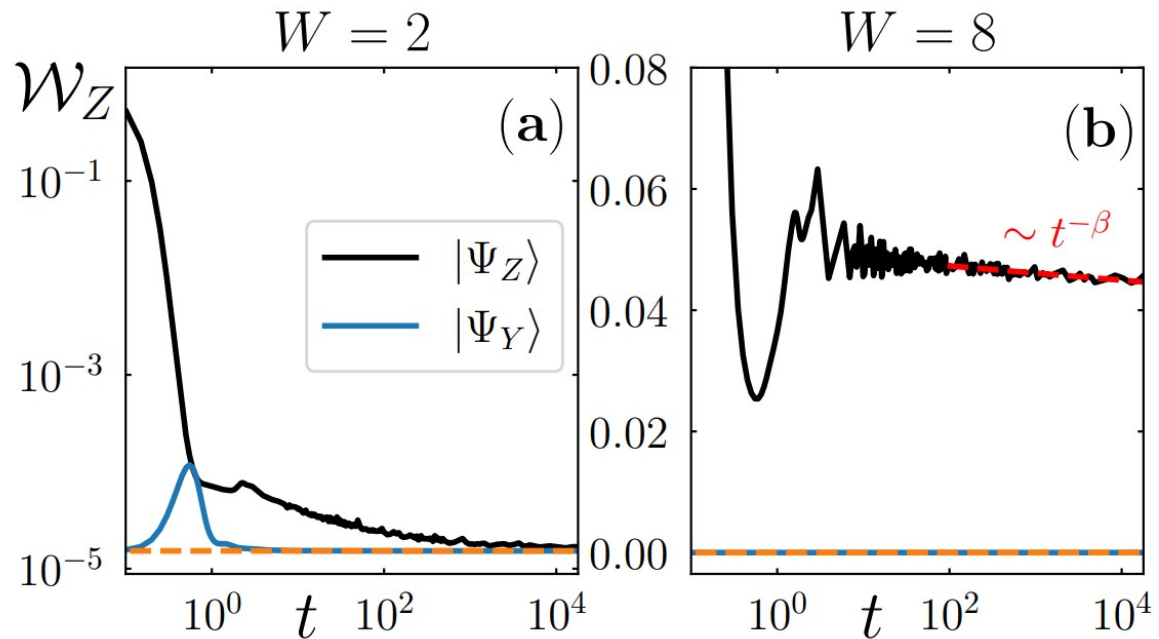
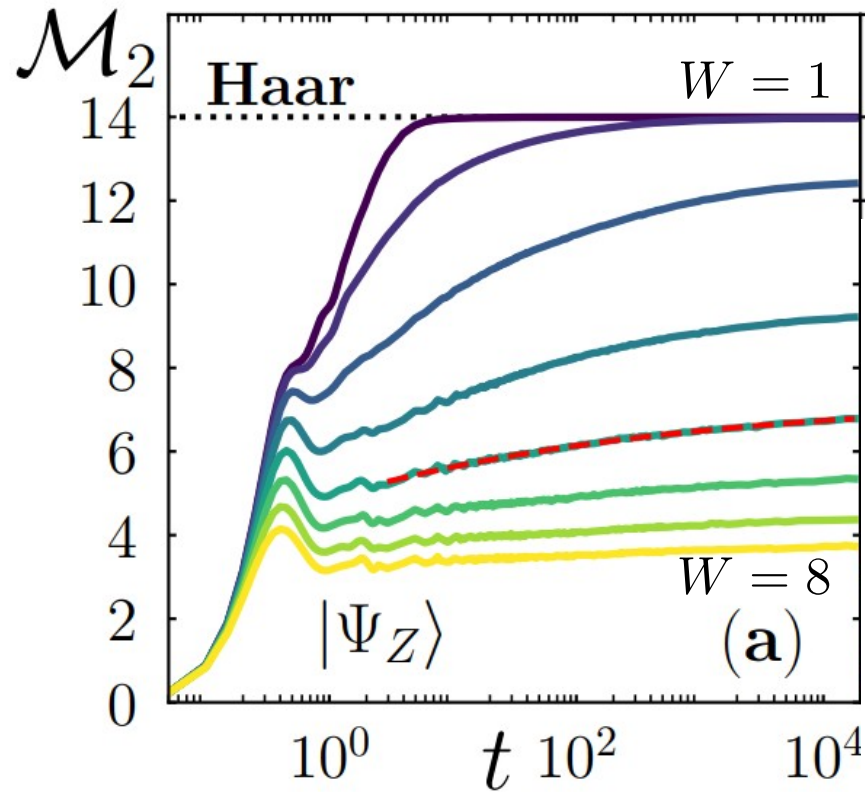
$$J_i \in [0.8, 1.2]$$

$$h_i \in [-W, W]$$

- SRE time evolution:

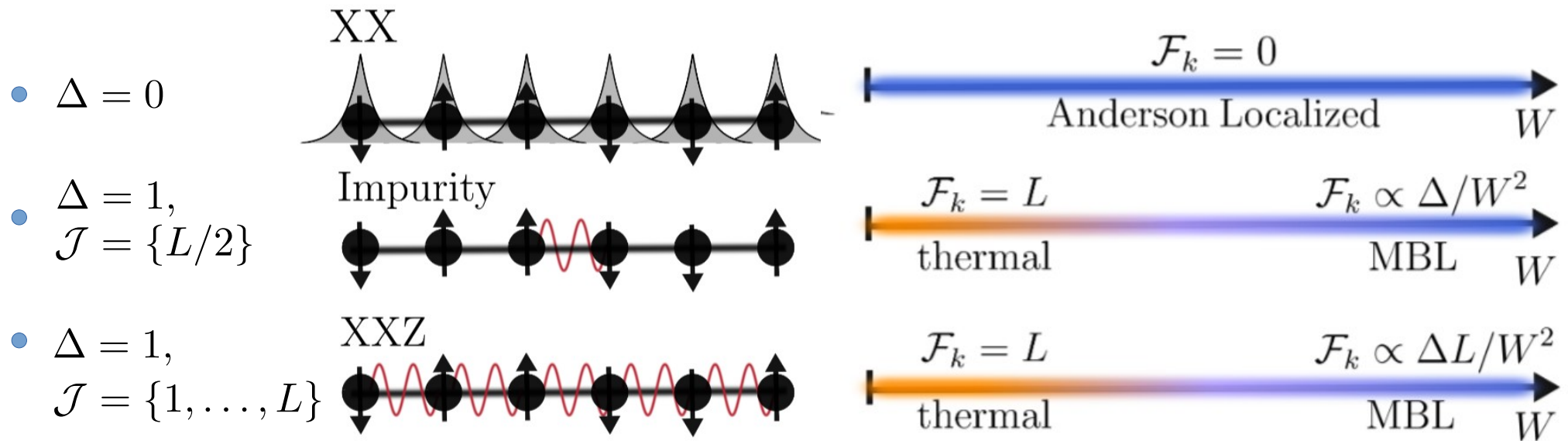
- The SRE:
$$M_2(|\psi\rangle) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{|\langle \psi | P | \psi \rangle|^4}{2^N} \right]$$

- The Z-sector:
$$\mathcal{W}_Z(|\Psi\rangle) = \sum_{P \in \mathcal{P}_{IZ}} \frac{|\langle \Psi | P | \Psi \rangle|^4}{D}$$



Fermionic non-Gaussianity and MBL

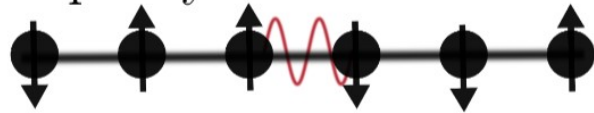
- Hamiltonian: $\hat{\mathcal{H}} = \frac{1}{4} \sum_{j=1}^L \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + \frac{1}{2} \sum_{j=1}^L h_j \hat{\sigma}_j^z + \sum_{i \in \mathcal{J}} \frac{\Delta}{4} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \quad h_j \in [-W, W]$



Fermionic non-Gaussianity and MBL

- Hamiltonian: $\hat{\mathcal{H}} = \frac{1}{4} \sum_{j=1}^L \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + \frac{1}{2} \sum_{j=1}^L h_j \hat{\sigma}_j^z + \sum_{i \in \mathcal{J}} \frac{\Delta}{4} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$ $h_j \in [-W, W]$

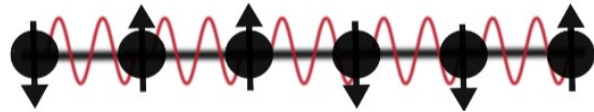
Impurity



$$\Delta = 1, \quad \mathcal{J} = \{L/2\}$$

$$\text{FAF: } \mathcal{F}_1 = N - \frac{1}{2} \text{tr} [M^T M]$$

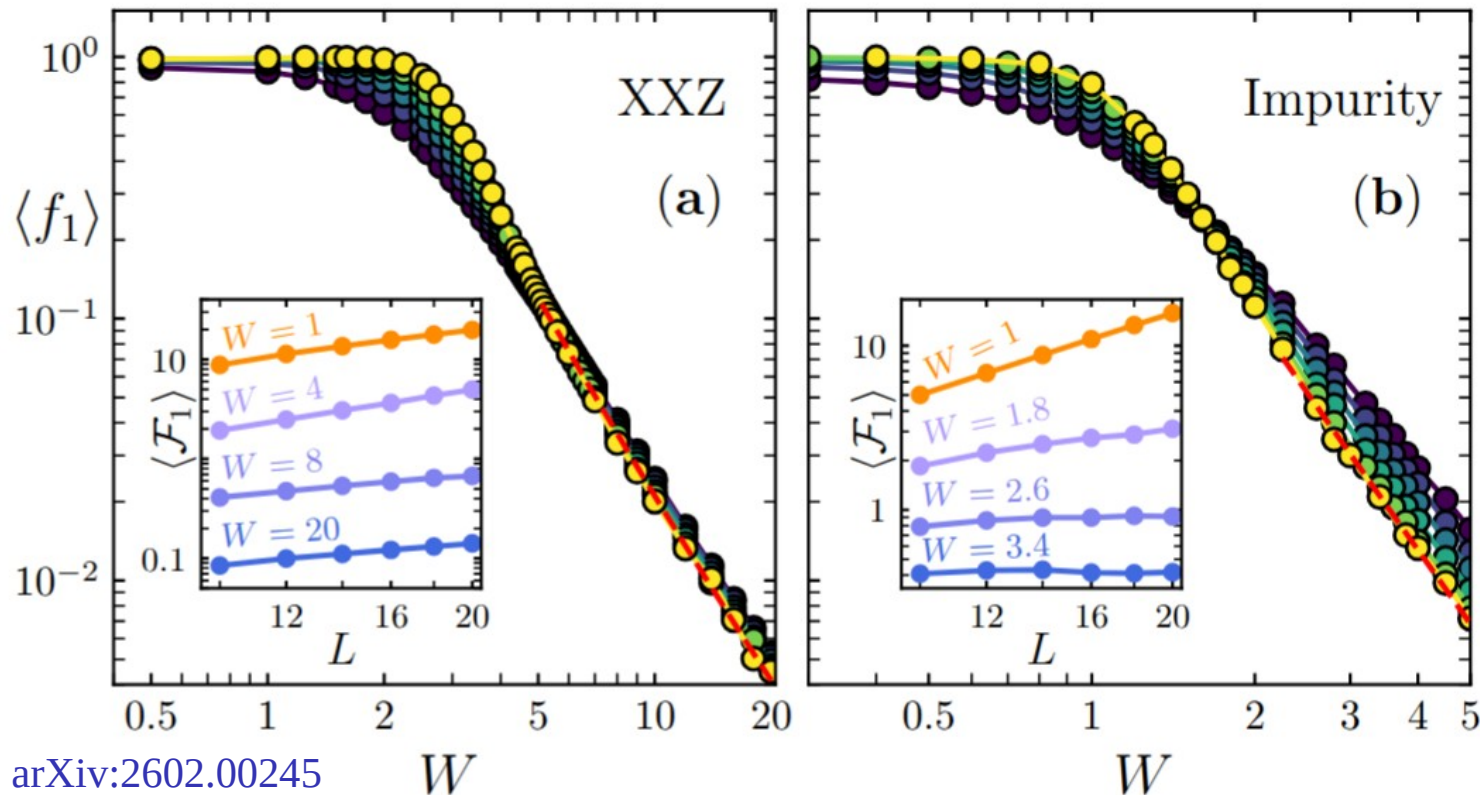
XXZ



$$\Delta = 1, \quad \mathcal{J} = \{1, \dots, L\}$$

$$f_1 = \mathcal{F}_1 / N$$

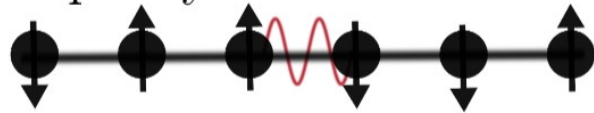
$L = 10$ $L = 12$ $L = 14$ $L = 16$ $L = 18$ $L = 20$



Fermionic non-Gaussianity and MBL

- Hamiltonian: $\hat{\mathcal{H}} = \frac{1}{4} \sum_{j=1}^L \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + \frac{1}{2} \sum_{j=1}^L h_j \hat{\sigma}_j^z + \sum_{i \in \mathcal{J}} \frac{\Delta}{4} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$ $h_j \in [-W, W]$

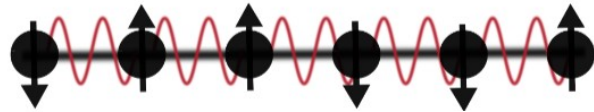
Impurity



$$\Delta = 1, \quad \mathcal{J} = \{L/2\}$$

$$\text{FAF: } \mathcal{F}_1 = N - \frac{1}{2} \text{tr} [M^T M]$$

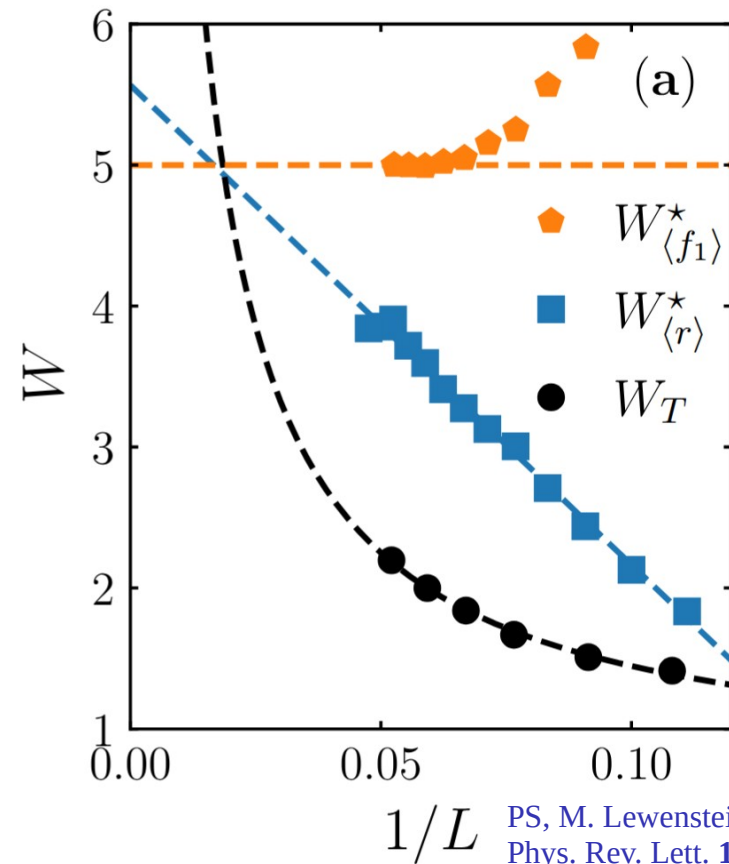
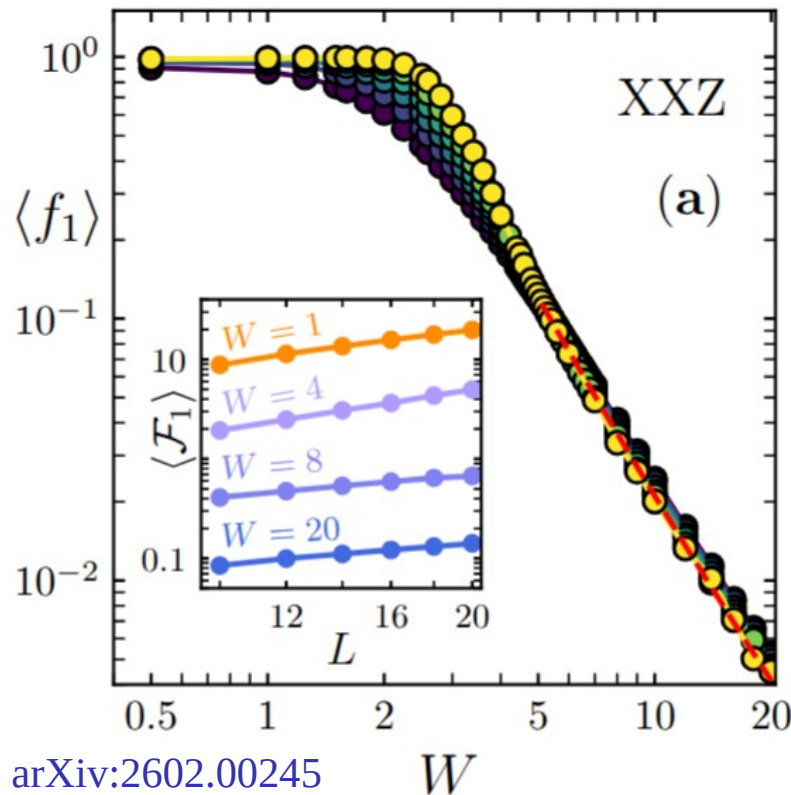
XXZ



$$\Delta = 1, \quad \mathcal{J} = \{1, \dots, L\}$$

$$f_1 = \mathcal{F}_1 / N$$

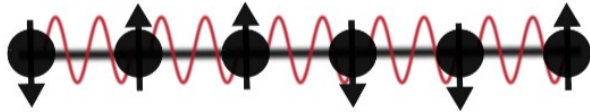
$L = 10$ $L = 12$ $L = 14$



Fermionic non-Gaussianity and MBL

- Hamiltonian: $\hat{\mathcal{H}} = \frac{1}{4} \sum_{j=1}^L \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + \frac{1}{2} \sum_{j=1}^L h_j \hat{\sigma}_j^z + \sum_{i \in \mathcal{J}} \frac{\Delta}{4} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \quad h_j \in [-W, W]$

XXZ

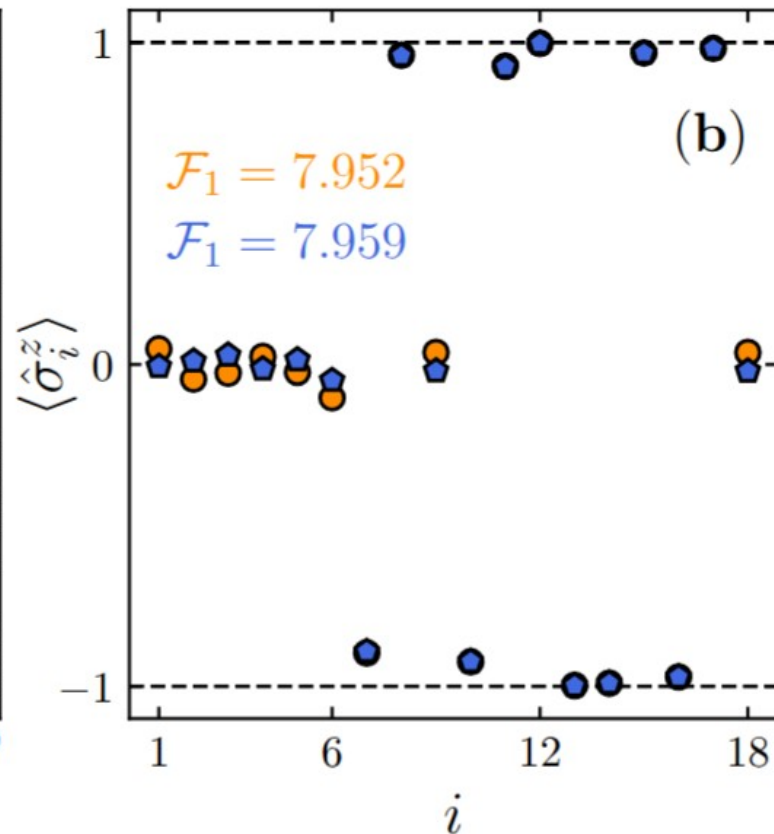
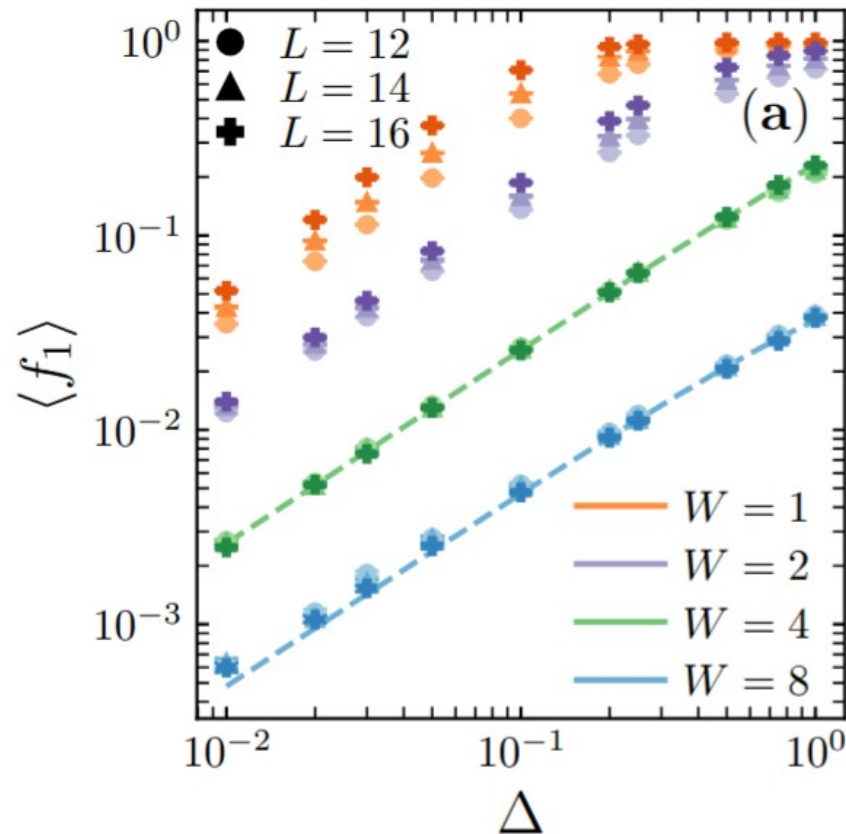


- Long-range cat-like resonant eigenstates

J. Colbois, F. Alet, N. Laflorencie, Phys. Rev. Lett. **133**, 116502 (2024)

J. Colbois, F. Alet, N. Laflorencie, Phys. Rev. B **110**, 214210 (2024)

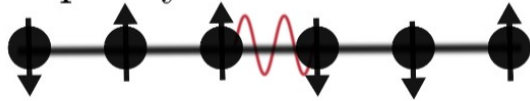
A. Padhan, J. Colbois, F. Alet, N. Laflorencie, Phys. Rev. Lett. **136**, 197101 (2026)



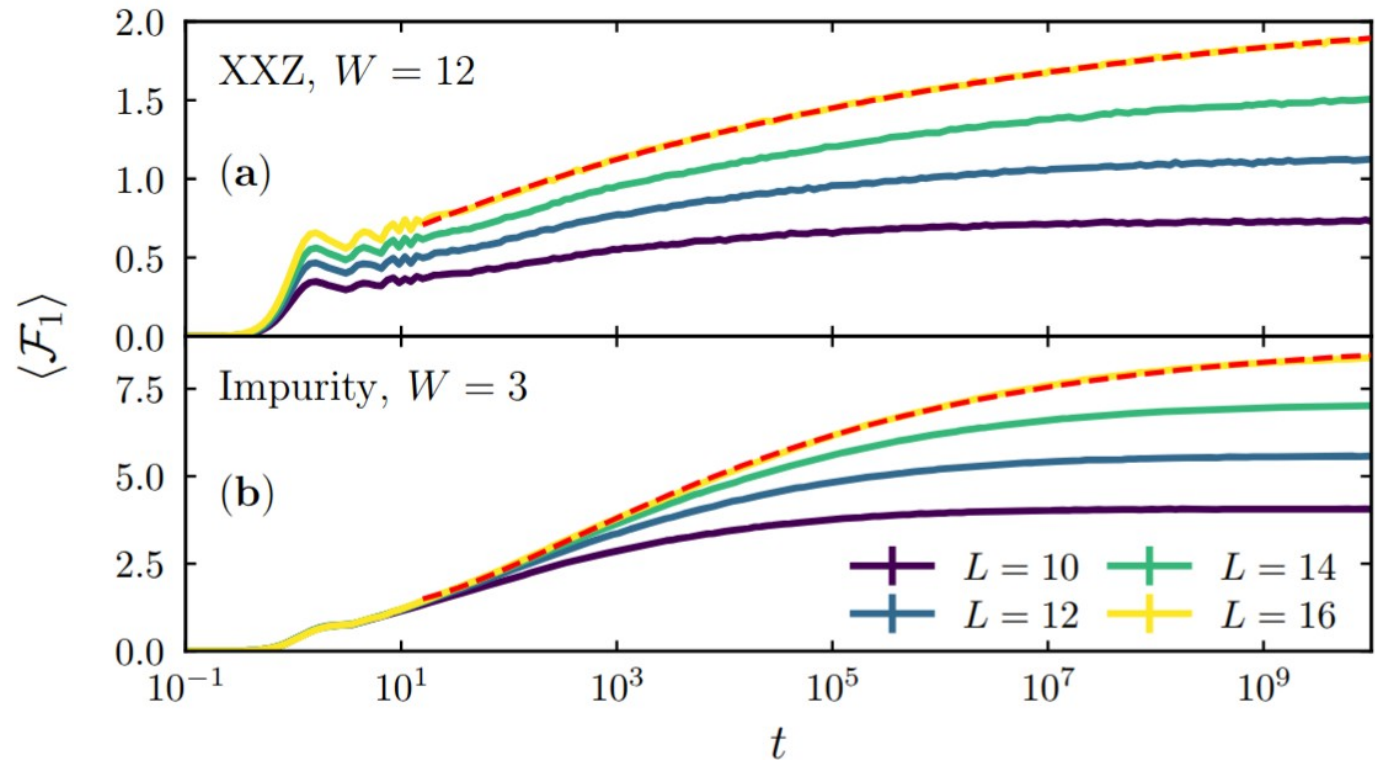
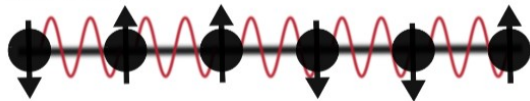
Fermionic non-Gaussianity and MBL – time evolution

- Hamiltonian:
$$\hat{\mathcal{H}} = \frac{1}{4} \sum_{j=1}^L \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + \frac{1}{2} \sum_{j=1}^L h_j \hat{\sigma}_j^z + \sum_{i \in \mathcal{J}} \frac{\Delta}{4} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \quad h_j \in [-W, W]$$

Impurity



XXZ



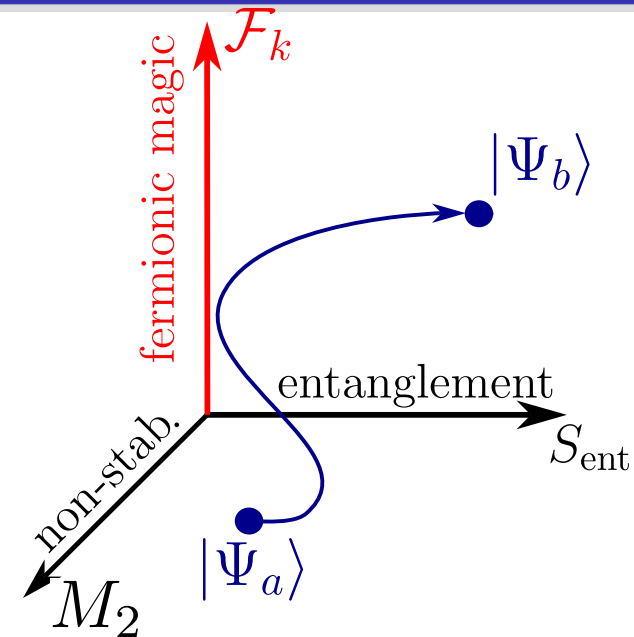
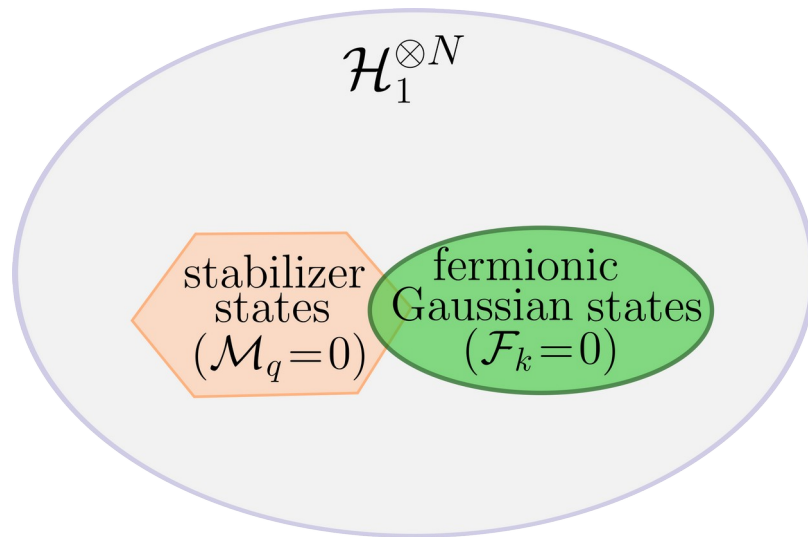
- $$\mathcal{F}_1(\Psi_t) = L - \sum_{m < n} |\langle \hat{\gamma}_m \hat{\gamma}_n \rangle_t|^2,$$

$$\mathcal{F}_1(\Psi_t) = \mathcal{F}_1^{\text{MBL}} - C t^{-\beta} + O(t^{-2\beta})$$

Dephasing of Pauli strings:

M. Serbyn, Z. Papić, D. A. Abanin, PRB **90**, 174302 (2014)

Conclusion



- Understanding resources in many-body phenomena

Fermionic Magic Resources of Quantum Many-Body Systems,
PS, P. Stornati, X. Turkeshi, PRX Quantum **7**, 010302 (2026)

Magic spreading in random quantum circuits,
X. Turkeshi, E. Tirrito, PS,
Nature Communications **16**, 2575 (2025)

Growth and spreading of quantum resources
under random circuit dynamics
S. Aditya, X. Turkeshi, PS,
Phys. Rev. Research **8**, 033062 (2026)

Anticoncentration and magic spreading
under ergodic dynamics,
E. Tirrito, X. Turkeshi, PS,
Phys. Rev. Lett. **135**, 220401 (2025)

Equivalence of quantum resources under ergodic
dynamics
S. Aditya, X. Turkeshi, PS, arXiv:2608.15197

Magic dynamics in many-body localized systems
P. Nicacio Falcao, PS, J. Zakrzewski, E. Tirrito,
Phys. Rev. Lett. **135**, 240404 (2025)

Fermionic magic resources in disordered
quantum spin chains
P. Falcão, J. Zakrzewski, PS, arXiv:2602.0024

- Improving classical methods to represent many-body states

Superpositions of
stabilizer states:
S. Bravyi, D. Gosset, PRL **116**, 250501 (2016)
S. Bravyi et al, Quantum **3**, 181 (2019)

fermionic Gaussian states:
O. Reardon-Smith, et al., Quantum **8**, 1549 (2024)
B. Dias, R. Koenig, Quantum **8**, 1350 (2024)

Augmenting tensor network states

G. Lami, T. Haug, J. De Nardis,
PRX Quantum **6**, 010345 (2025)

S. Masot-Llima, PS, P. Stornati, A. Garcia-Saez,
Phys. Rev. B **114**, 024311 (2026)

Random Gaussian Augmented MPS
S. Puri, P. Tarabunga, PS, arXiv:2609.25458

Fermionic antifattness (FAF) of simple states

- Product of fermionic magic states: in $\mathcal{P} = 1$ sector, all states are FGS for $N = 2, 3$

$$|\Psi_{\theta,N}\rangle = \left[\frac{1}{2} (|0000\rangle + |0011\rangle + |1100\rangle + e^{i\theta}|1111\rangle) \right]^{\otimes \frac{N}{4}} \longrightarrow \mathcal{F}_k(|\Psi_{\theta,N}\rangle) = N \left(1 - \cos^{2k} \left(\frac{\theta}{2} \right) \right)$$

- Action of a local non-Gaussian gate O_a on FGS $|\Psi_G\rangle = U_G|0\rangle^{\otimes N}$

$$\text{covariance matrix } M_{mn} = -\frac{i}{2} \langle \Psi | [\gamma_m, \gamma_n] | \Psi \rangle \quad U_G^\dagger O_a^\dagger \gamma_m O_a U_G = U_G^\dagger \gamma_m U_G + C_m,$$

$$\text{with } C_m = U_G^\dagger [O_a^\dagger, \gamma_m] O_a U_G$$

$$\longrightarrow \mathcal{F}_k(O_a |\Psi_G\rangle) \leq 2a,$$

- Haar-random states $|\Psi\rangle = U|0\rangle^{\otimes N}$ where $U \in \text{U}(2^N)$

$$\mathcal{F}_1^{\text{typ}} = N - \frac{1}{d+1} N(2N-1) \quad \text{where } d = 2^N \text{ or } d = 2^{N-1} \text{ in } \mathcal{P} = 1 \text{ sector}$$

$$\mathcal{F}_k^{\text{typ}} = N - C_k \frac{2^k N^{k+1}}{d^k} + O\left(\frac{N^k}{d^k}\right) \quad (\text{can be obtained e.g. by seeing } M \text{ as a random matrix})$$

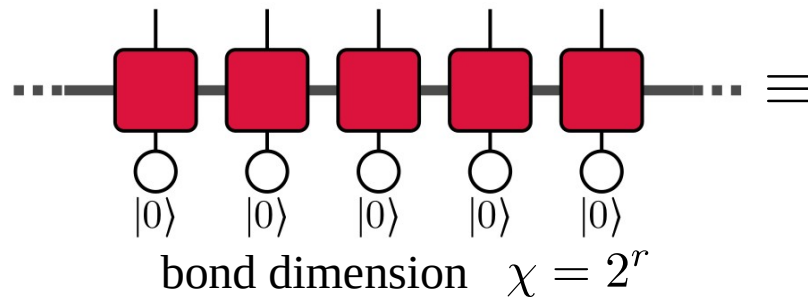
cf. SRE behavior in typical states $M_2 = N - c_2$

X. Turkeshi, A. Dymarsky, PS, PRB **111**, 054301 (2025)

Fermionic antiflatness (FAF) in model systems

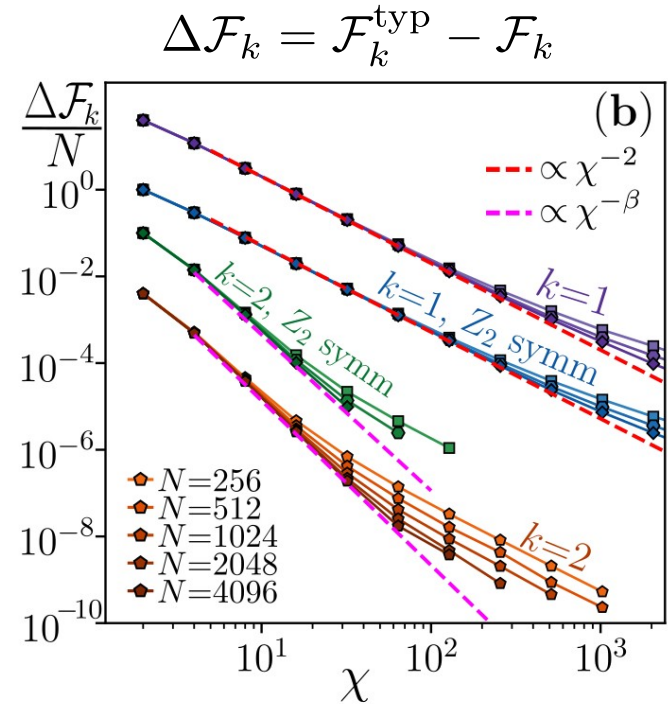
Random Matrix Product States (RMPS)

$U \in U(2^{r+1})$ reshaped as $U_{p,\alpha}^{s,\beta}$

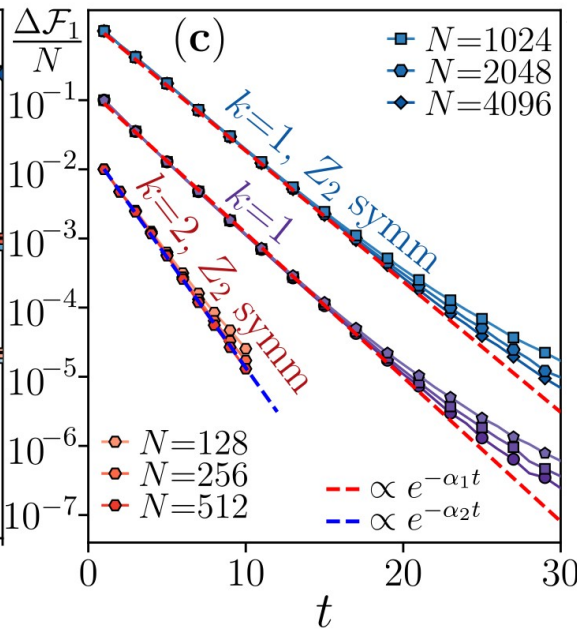
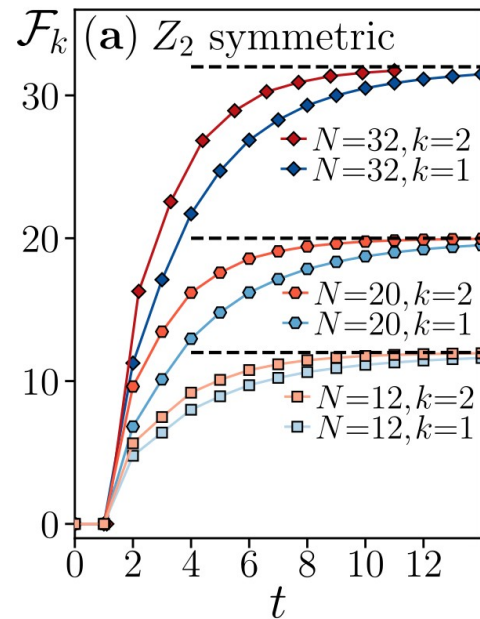
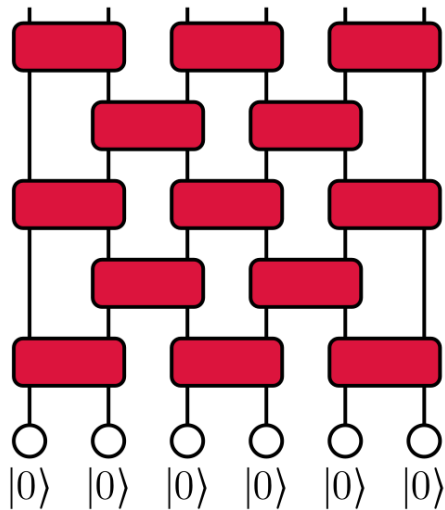


$$\mathcal{F}_k = N - \frac{1}{2} \text{tr} [(M^T M)^k] \quad \text{for } k=1 \text{ Clifford circuits reproduce Haar averages}$$

FAF of RMPS: $\mathcal{F}_k = \left(1 - \frac{a_k}{\chi^\beta}\right) N + b_k,$



Random quantum circuits



saturation at
 $t_{\text{sat}} \propto \log(N)$

similar to SRE
X. Turkeshi, E. Tirrito, PS,
Nat. Comm. 16, 2575 (2025)

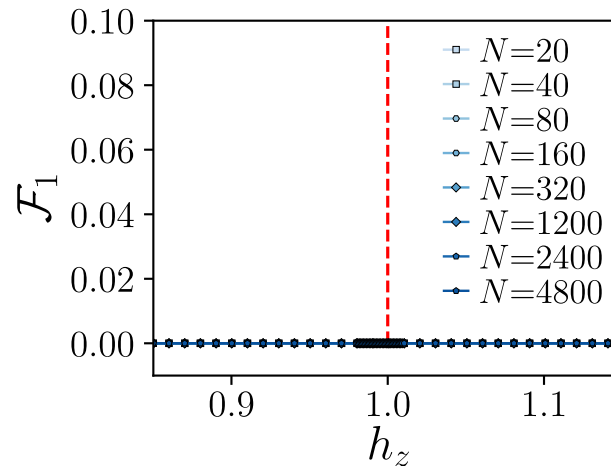
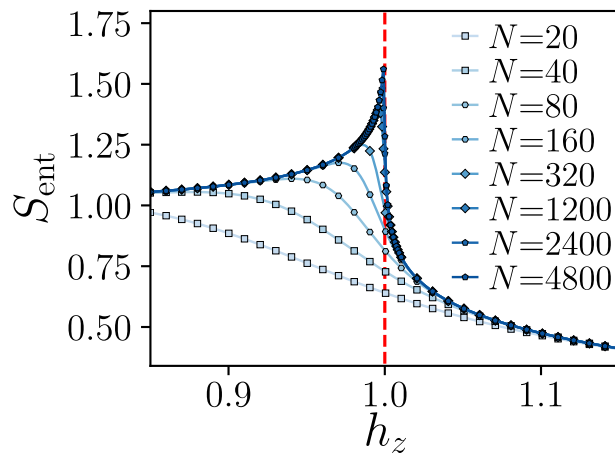
Fermionic antiflatness across Ising phase transition

- Baseline behavior of the FAF

$$\mathcal{F}_k = N - \frac{1}{2} \text{tr} [(M^T M)^k]$$

- The Ising model

$$H = -h_z \sum_m Z_m - \sum_m X_m X_{m+1} \stackrel{\mathcal{P} \equiv 1}{=} i h_z \sum_{m=1}^N \gamma_{2m-1} \gamma_{2m} + i \sum_{m=1}^{N-1} \gamma_{2m} \gamma_{2m+1}$$

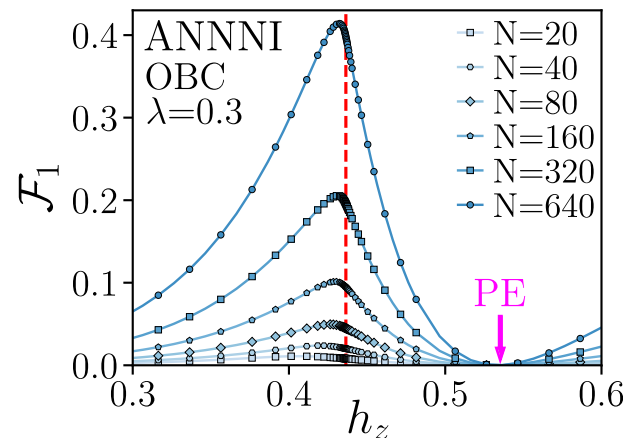
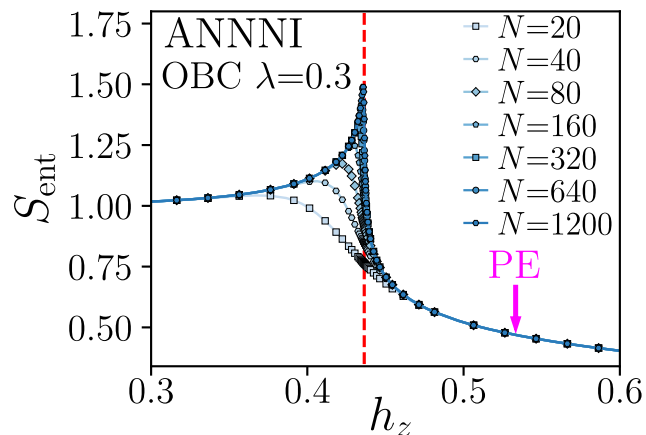


TFIM's ground state is
fermionic Gaussian state

↓
 $\mathcal{F}_k = 0$

- The ANNNI model

$$H_{\text{ANNNI}} = -h_z \sum_m Z_m - \sum_m X_m X_{m+1} - \lambda \sum_m X_m X_{m+2}$$



Extensive scaling

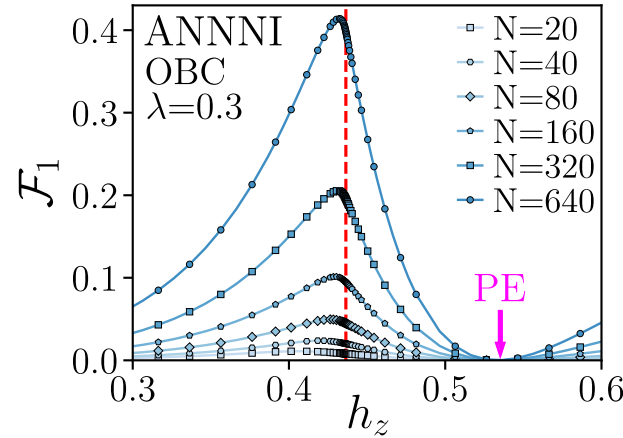
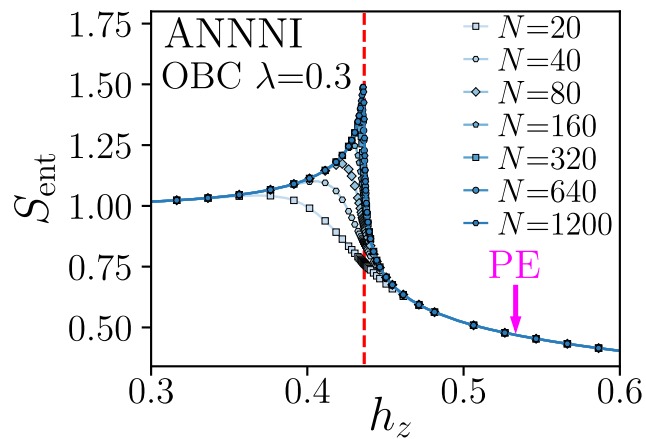
$$\mathcal{F}_k = D_k N + f_k$$

Fermionic antifattness across Ising phase transition

- Baseline behavior of the FAF

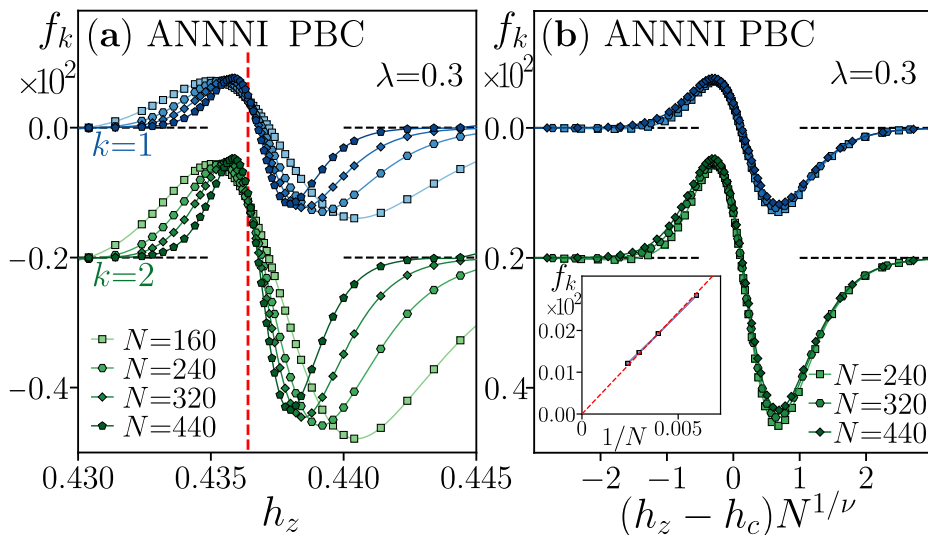
$$\mathcal{F}_k = N - \frac{1}{2} \text{tr} [(M^T M)^k]$$

- The ANNNI model $H_{\text{ANNNI}} = -h_z \sum_m Z_m - \sum_m X_m X_{m+1} - \lambda \sum_m X_m X_{m+2}$

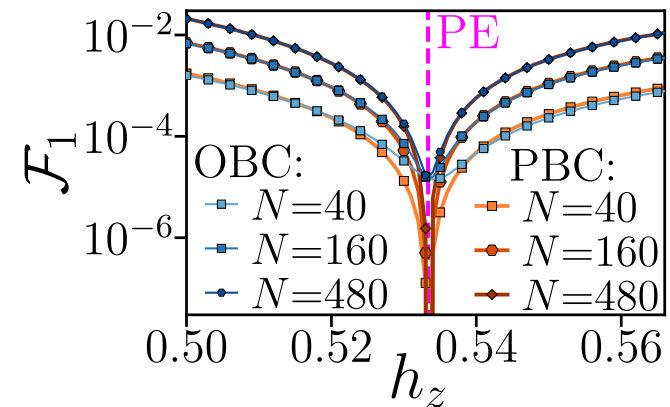


Extensive scaling
 $\mathcal{F}_k = D_k N + f_k$

- At the critical point:



- At the PE point:



I. Peschel, V. Emery, Z. Phys. B **43**, 241 (1981)
 H. Katsura, D. Schuricht, M. Takahashi, PRB **92**, 115137 (2015)

FAF: impurity and ANNNI models

$$H_{ANNNI} = -h_z \sum_m Z_m - \sum_m X_m X_{m+1} - \lambda \sum_m X_m X_{m+2}$$

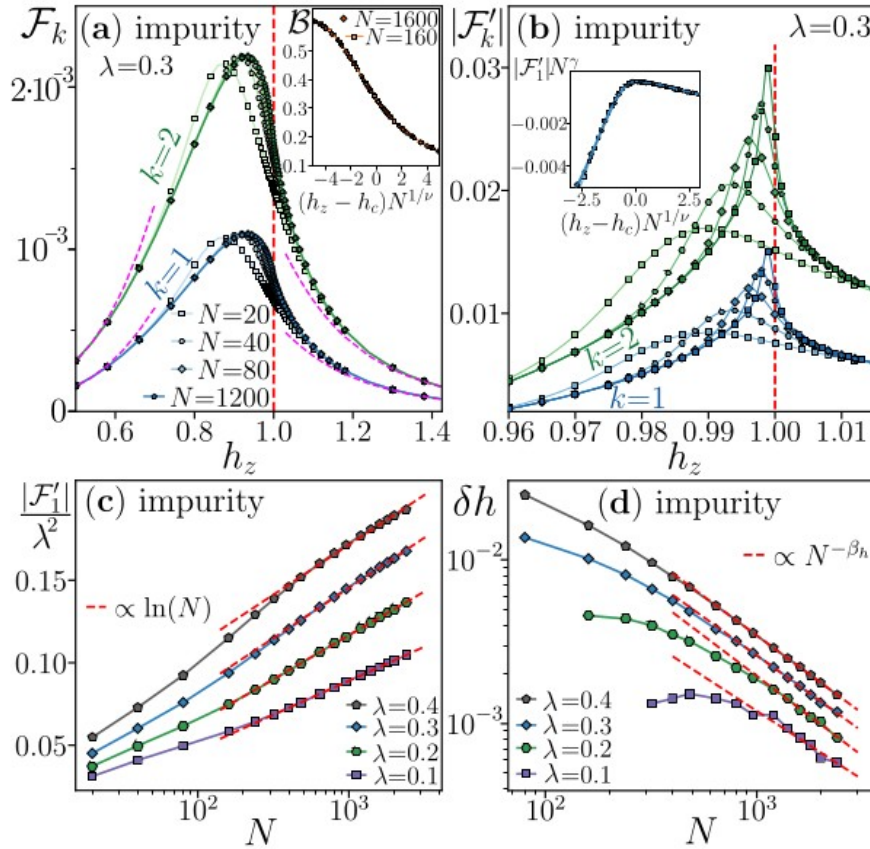


Figure 4. Fermionic antiflatness $\mathcal{F}_k$ in the impurity model Eq. (81).

$$H_+ = ih_z \sum_{m=1}^N \gamma_{2m-1} \gamma_{2m} + i \sum_{m=1}^{N-1} \gamma_{2m} \gamma_{2m+1}$$

$$H_{\text{imp}} = H_+ - \lambda \gamma_{2\ell_0} \gamma_{2\ell_0+1} \gamma_{2\ell_0+2} \gamma_{2\ell_0+3}$$

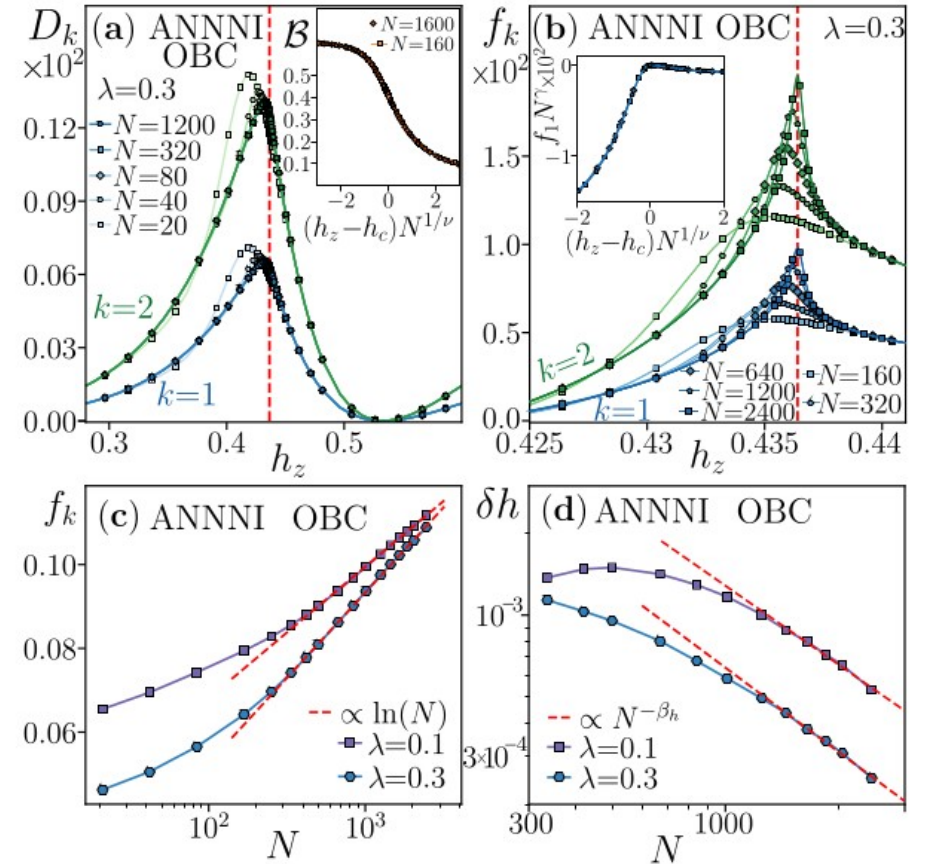


Figure 5. Fermionic antiflatness $\mathcal{F}_k$ in ANNNI model Eq. (88) with OBC.

FAF: highly excited eigenstates

$$H_{\text{ANNNI}} = -h_z \sum_m Z_m - \sum_m X_m X_{m+1} - \lambda \sum_m X_m X_{m+2}$$

$$H_+ = ih_z \sum_{m=1}^N \gamma_{2m-1} \gamma_{2m} + i \sum_{m=1}^{N-1} \gamma_{2m} \gamma_{2m+1}$$

$$H_{\text{imp}} = H_+ - \lambda \gamma_{2\ell_0} \gamma_{2\ell_0+1} \gamma_{2\ell_0+2} \gamma_{2\ell_0+3}$$

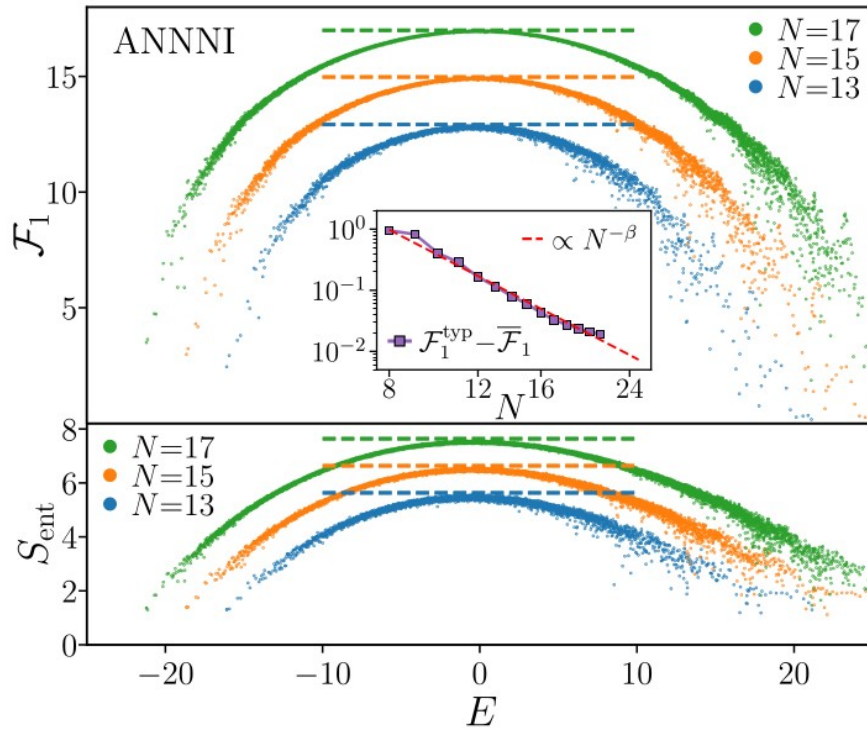


Figure 8. Fermionic antiflatness $\mathcal{F}_1$ (top panel) as function of energy E of eigenstates $|\Psi_n\rangle$ of the ANNNI model

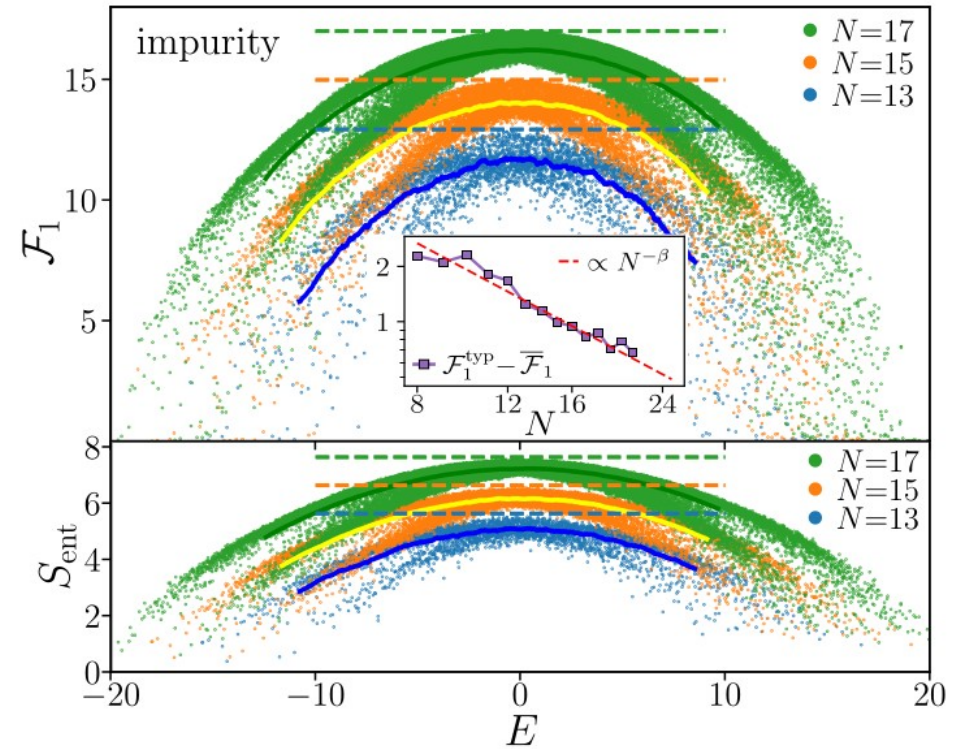


Figure 9. Fermionic antiflatness $\mathcal{F}_1$ (top panel) across the spectrum of the impurity model

FAF: under ergodic dynamics

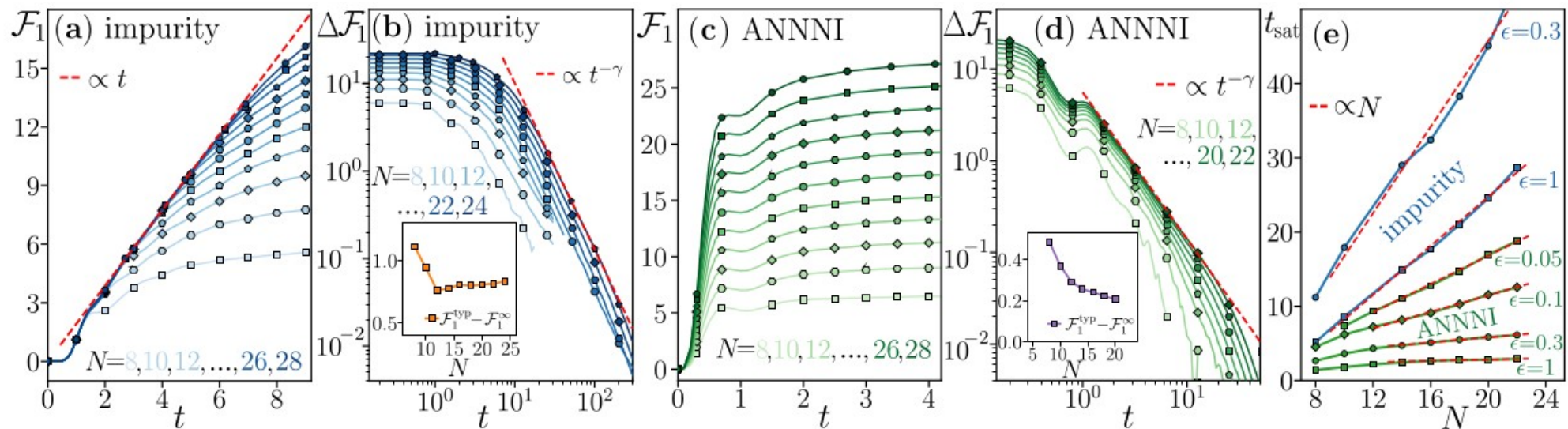
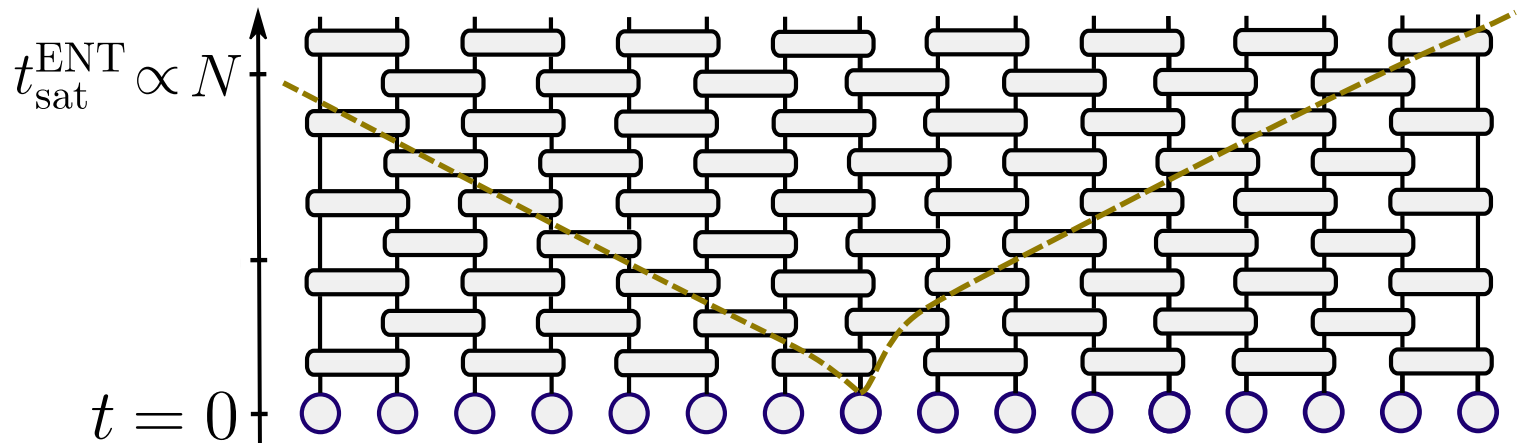


Figure 10. Fermionic antiflatness $\mathcal{F}_k$ in ergodic many-body dynamics.

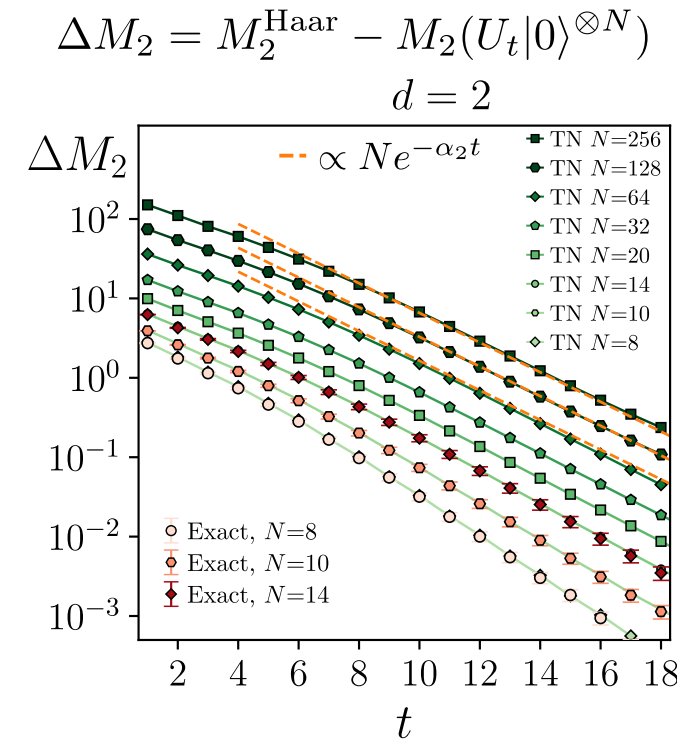
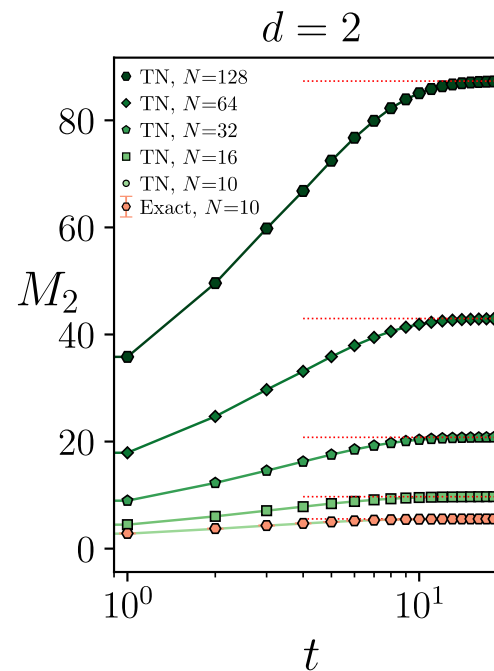
Nonstabilizerness in brick-wall Haar-random circuits

- 2-qubit gates with Haar measure on $SU(2)$
- Minimal model of unitary and local many-body dynamics

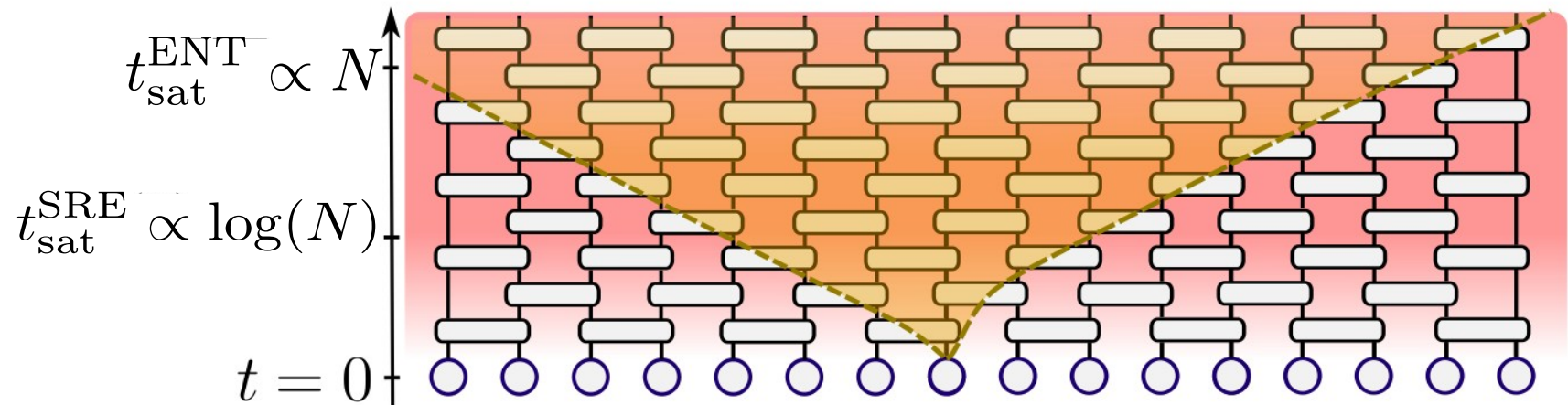


- We calculate the SRE

$$M_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$

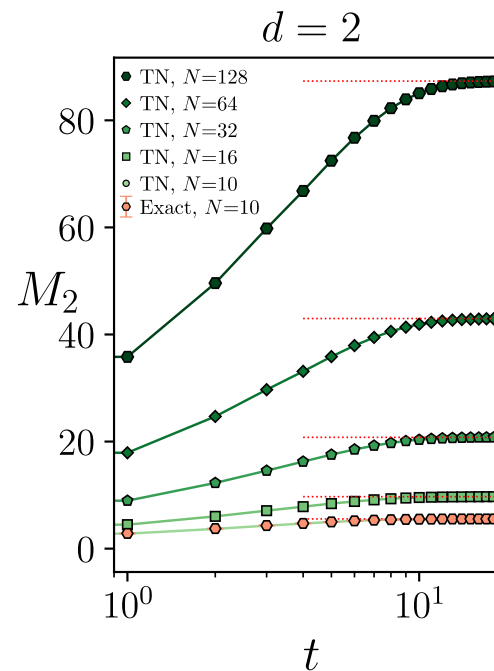


Nonstabilizerness in brick-wall Haar-random circuits

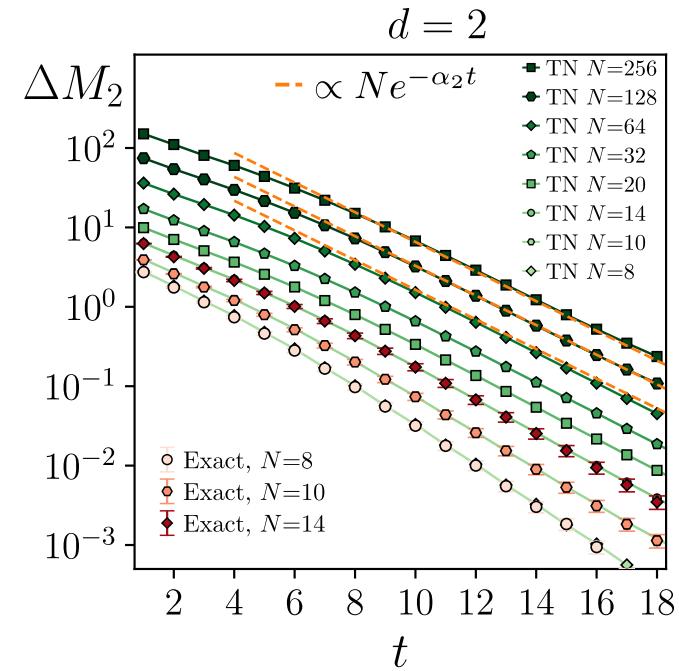


- We calculate the SRE

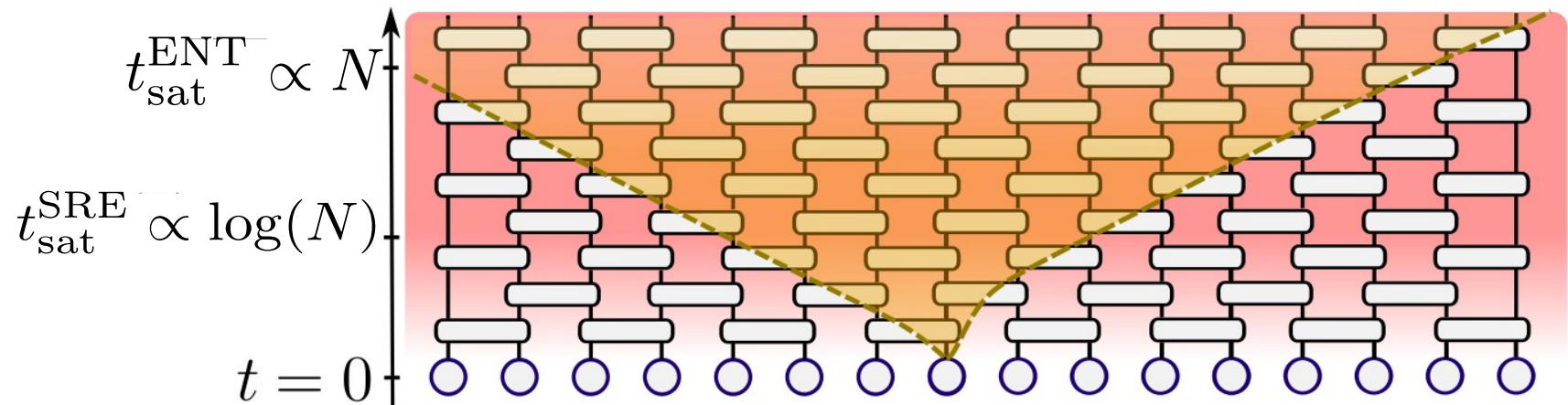
$$M_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$



$$\Delta M_2 = M_2^{\text{Haar}} - M_2(U_t | 0\rangle^{\otimes N})$$



Nonstabilizerness in brick-wall Haar-random circuits



- We calculate the SRE

$$M_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$

- Ergodic many-body systems:

E. Tirrito, X. Turkeshi, PS, arXiv:2412.10229

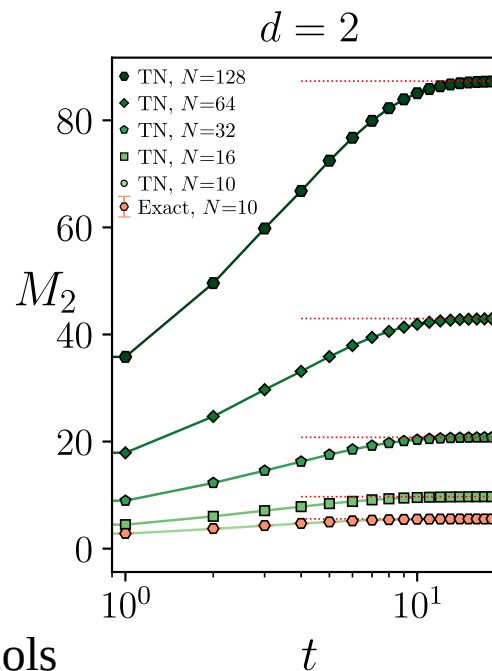
$$t_{\text{sat}}^{\text{SRE}} \sim N$$

- Non-ergodic disordered systems

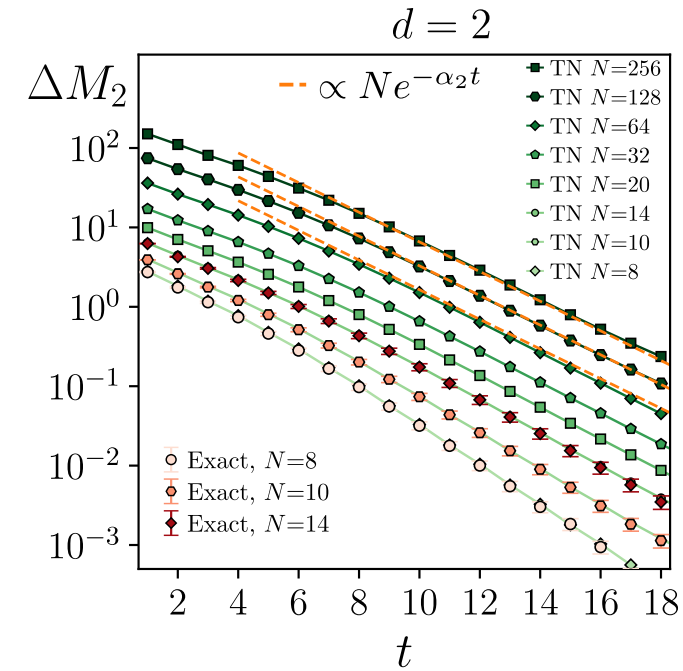
PRN Falcão, PS, J. Zakrzewski, E. Tirrito, arXiv:2503.07468

- Quantum enhanced metrological protocols

T. Hernández Yanes, PS, J. Zakrzewski, M. Płodzień, in prep.

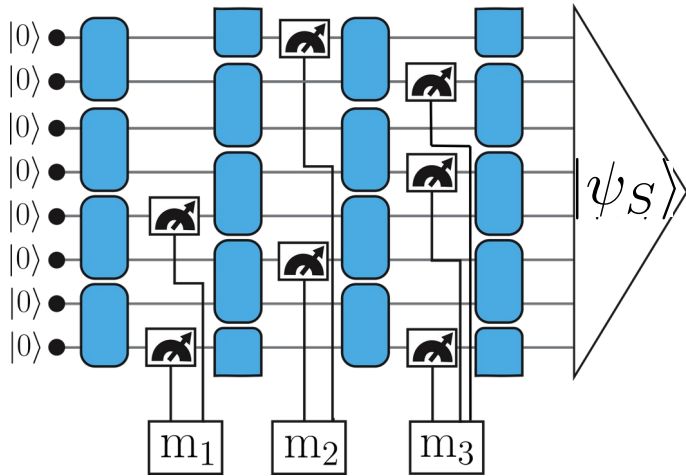


$$\Delta M_2 = M_2^{\text{Haar}} - M_2(U_t | 0\rangle^{\otimes N})$$



X. Turkeshi, E. Tirrito, PS, Nature Communications **16**, 2575 (2025)

Clifford circuits and stabilizer states



- Pauli operators X, Y, Z acting on qubit
- Pauli string = tensor product of $1, X, Y, Z$

$$P = \prod_{k=1}^N X_k^{n_k^j} Z_k^{m_k^j}$$

- Clifford unitary U_C maps Pauli string to Pauli string

$$\begin{aligned} Z_j |0\rangle^{\otimes N} &= |0\rangle^{\otimes N} \\ (U_C Z_j U_C^\dagger) U_C |0\rangle^{\otimes N} &= U_C |0\rangle^{\otimes N} \end{aligned}$$

- Stabilizer state $|\psi_S\rangle = U_C |0\rangle^{\otimes N}$

fulfills $P_j |\psi_S\rangle = |\psi_S\rangle$

$$M_X = [n_k^j] \quad M_Z = [m_k^j] \quad a = [a_j]$$

and is completely determined by the matrices (with entries=0,1)

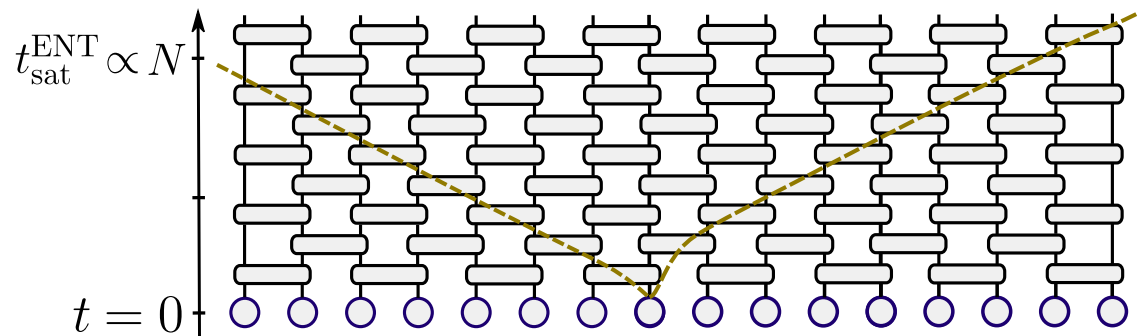
- Stabilizer protocols involve also computational basis measurements etc.
- Simulation in time *polynomial* in N (*Gottesman-Knill theorem*)
S. Aaronson, D. Gottesman, Phys. Rev. A **70**, 052328 (2004)
- Quantum error correction and, e.g., measurement-induced transitions
Y. Li, X. Chen, M. P. A. Fisher, PRB **100**, 134306 (2019)

SRE in brick-wall Haar-random circuits

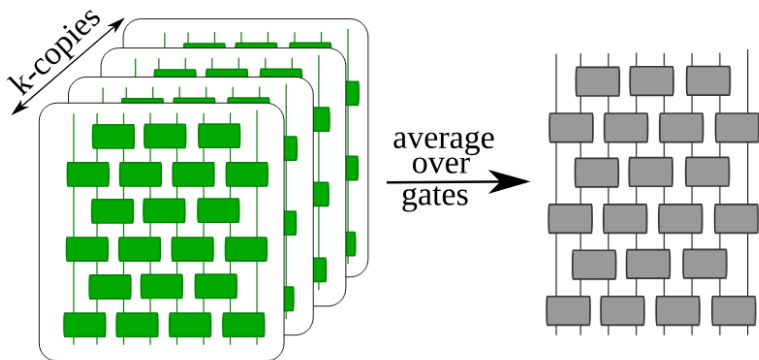
- Stabilizer Renyi entropy

$$M_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$

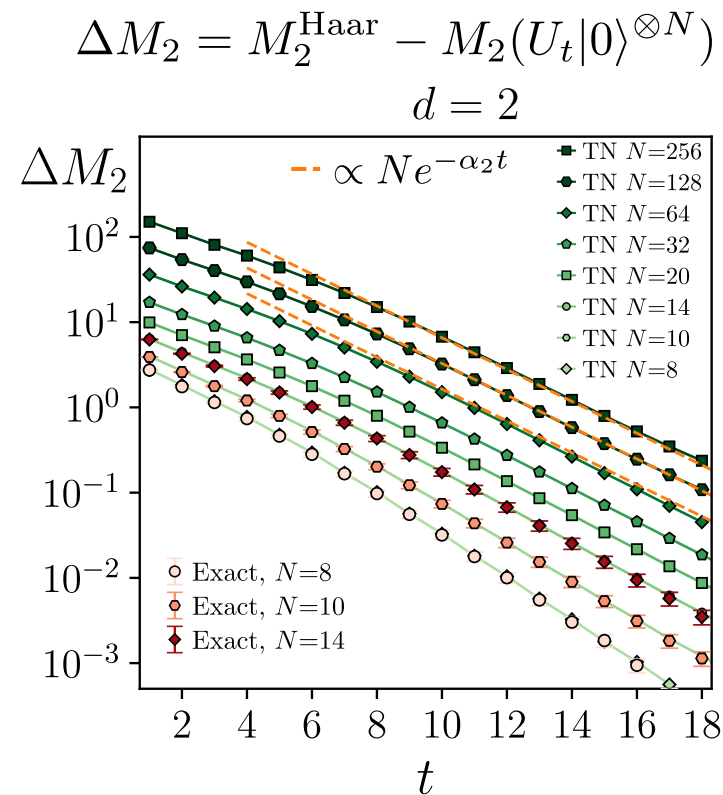
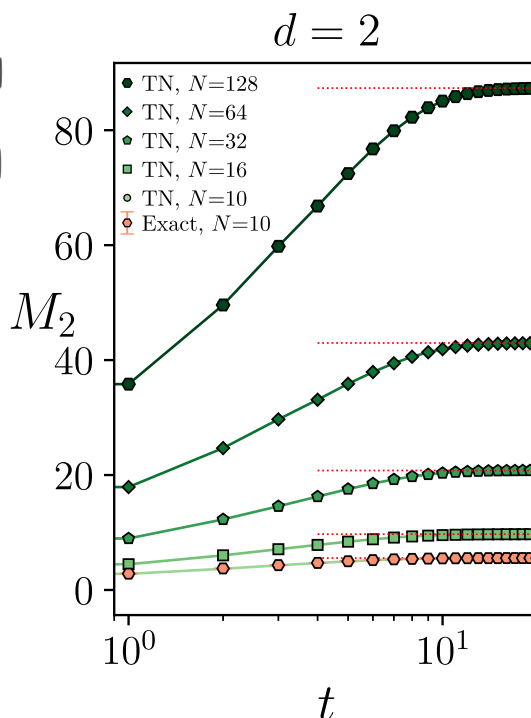
$$M_2 = -\log \left[\text{Tr} \left(Q_2^{\otimes N} (|\psi\rangle\langle\psi|)^{\otimes 4} \right) \right]$$



- Replica trick



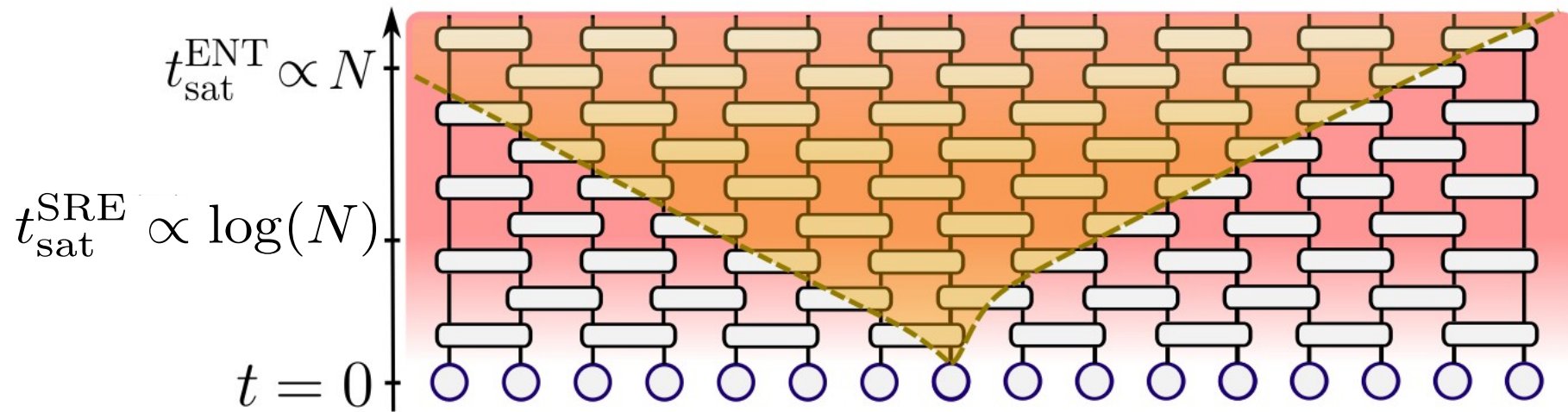
T. Zhou and A. Nahum, PRB **99**, 174205 (2019)
 M. Fisher, et al., Annu. Rev. Cond. Matt. Phys. **14**, 335 (2023)
 P. Braccia, et al., Quantum Mach. Intell. **6**, 54 (2024)



SRE in brick-wall Haar-random circuits

- We find $\Delta M_2 \equiv M_2^{\text{Haar}} - M_2(U_t|0\rangle^{\otimes N}) = aNe^{-\alpha t}$

The saturation to Haar state value up to fixed accuracy occurs at time $t_{\text{sat}}^{\text{SRE}} = \frac{1}{\alpha} \log(N)$



- SRE in ergodic many-body dynamics

E. Tirrito, X. Turkeshi, PS, [arXiv:2412.10229](#)

Floquet dynamics

$$t_{\text{sat}}^{\text{SRE}} \propto \log(N)$$

cf., log-depth approximate unitary designs

N. LaRacuente, F. Leditzky, [arXiv:2407.07876](#)

T. Schuster, J. Haferkamp, H.-Y. Huang, [arXiv:2407.07754](#)

Hamiltonian dynamics

$$t_{\text{sat}}^{\text{SRE}} \sim N$$

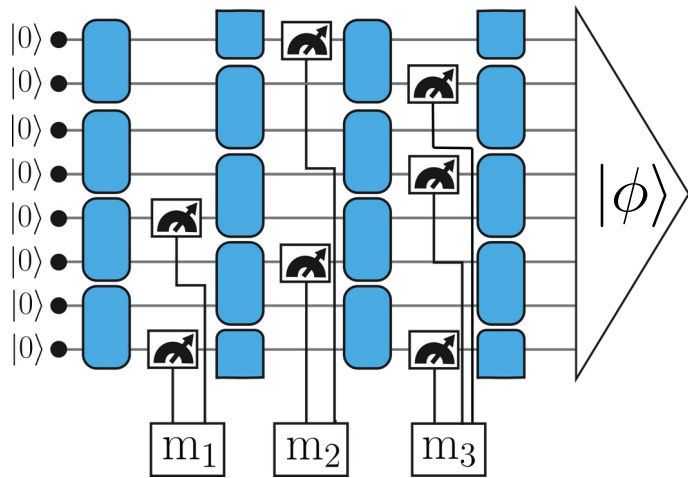
due to conservation laws, cf.,

J. Haah, [arXiv:2504.14005](#)

- SRE in non-ergodic dynamics of strongly disordered systems

PRN Falcão, PS, J. Zakrzewski, E. Tirrito, [arXiv:2503.07468](#)

Clifford circuits and nonstabilizerness



- Stabilizer state $P_j |\phi\rangle = |\phi\rangle \in \text{STAB}$
where

$$P_j = e^{i\pi a_j} \prod_{k=1}^L X_k^{n_k^j} Z_k^{m_k^j}$$

is completely determined by the matrices (with entries=0,1)

$$M_X = [n_k^j] \quad M_Z = [m_k^j] \quad a = [a_j]$$

- Simulation in time *polynomial* in L
S. Aaronson, D. Gottesman, Phys. Rev. A **70**, 052328 (2004)

- Given $|\psi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |i_1 i_2 \dots i_N\rangle$ how to check if it is a stabilizer state?

Stabilizer rank: minimal k such that $|\psi\rangle = \sum_{\alpha=1}^k b_\alpha |\phi_\alpha\rangle$
S. Bravyi et al., Phys. Rev. X **6**, 021043 (2016)

it requires costly minimization procedure, similarly, e.g., robustness of magic

H. Hamaguchi et al., Quantum **8**, 1461 (2024) $L = 9$

- Stabilizer Renyi entropy (SRE)
L. Leone, S. Oliviero, A. Hamma, PRL **128**, 050402 (2022)

$$M_2(\psi) \equiv -\log_2 \left[\sum_{P \in \mathcal{P}_N} \frac{\langle \psi | P | \psi \rangle^4}{2^N} \right]$$

$\mathcal{P}_N$ - group of all 4^N Pauli strings

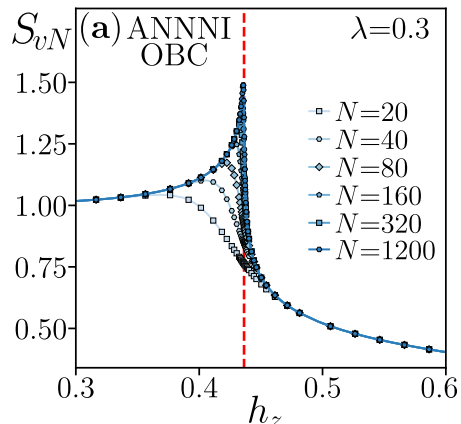
Parameterizing the Many-Body Hilbert space

$$\mathcal{H}_1^{\otimes N} \ni |\psi\rangle = \sum_{i_1, i_2, \dots, i_N} c_{i_1, i_2, \dots, i_N} |i_1 i_2 \dots i_N\rangle$$

- 1D: Matrix Product States (MPS)

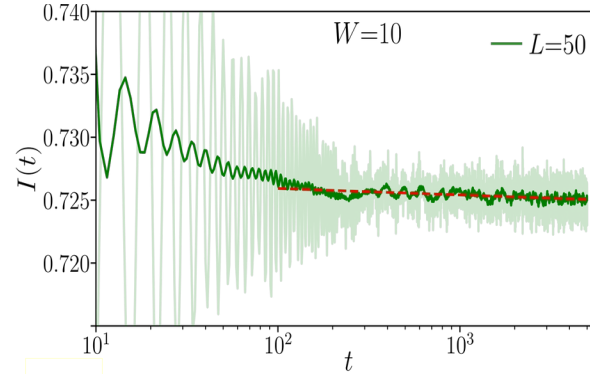
$$|\psi_{\text{MPS}}\rangle = \sum_{i_1, i_2, \dots, i_N} A^{i_1} A^{i_2} \dots A^{i_N} |i_1 i_2 \dots i_N\rangle$$

ground states and phase transitions

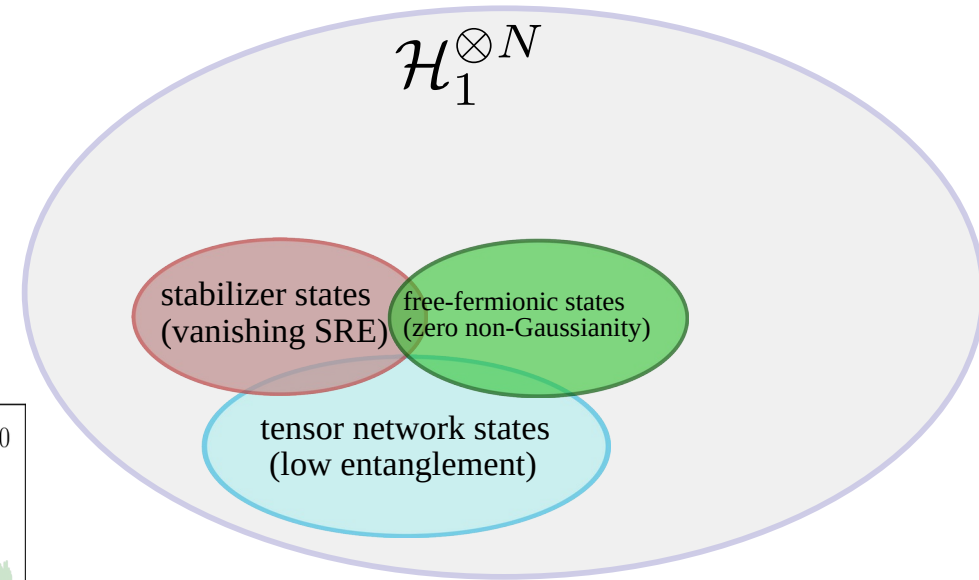


area-law for entanglement entropy
J. Eisert, M. Cramer, M. B. Plenio,
Rev. Mod. Phys. **82**, 277 (2010)

time evolution in disordered systems



PS, J. Zakrzewski, PRB **105**, 224203 (2022)



- fermionic Gaussian states
(or free-fermionic states)

$$|\Psi_G\rangle = \exp \left[\frac{1}{4} \sum_{m,n=1}^{2N} H_{m,n} \gamma_m \gamma_n \right] |0\rangle^{\otimes N}$$

S. Bravyi, QI&C, **5**, 3, 216-238 (2005)

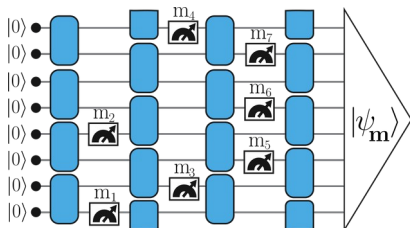
e.g., transverse field Ising model

$$H = -h_z \sum_{m=1}^N Z_m - J \sum_{m=1}^{N-1} X_m X_{m+1}$$

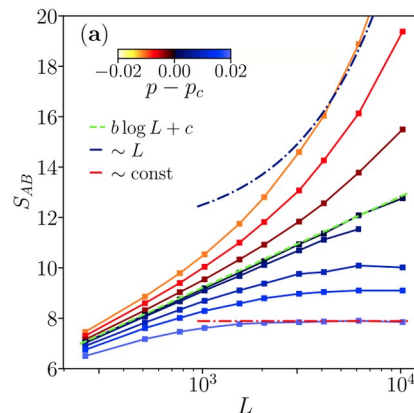
J. Surace, L. Tagliacozzo,
SciPost Phys. Lect. Notes **54** (2022)
G. Mbeng, A. Russomanno, G. Santoro,
SciPost Phys. Lect. Notes **82** (2024)

- Stabilizer states $g_j |\Psi_S\rangle = |\Psi_S\rangle$

S. Aaronson, D. Gottesman, PRA **70**, 052328 (2004)



Measurement-induced phase transitions



Y. Li, X. Chen, M. P. A. Fisher, PRB **100**, 134306 (2019)

Fermionic antiflatness (FAF) vs other measures

- Fermionic antiflatness (**FAF**) defined as $\mathcal{F}_k = N - \frac{1}{2} \text{tr} [(M^T M)^k]$

since: $M = Q^T \bigoplus_{m=1}^N \begin{pmatrix} 0 & \lambda_m \\ -\lambda_m & 0 \end{pmatrix} Q$ $\mathcal{F}_k = N - \sum_i \lambda_i^{2k}$

- Non-Gaussian entropy (**NGE**) [X. Lyu, K. Bu, arXiv:2409.08180](#)
[L. Coffman, G. Smith, X. Gao, arXiv:2501.06179](#)

Convolution operation: $\boxtimes^1 \rho = \text{tr}_2 [W(\rho \otimes \rho) W^\dagger]$, $W = \exp \left(\frac{\pi}{8} \sum_{j=1}^{2N} \gamma_j \gamma_{2N+j} \right)$,

$NGE_q(\Psi) = -\log_2 \left(\text{tr} [(\boxtimes^q \rho)^2] \right) \xrightarrow{q \rightarrow \infty}$

$NGE_\infty(\Psi) = S(|\Psi_F\rangle) = -\sum_i \log_2 \left(\left(\frac{1+\lambda_i}{2} \right)^2 + \left(\frac{1-\lambda_i}{2} \right)^2 \right)$

[A. Gottlieb, N. Mauser, PRL **95**, 123003 \(2005\)](#)

[L. Lumia, E. Tirrito, R. Fazio, M. Collura, PRR **6**, 023176 \(2024\)](#)

- NGE vs FAF

Close to FGS limit $\lambda_i = 1 - \epsilon_i$

$NGE_\infty(\Psi) = \sum_i \epsilon_i + O(\epsilon_i^2) = \frac{1}{2} \mathcal{F}_1 + O(\epsilon_i^2),$

Close to Haar random limit $\lambda_i = \epsilon_i$

$NGE_\infty(\Psi) = N - \sum_i \lambda_i^2 + O(\lambda_i^4) = \mathcal{F}_1 + O(\lambda_i^4).$

- In contrast to NGE, FAF is linear in $(|\psi\rangle\langle\psi|)^{\otimes 2k}$, and has direct interpretation in terms of

$\langle \Psi | \gamma_m \gamma_n | \Psi \rangle$

