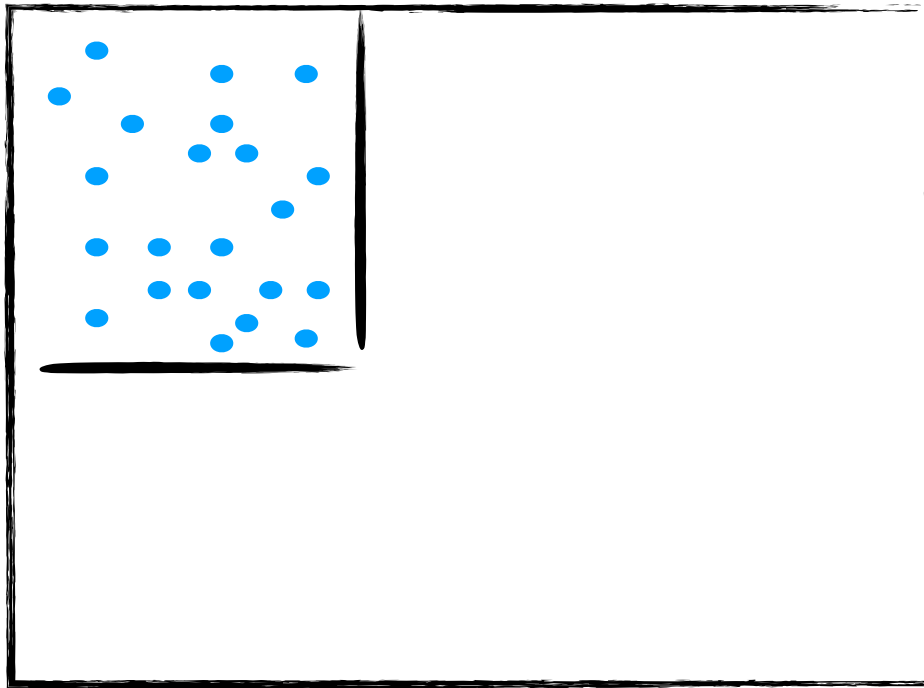
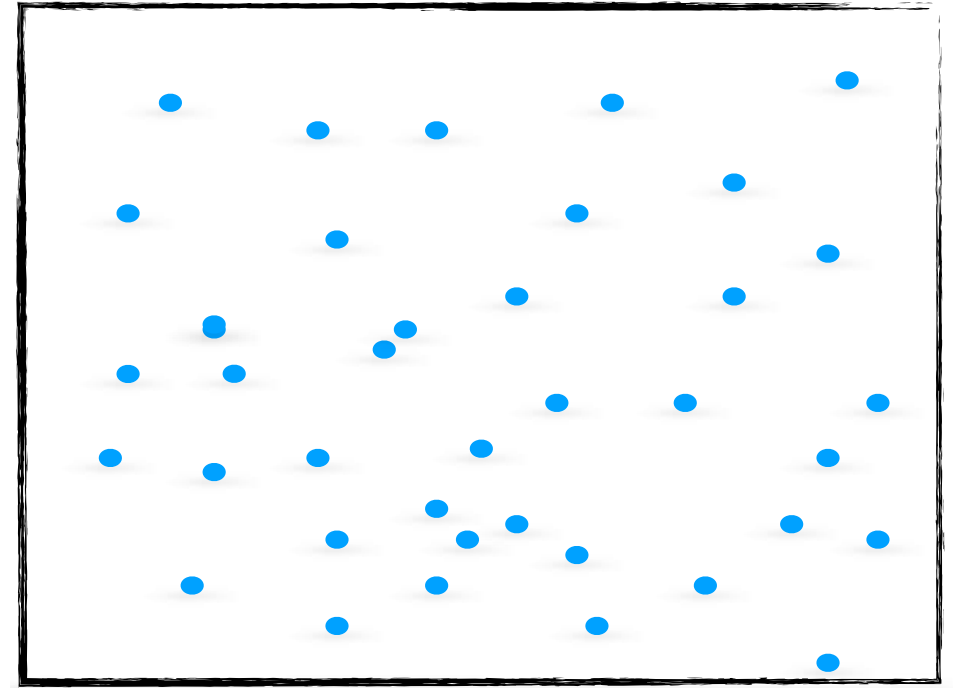


Ergodicity in Hilbert Space: From Porter–Thomas to Entanglement and Quantum Designs

Giuseppe De Tomasi

Classical Thermalization:



Time →

$$A(P(t), Q(t)) \rightarrow A_{eq}$$

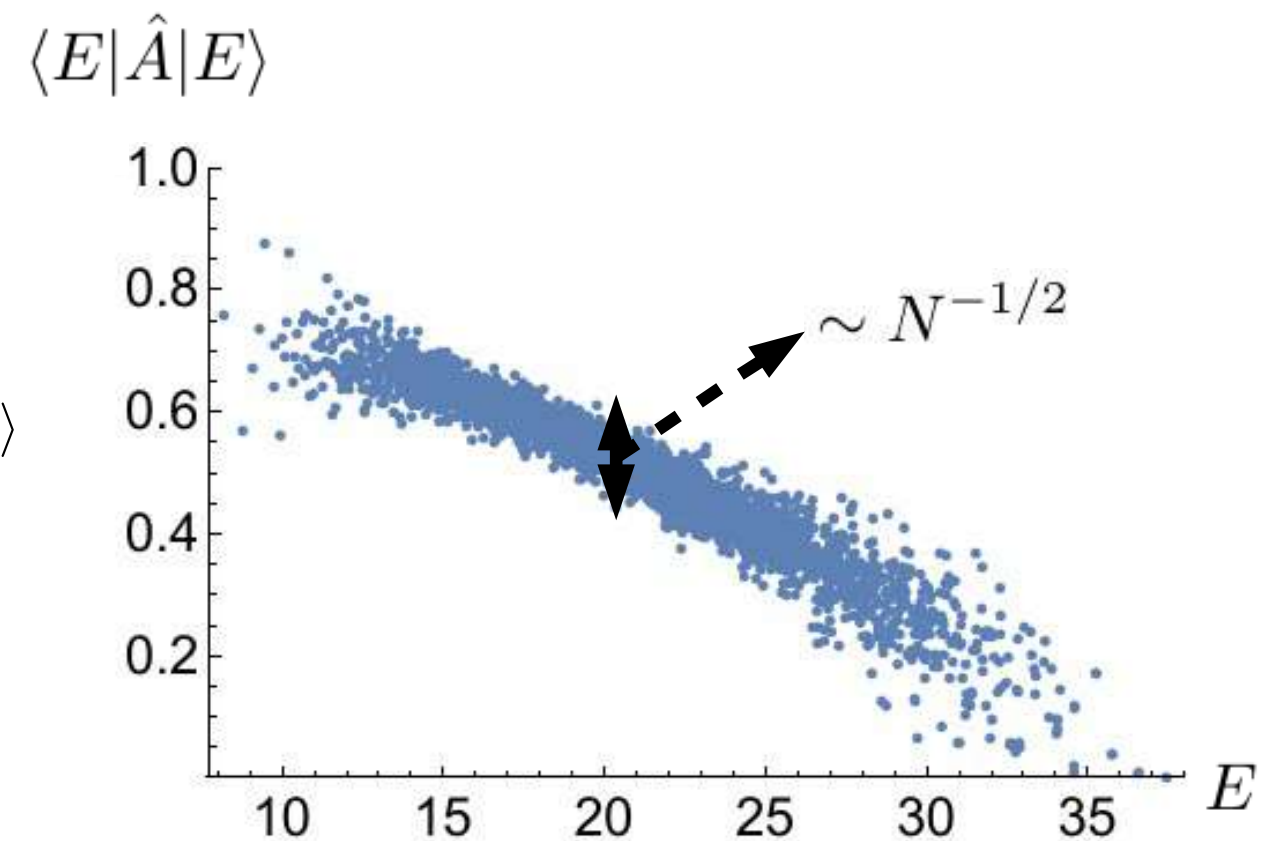
$$A_{eq} = \frac{1}{Z} \int dP dQ A(P, Q) e^{-\beta H(P, Q)}$$

**Eigenstate
Thermalization
Hypothesis**

$$\xrightarrow[\hat{\rho}^{eq} \propto \hat{\mathbb{1}}]{\beta \rightarrow 0}$$

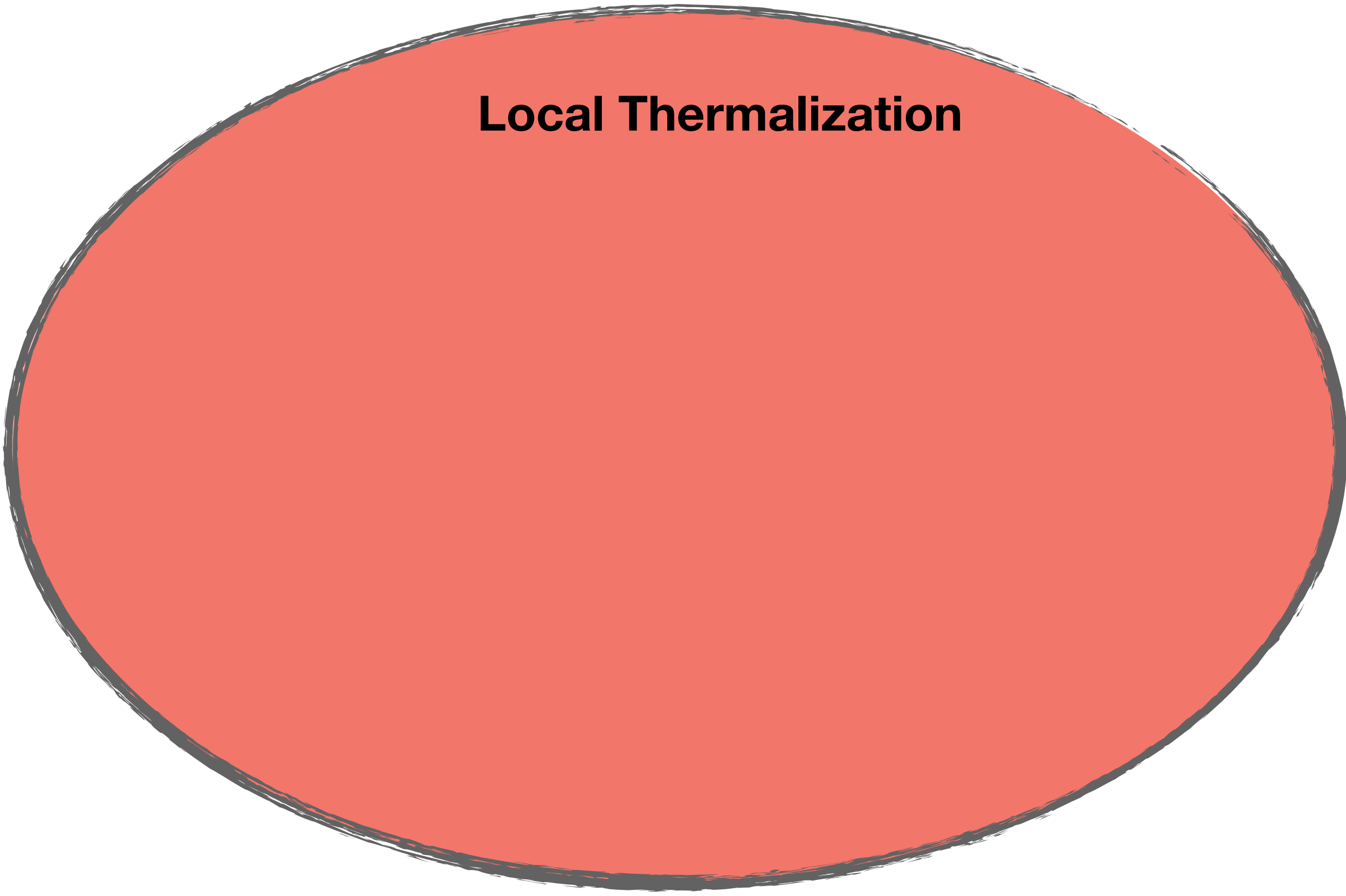
Random Matrix Theory

$$\overline{A} = \text{Tr}[\hat{\rho}^{eq} \hat{A}] = \frac{\text{Tr}[\hat{A}]}{N} \approx \langle \psi_{\text{Random}} | \hat{A} | \psi_{\text{Random}} \rangle$$

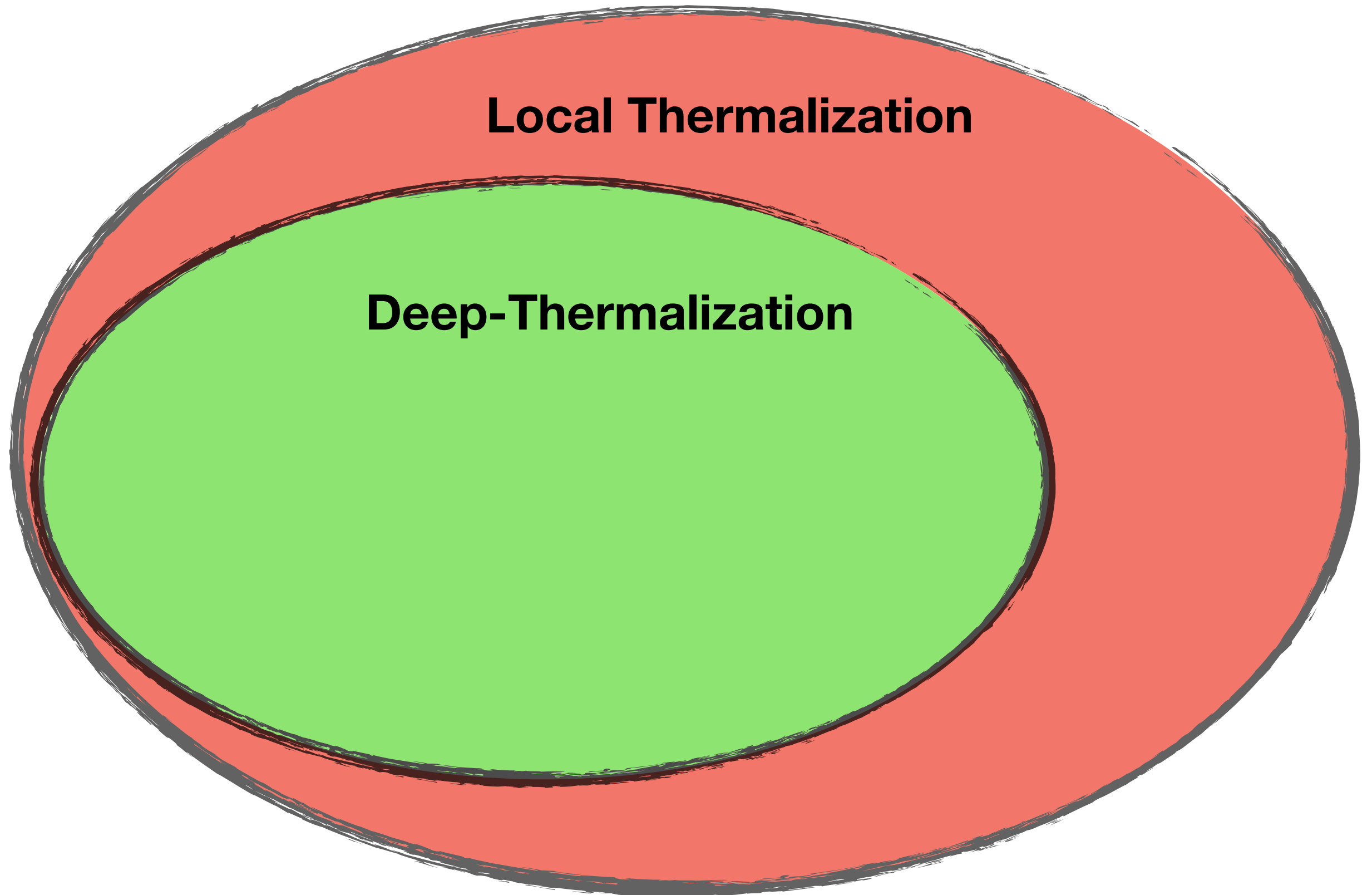


Hierarchical Structure

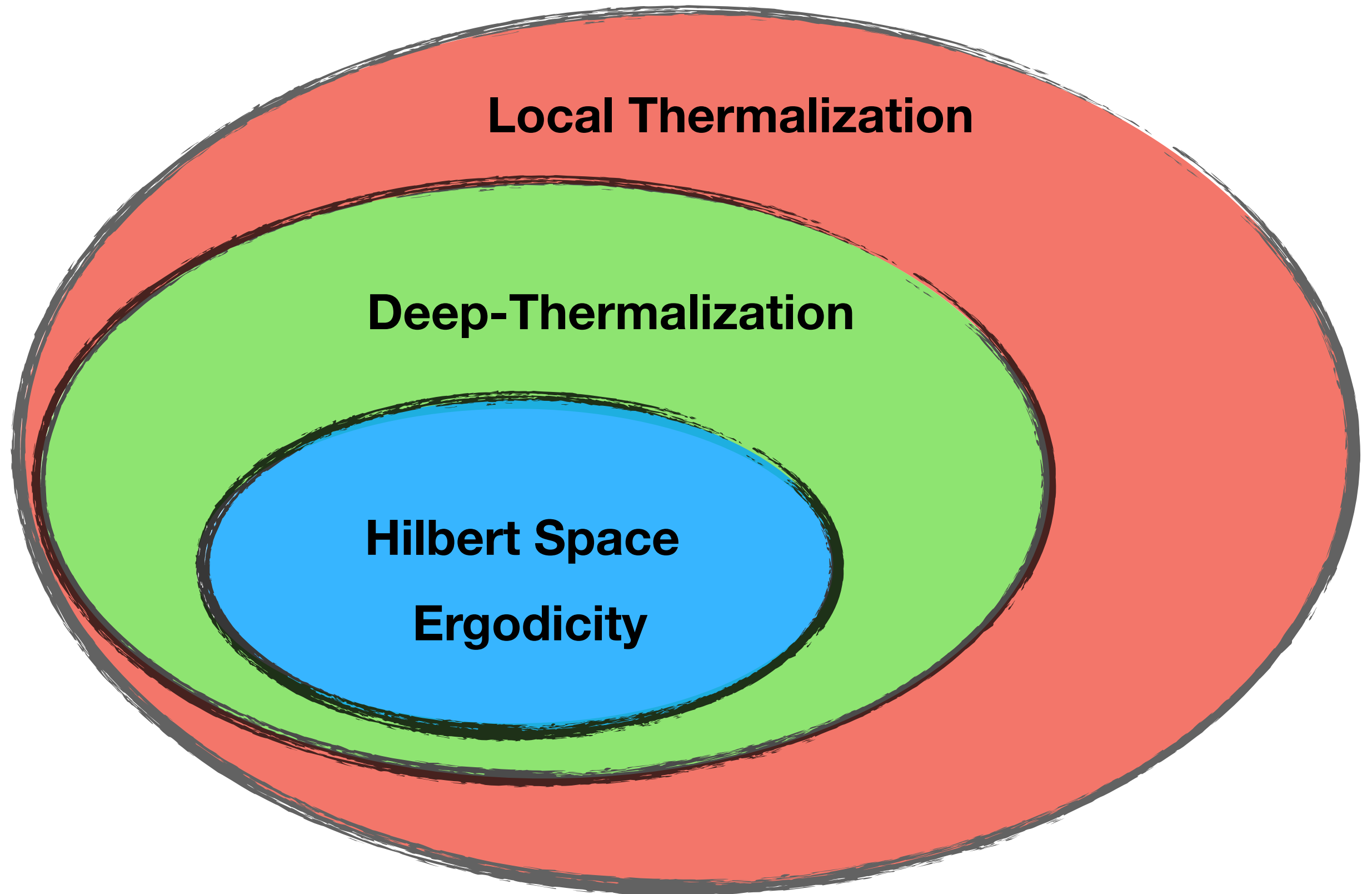
Local Thermalization

A large, horizontally-oriented oval filled with a solid red color. It has a thick, dark grey border that appears hand-drawn or sketched. The oval is centered on the page, below the title 'Hierarchical Structure' and above the text 'Local Thermalization'.

Hierarchical Structure



Hierarchical Structure



Delocalization in the Fock-Space

$$I_q = \sum_n |c_n|^{2q} \sim N^{(1-q)D_q}$$

Inverse Participation Ratio

$$|c_n|^2 = |\langle E|n\rangle|^2$$

$$N = \text{Dim}\mathcal{H}$$

$$1 = \sum_n^N |c_n|^2$$

Normalization

$$\beta \rightarrow 0$$

Ergodic

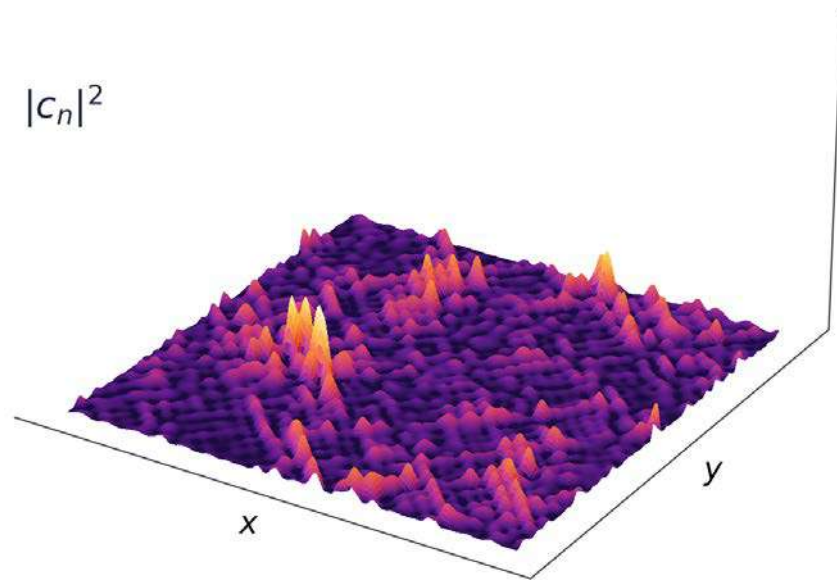
$$|c_n|^2 \sim N^{-1}$$

$$I_q \sim \sum_n^N N^{-q} \sim N^{(1-q)}$$

$$D_q = 1$$

Fractal Dimension

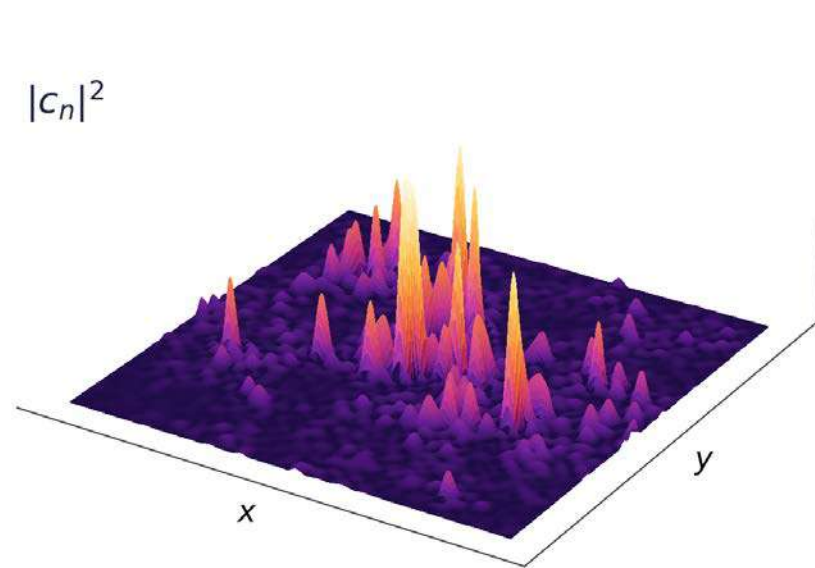
Ergodic



$$|c_n|^2 \sim \frac{1}{N}$$

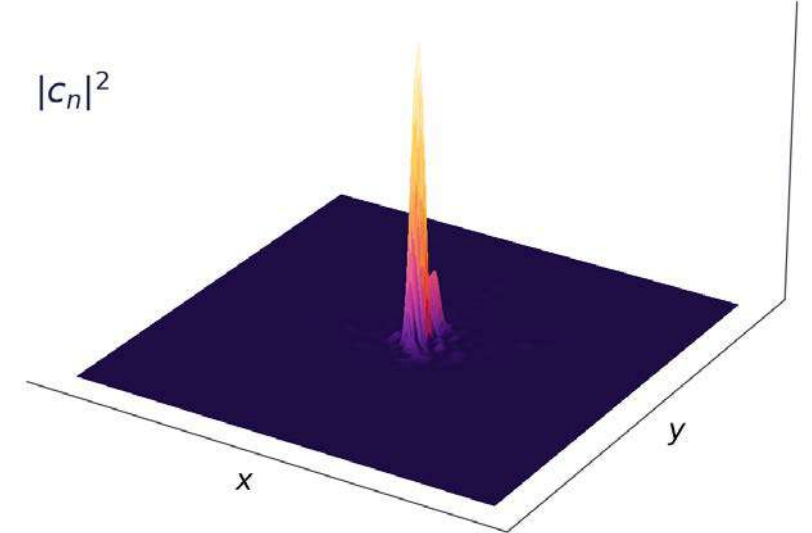
$$D_q = 1$$

Multifractal



$$0 < D_q < 1$$

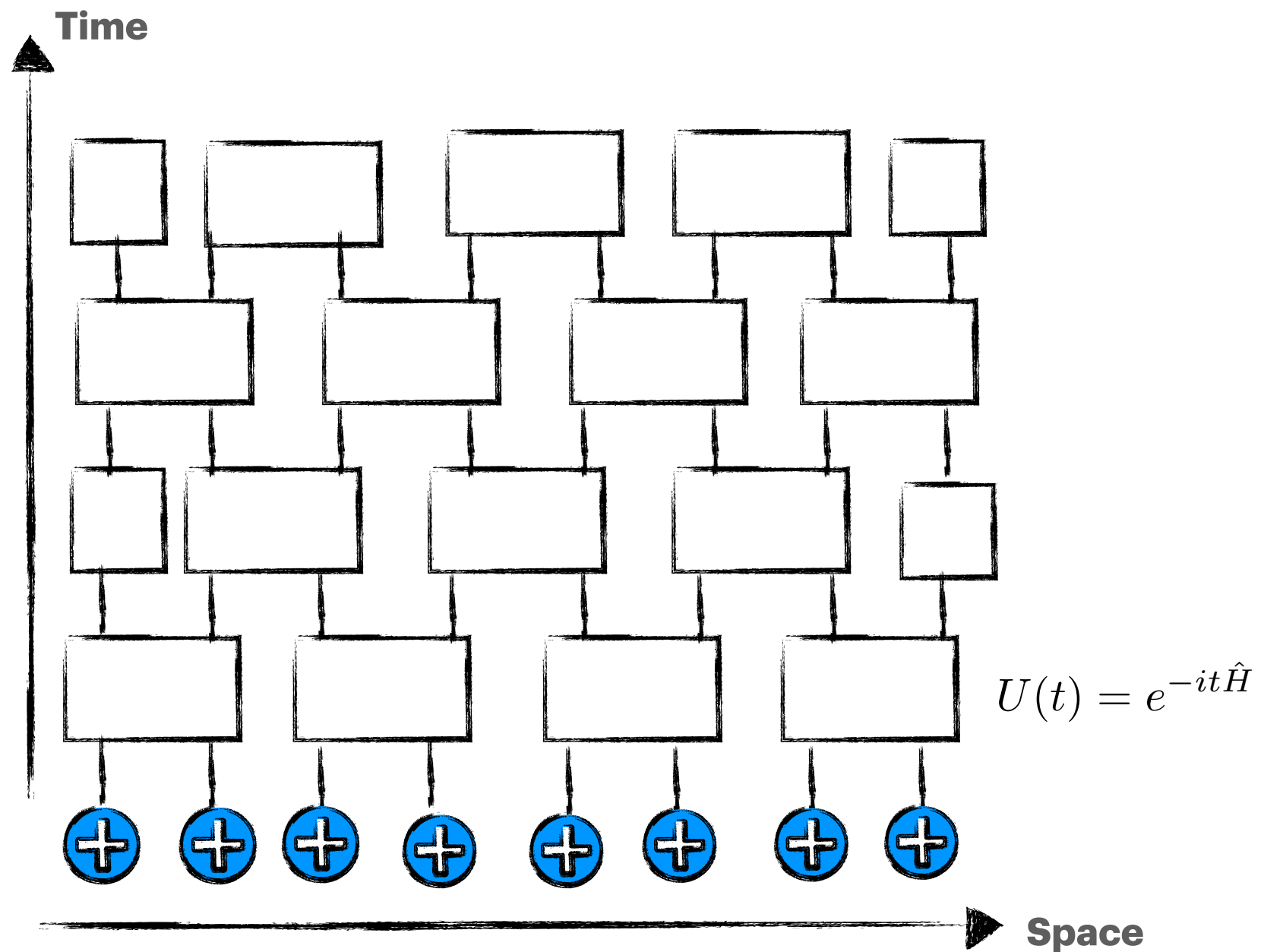
Localized

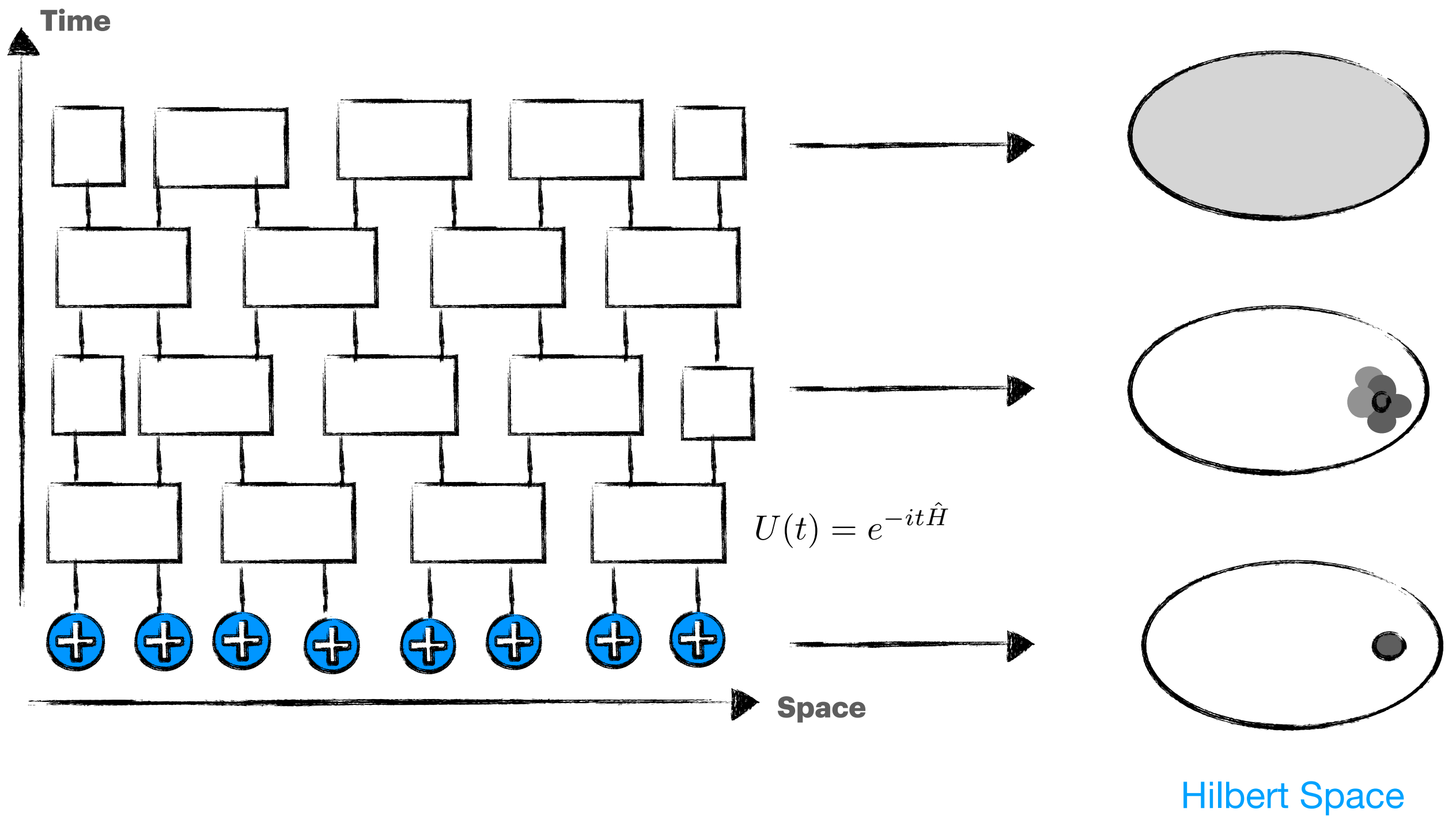


$$|c_n|^2 \sim \delta_{nn_0}$$

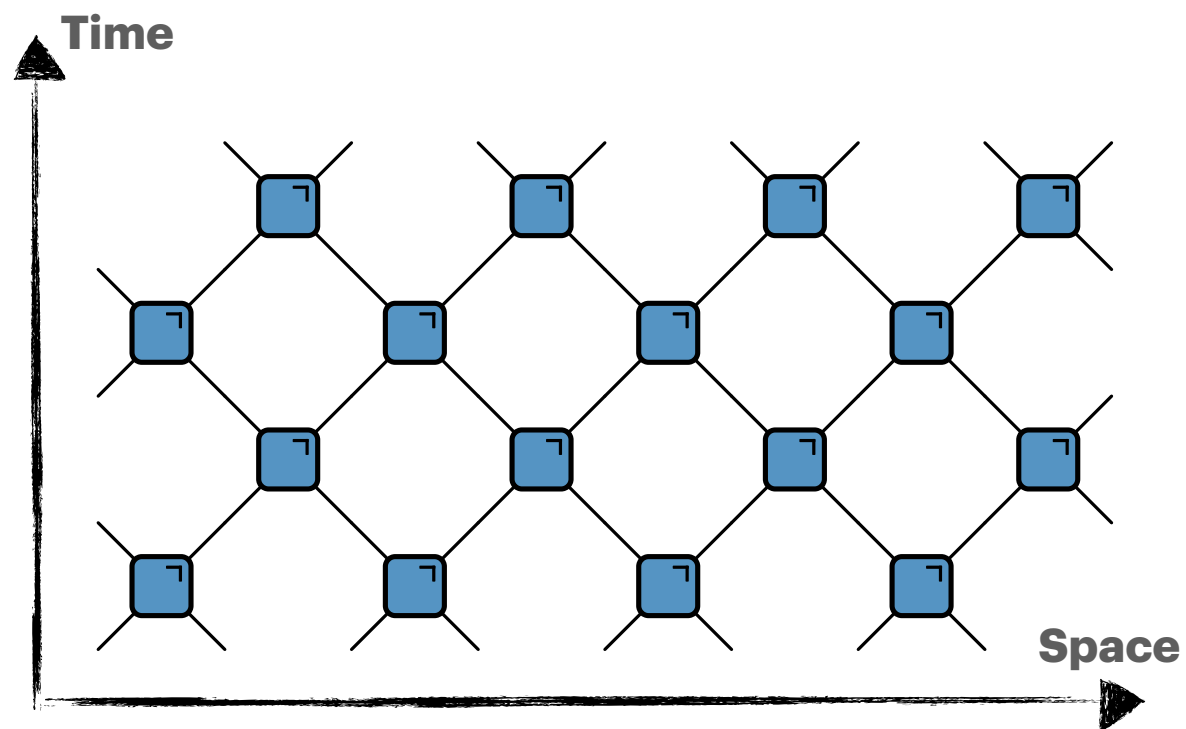
$$D_q = 0$$

Delocalization in the Fock-Space under time evolution





Dual-Unitary Circuits

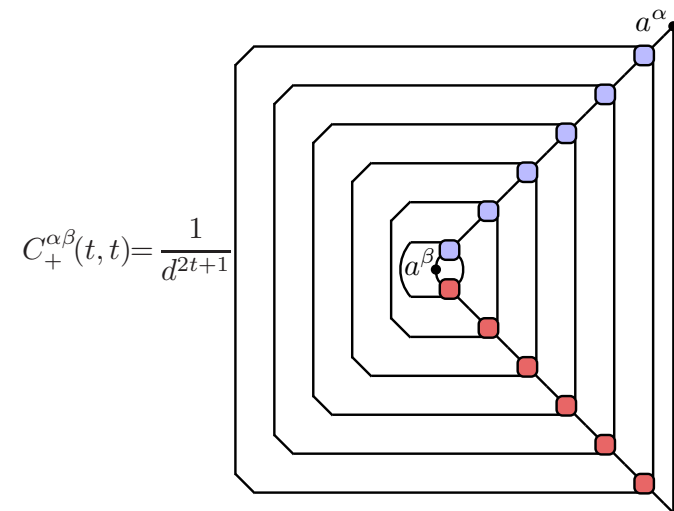


$$U_{ab,cd} = \text{blue square with inputs } a, b \text{ and outputs } c, d$$

$$U_{ab,cd}^\dagger = \text{red square with inputs } a, b \text{ and outputs } c, d$$

$$UU^\dagger = \begin{array}{c} \text{blue square} \\ \text{red square} \end{array} = \begin{array}{c} \text{left vertical line} \\ \text{right vertical line} \end{array} = \mathbb{1}$$

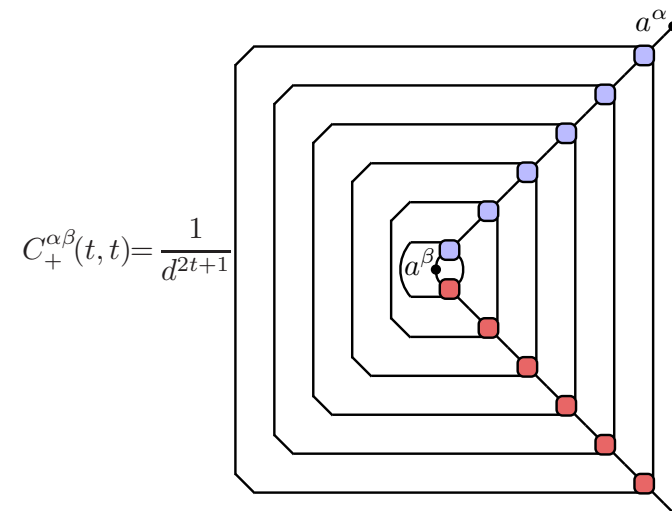
$$\tilde{U}\tilde{U}^\dagger = \begin{array}{c} \text{blue square} \\ \text{red square} \end{array} = \begin{array}{c} \text{left vertical line} \\ \text{right vertical line} \end{array} = \mathbb{1}$$



Exact Time Correlators

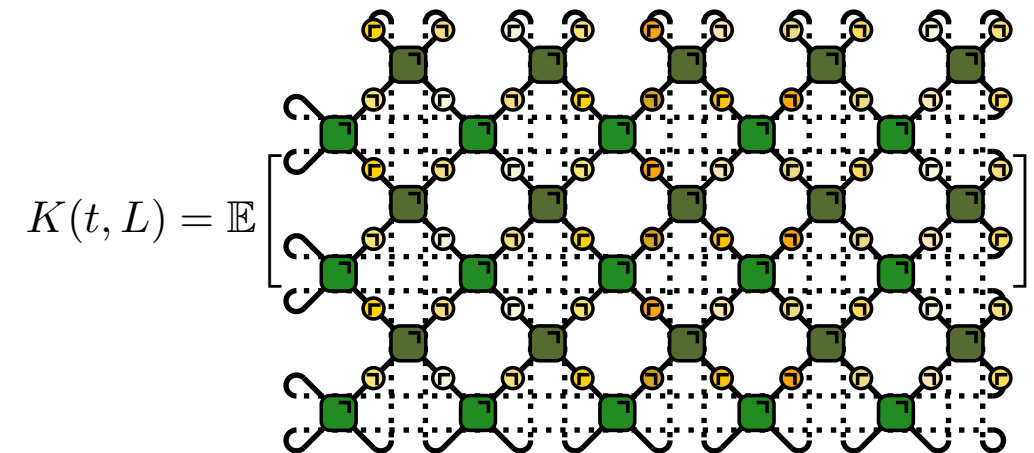
B. Bertini et al., PRL 123 (21), 210601

Dual-Unitary



Exact Time Correlators

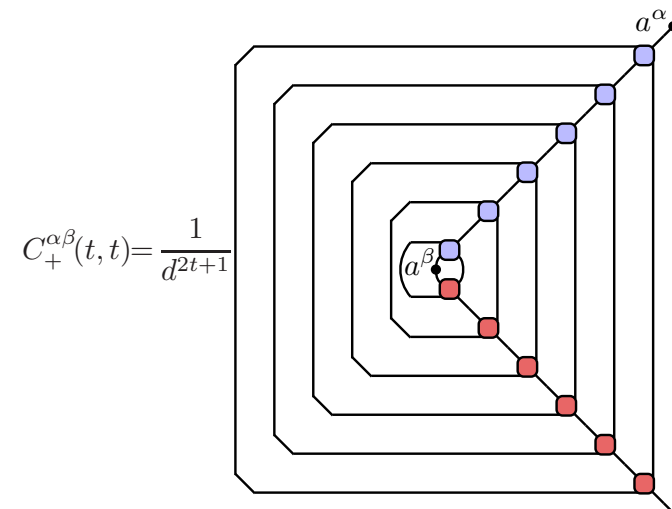
B. Bertini et al., PRL 123 (21), 210601



Spectral Formal Factor

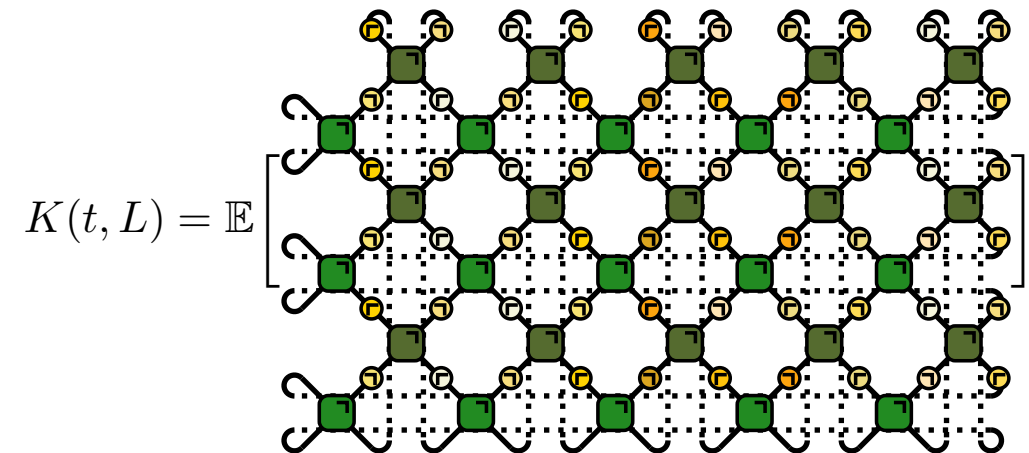
B. Bertini et al., PRL 121 (26), 264101

Dual-Unitary



Exact Time Correlators

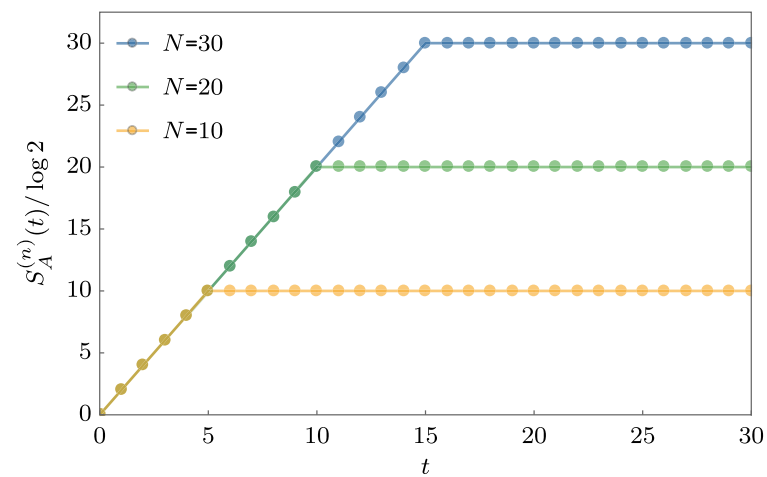
B. Bertini et al., PRL 123 (21), 210601



Spectral Formal Factor

B. Bertini et al., PRL 121 (26), 264101

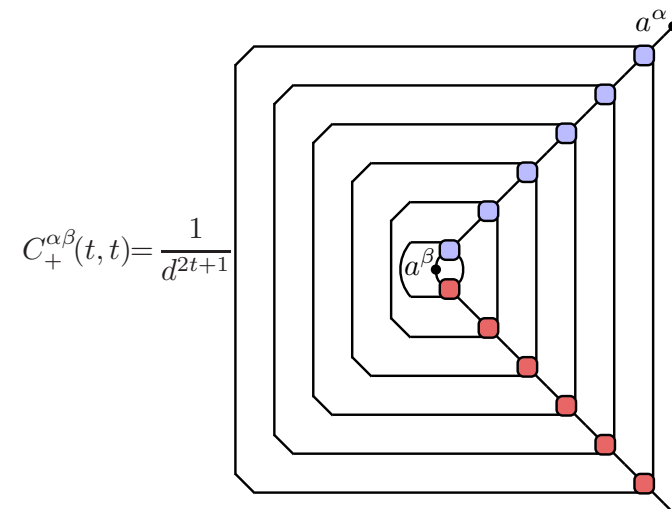
Dual-Unitary



Entanglement

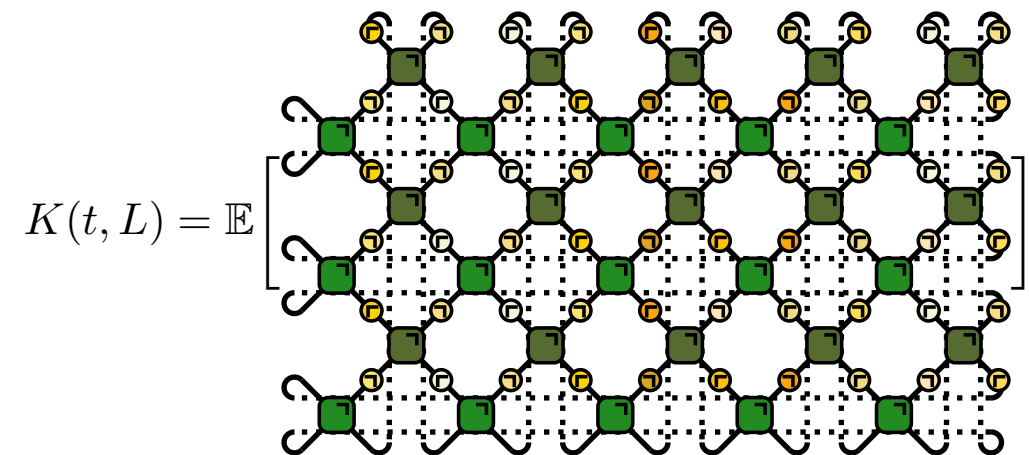
L. Piroli et al. PRB 101 (9), 094304,

B. Bertini PR X 9 (2), 021033



Exact Time Correlators

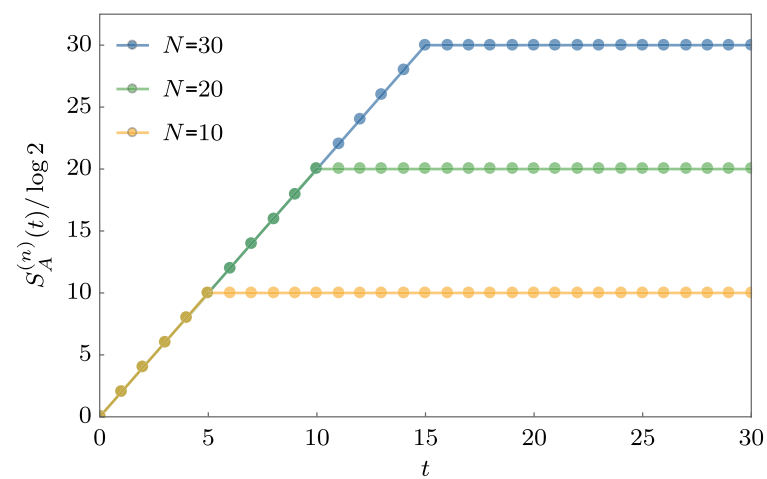
B. Bertini et al., PRL 123 (21), 210601



Spectral Formal Factor

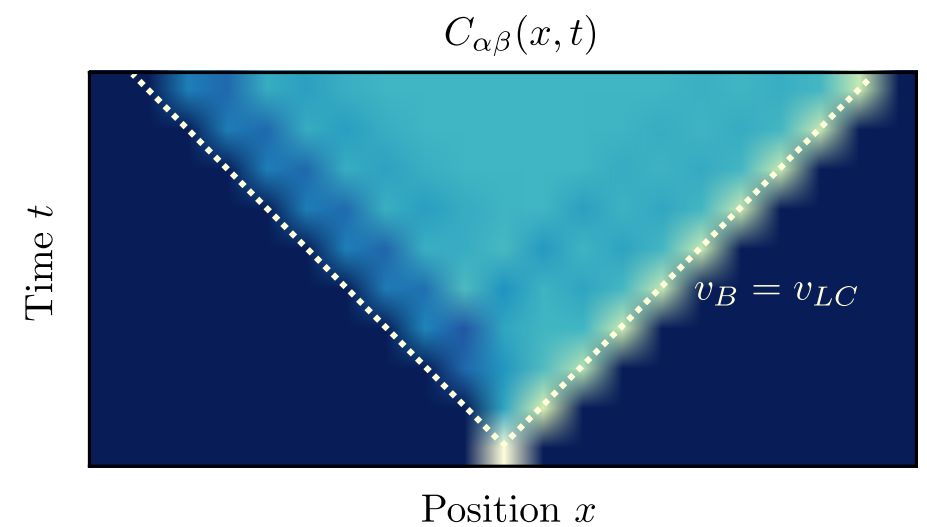
B. Bertini et al., PRL 121 (26), 264101

Dual-Unitary



Entanglement

L. Piroli et al. PRB 101 (9), 094304,
B. Bertini PR X 9 (2), 021033



Scrambling

B. Bertini SciPost 8 (4), 067
P.W. Claeys et al. PRR 2 (3), 033032

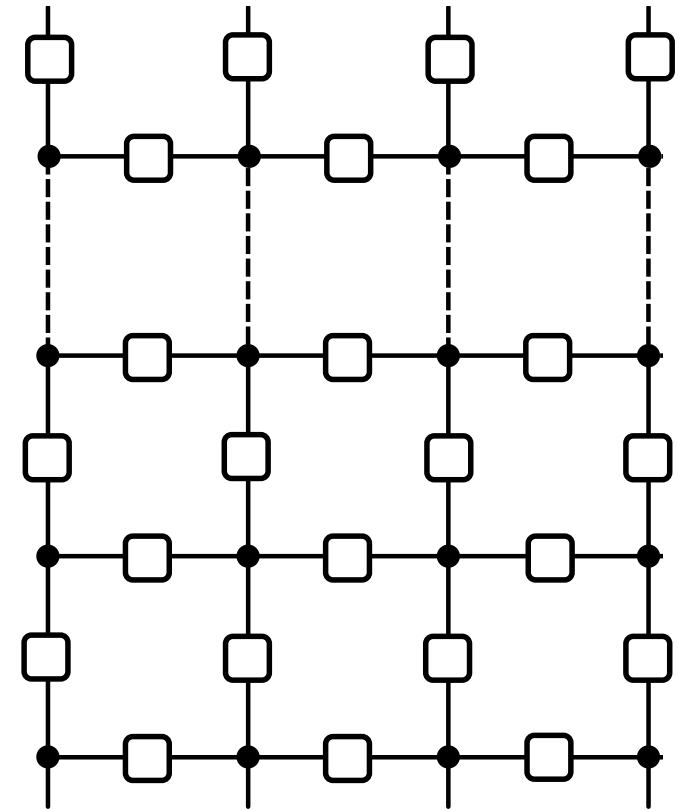
Self-Dual Kicked Ising Model

$$U_F = e^{-iH_K} e^{-iH_I}$$

$$H_I = \frac{\pi}{4} \sum_j^{L-1} \sigma_j^z \sigma_{j+1}^z + \sum_j h_j \sigma_j^z$$

$$H_K = \frac{\pi}{4} \sum_j \sigma_j^x$$

$$U_F = \frac{1}{2^{tL/2}}$$



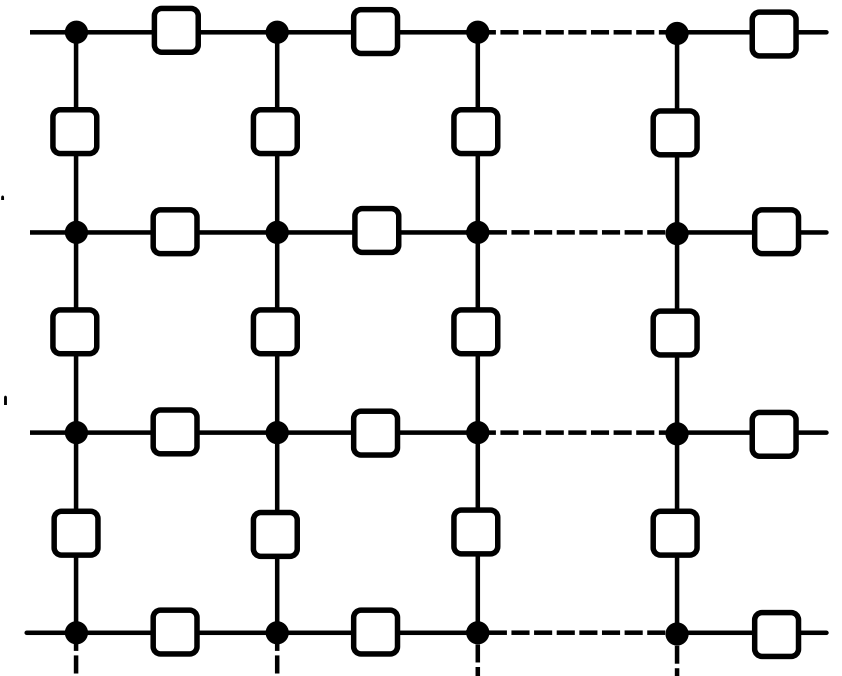
Self-Dual Ising Model

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$$H_K = \frac{\pi}{4} \sum_j \sigma_j^x$$

$$\tilde{U}_F = \frac{1}{2^{tL/2}}$$



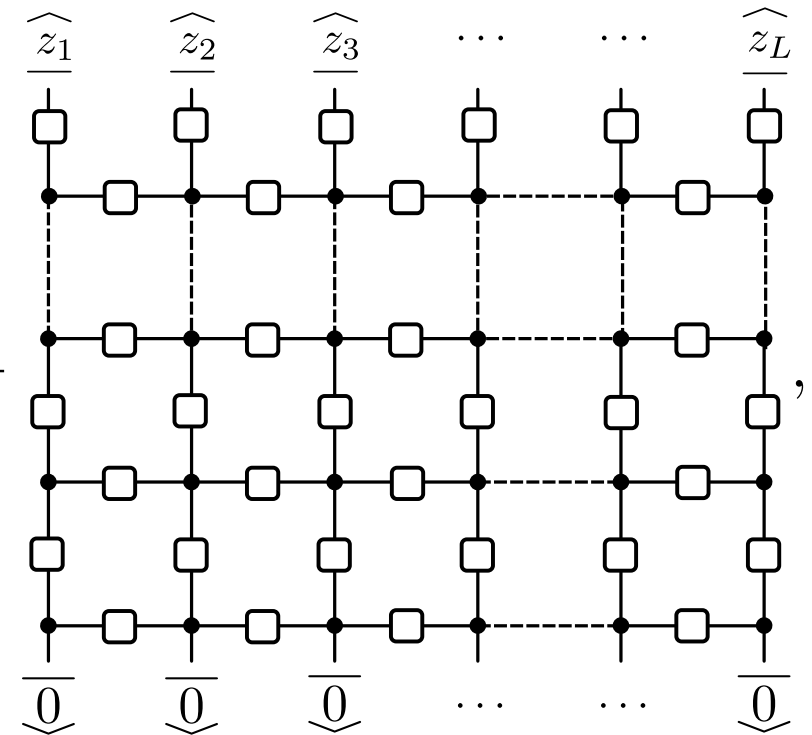
Self-Dual Ising Model

$$U_F = e^{-iH_K} e^{-iH_I}$$

$$H_I = \frac{\pi}{4} \sum_j^{L-1} \sigma_j^z \sigma_{j+1}^z + \sum_j h_j \sigma_j^z$$

$$H_K = \frac{\pi}{4} \sum_j \sigma_j^x$$

$$\langle z | \psi(t) \rangle = \frac{1}{2^{tL/2}}$$



$$|\psi(t)\rangle = U_F^t \bigotimes_{j=1}^L |0\rangle$$

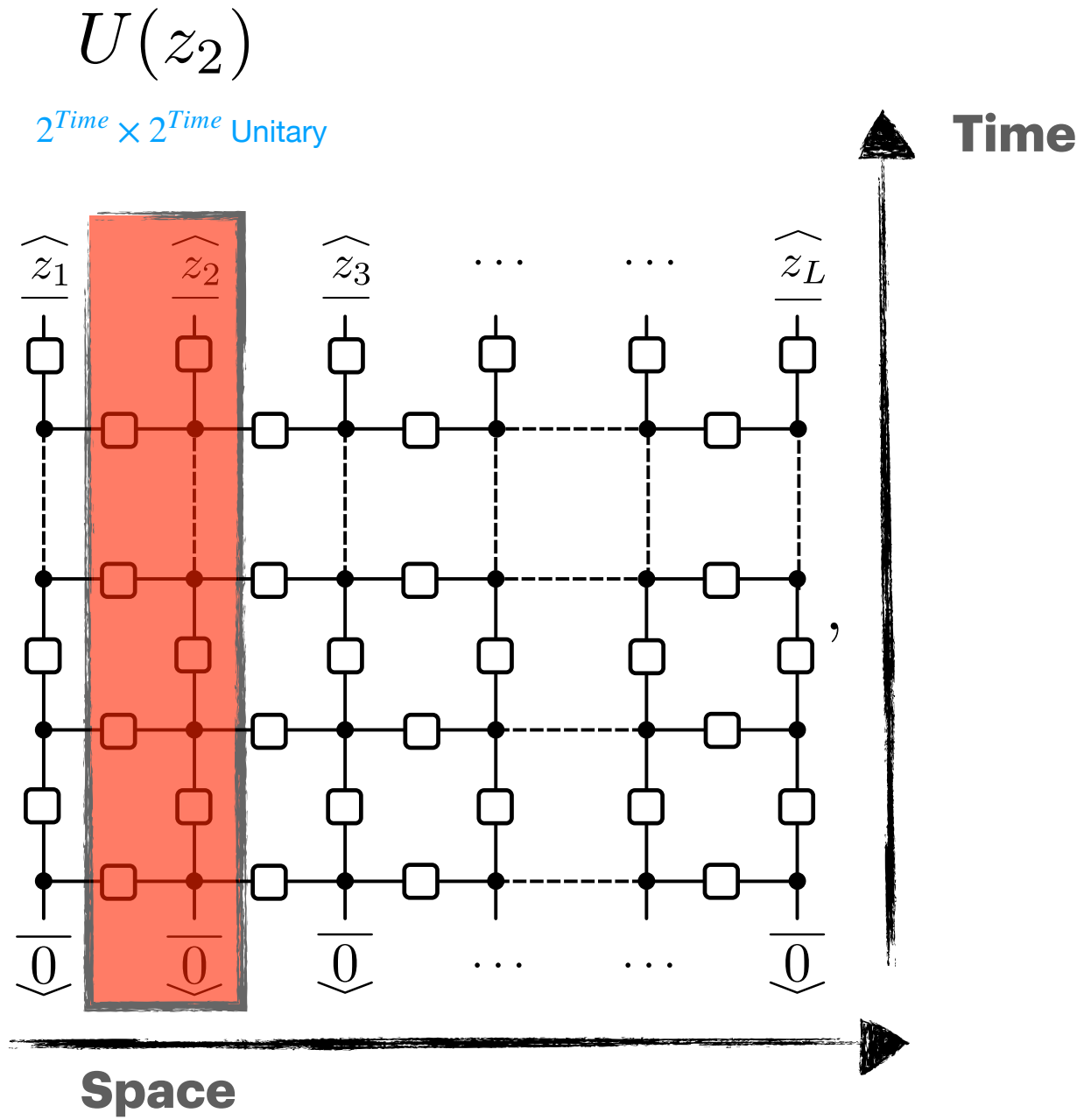
$$N = 2^L$$

Hilbert Space Dimension

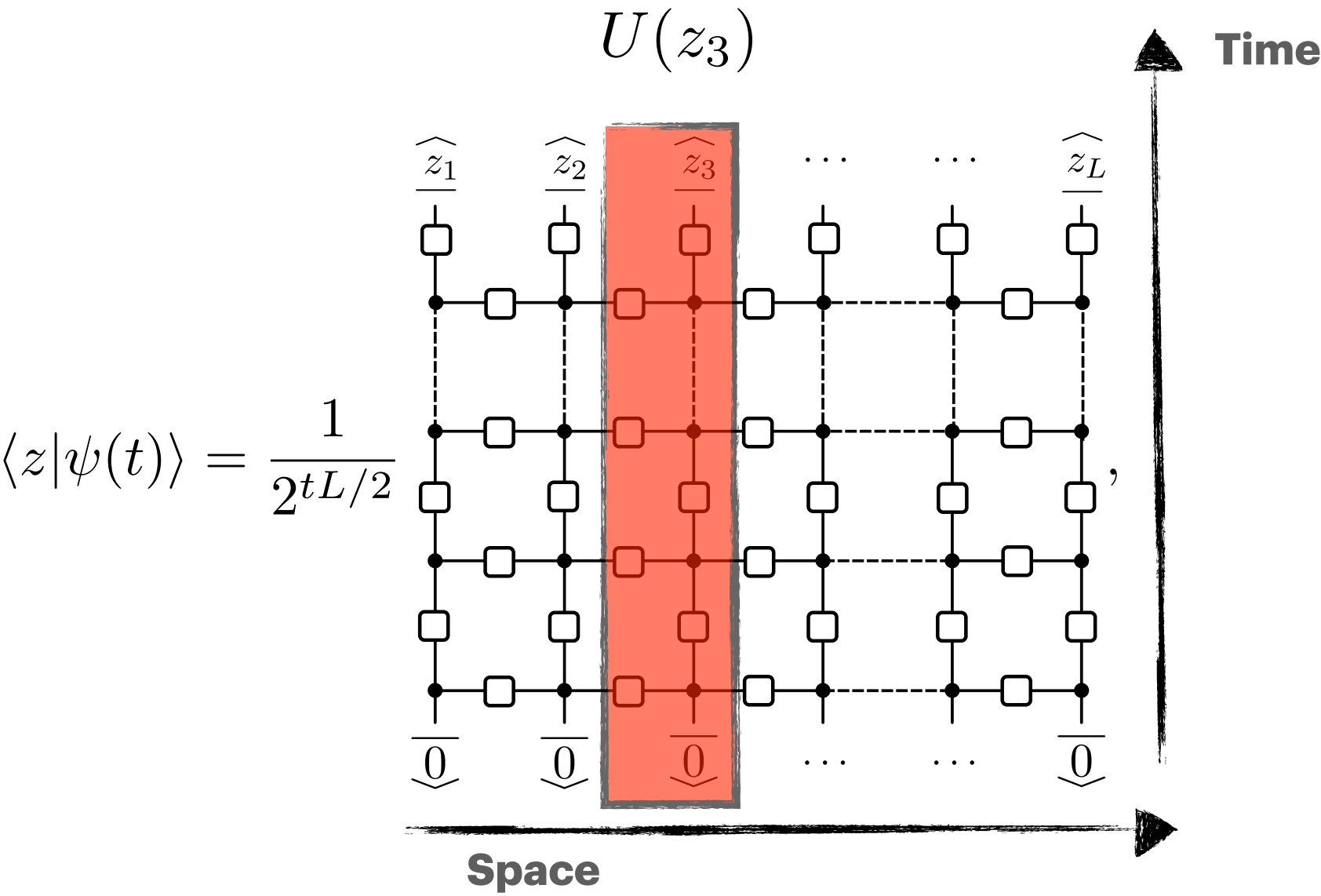
M. Akila, et al J. Phys. A.. 49 375101
 W.W. Ho & S. Choi, PRL 128, 060601 (2022)
 P.W .Claeys, **GDT**, PRL 134 (5), 050405 (2025)

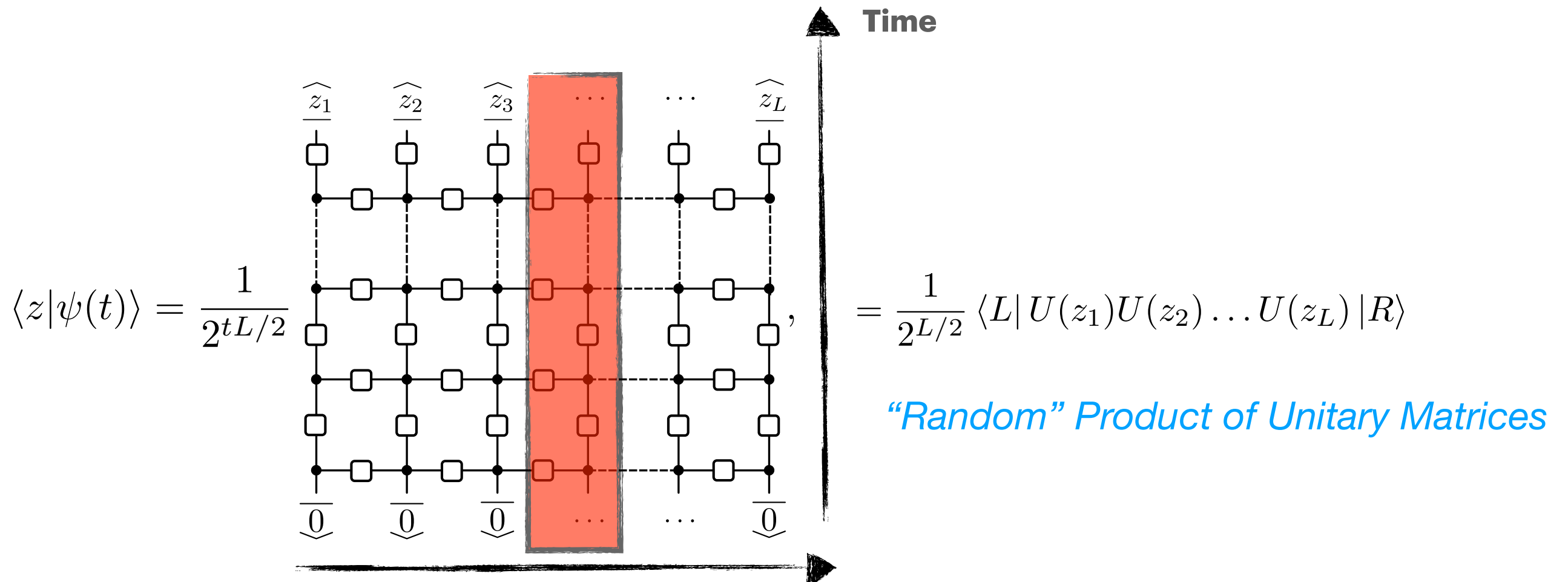
Transfer-Matrix

$$\langle z | \psi(t) \rangle = \frac{1}{2^{tL/2}}$$



Transfer-Matrix

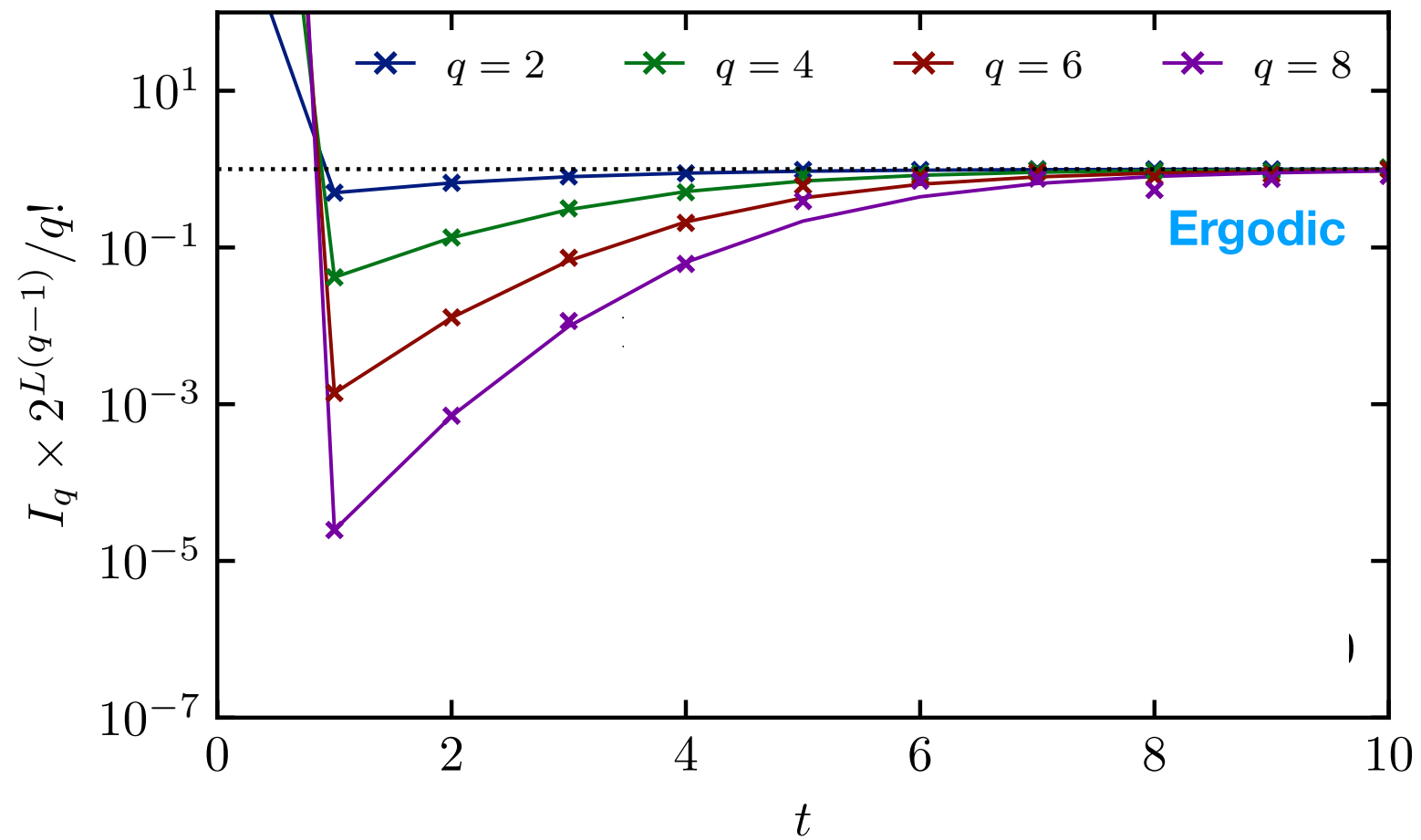




$$\frac{1}{2^{L/2}} \langle L | \prod_i U(z_i) | R \rangle \underset{Haar}{\sim} \frac{1}{2^{L/2}} \langle L | U | R \rangle \longrightarrow$$

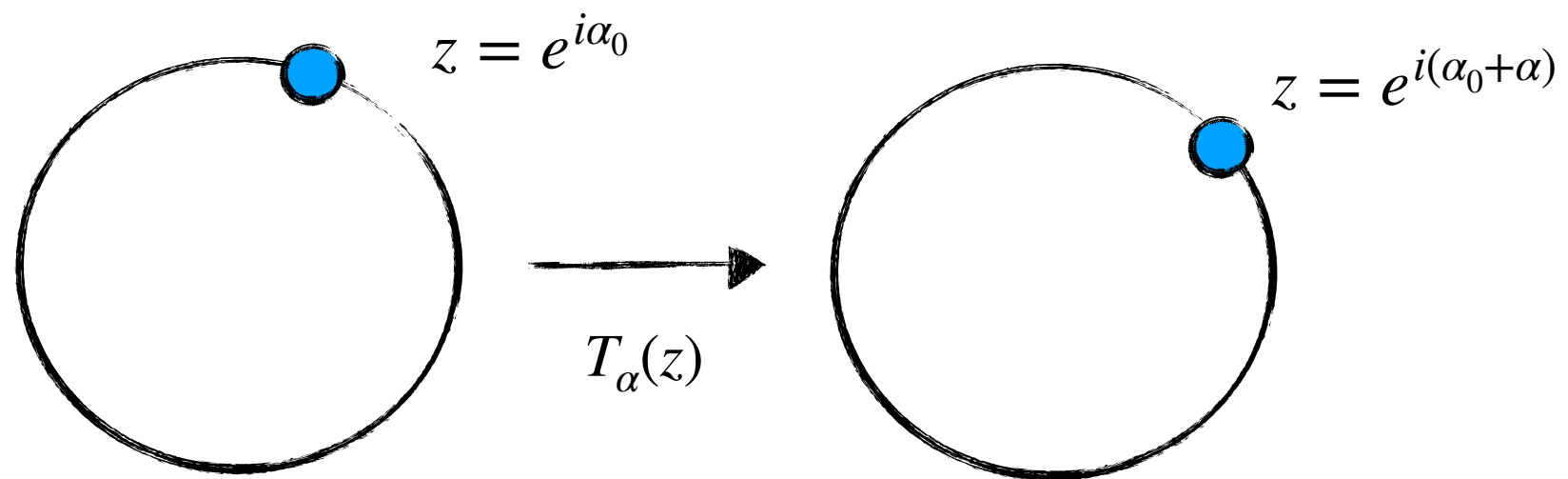
$$I_q = \sum_{z \in \mathcal{B}} |\langle z | \psi(t) \rangle|^{2q} = 2^{L(1-q)} \times \mathbb{E}_{Haar} [|U_{LR}|^{2q}]$$

$$I_q(\tau + 1) = 2^{L(1-q)} \frac{q! 2^{q\tau}}{2^\tau (2^\tau + 1) \dots (2^\tau + q - 1)}$$

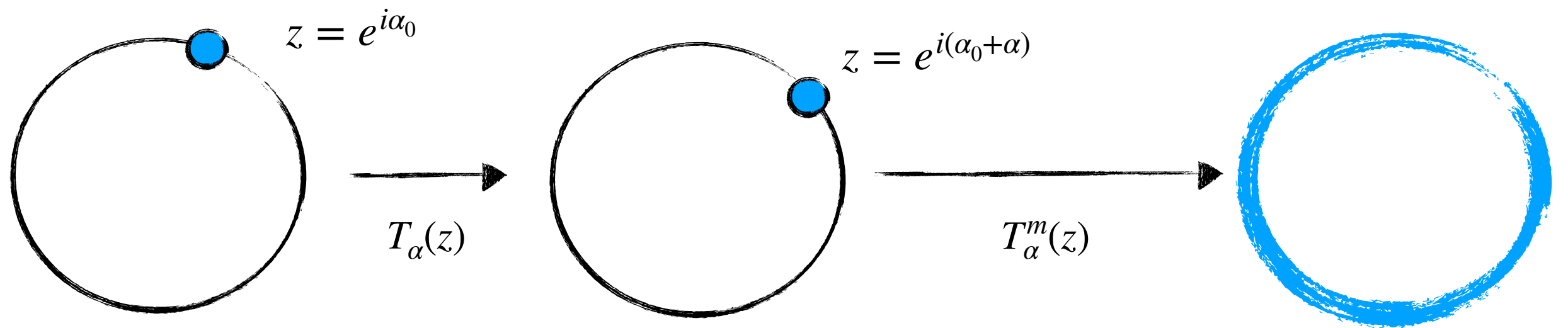


$$I_q(\tau + 1) \sim I_q^{Haar} \frac{1}{1 + \frac{q(q-1)}{2} e^{-\tau} + o(e^{-\tau})}$$

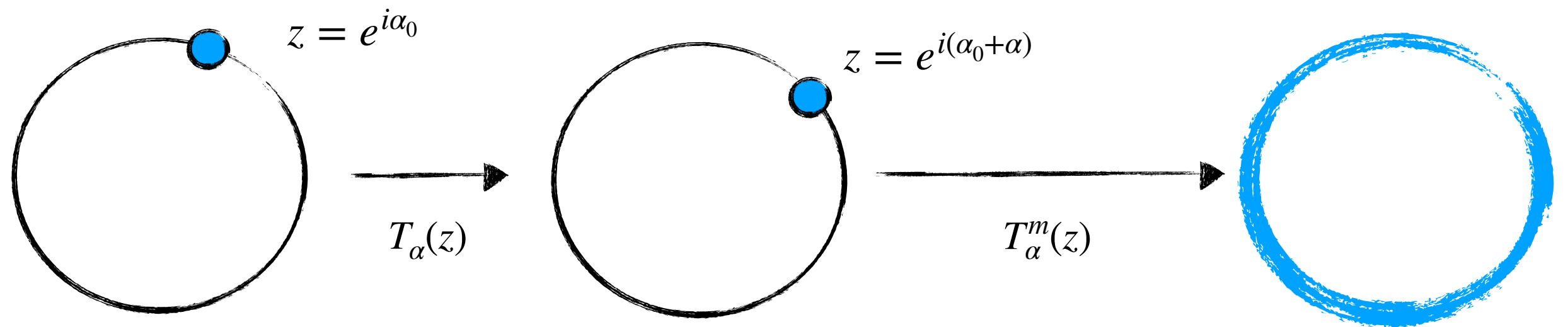
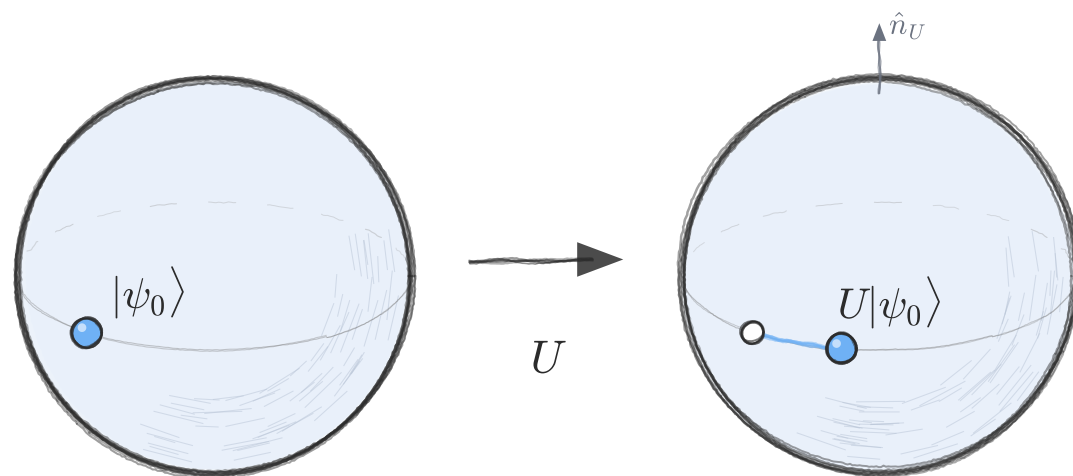
Intermezzo:



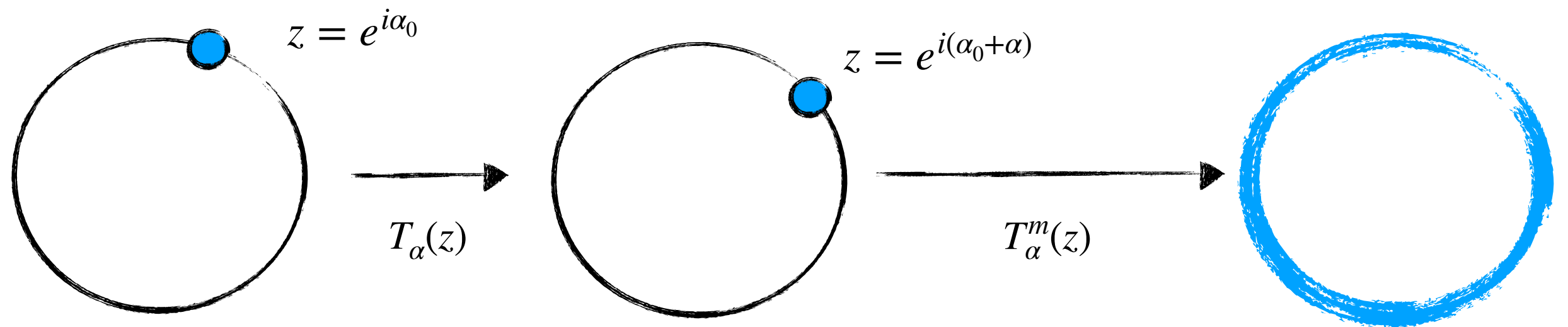
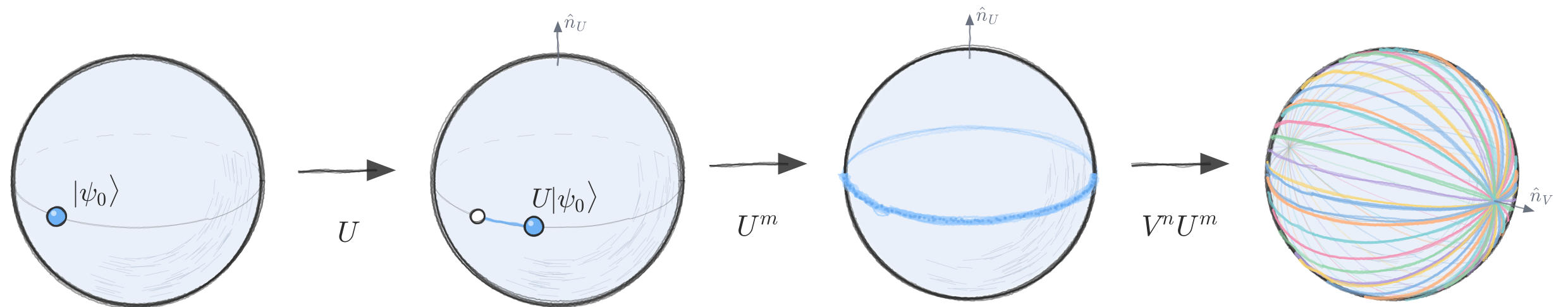
Intermezzo:

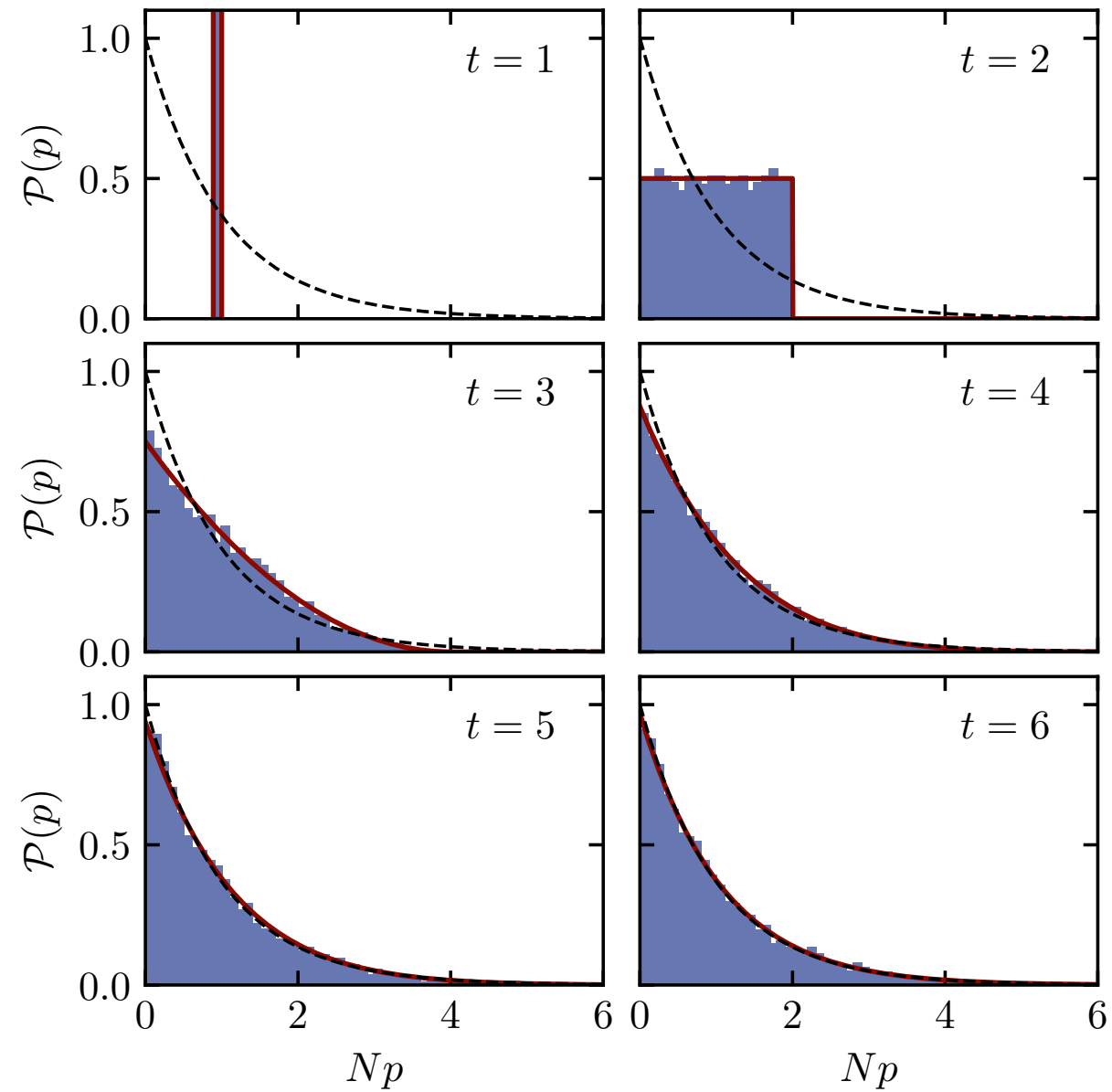


Intermezzo:



Intermezzo:





$$I_q = \sum_{z \in \mathcal{B}} |\langle z | \psi \rangle|^{2q} = 2^L \int_0^1 dp \mathcal{P}(p) p^q$$

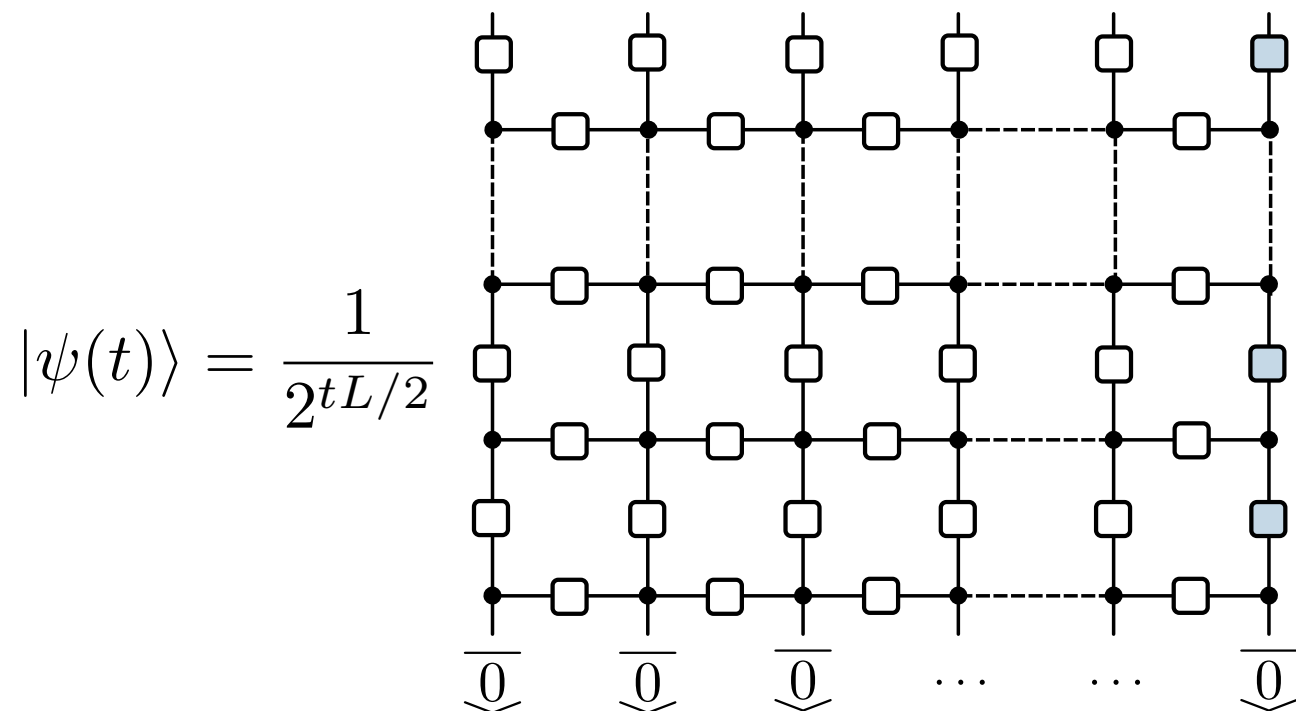
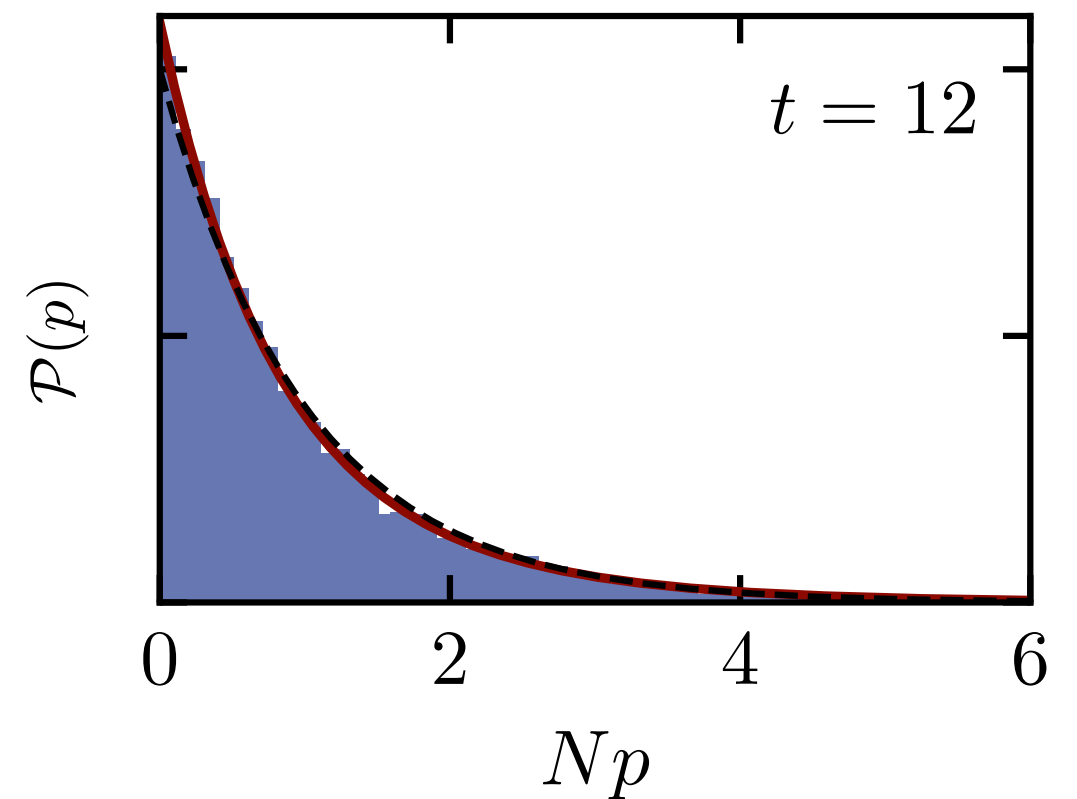
Inverse Participation Ratio

$$\begin{aligned} \mathcal{P}_{\text{DU}}(p; t) &= N(1 - 2^{-\tau}) \left(1 - \frac{Np}{2^\tau}\right)^{2^\tau - 2} \theta(2^\tau - Np) \\ &\propto \theta(2^\tau - \tilde{p}) \left(1 - \frac{\tilde{p}}{2^\tau}\right)^{2^\tau - 2} \quad \tilde{p} = Np \end{aligned}$$

Rescaled Porter-Thomas Distribution

Stability to Perturbations

$$I_q(t) = I_q^{\text{DU}}(t) \times \frac{[1 + \cos^t(2\theta)]^q + [1 - \cos^t(2\theta)]^q}{2}$$



$\square \quad u = e^{-i\theta\sigma^x}$

Dual-Unitary Breaking Perturbation

Recap!

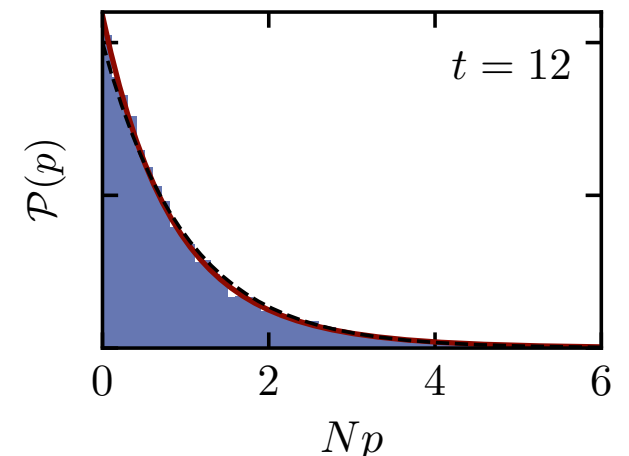
- **Emergence of the Porter-Thomas Distribution**

- **The approach is exponentially fast and time scales do not depend on the system size**

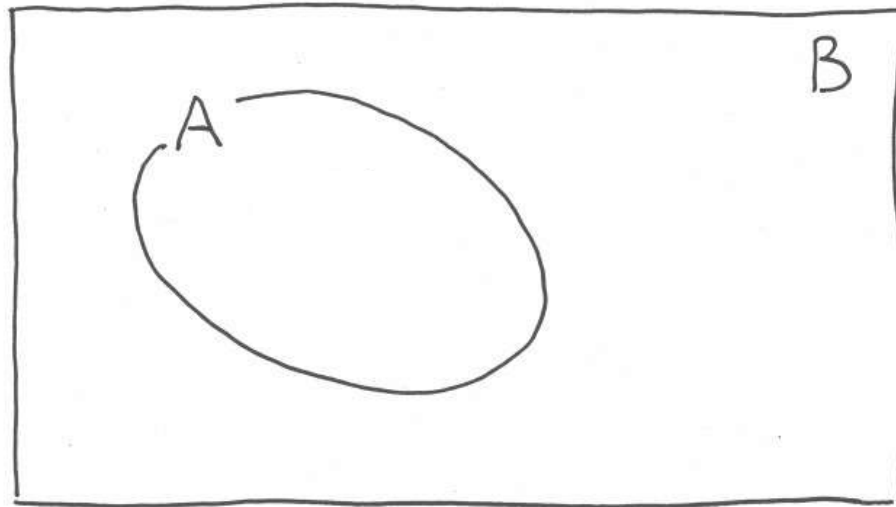
Random Unitary $\sim \log L$ X. Turkeshi, P. Sierant Entropy 2024, 26, 471 (2024)

- **No need of disorder**

- **Stability to “some” Perturbations**



Entanglement Entropy



$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$$

$$\text{Dim}(\mathcal{H}) = \text{Dim}(\mathcal{H}_A)\text{Dim}(\mathcal{H}_B) \rightarrow N = N_A N_B$$

Ergodic: Page value

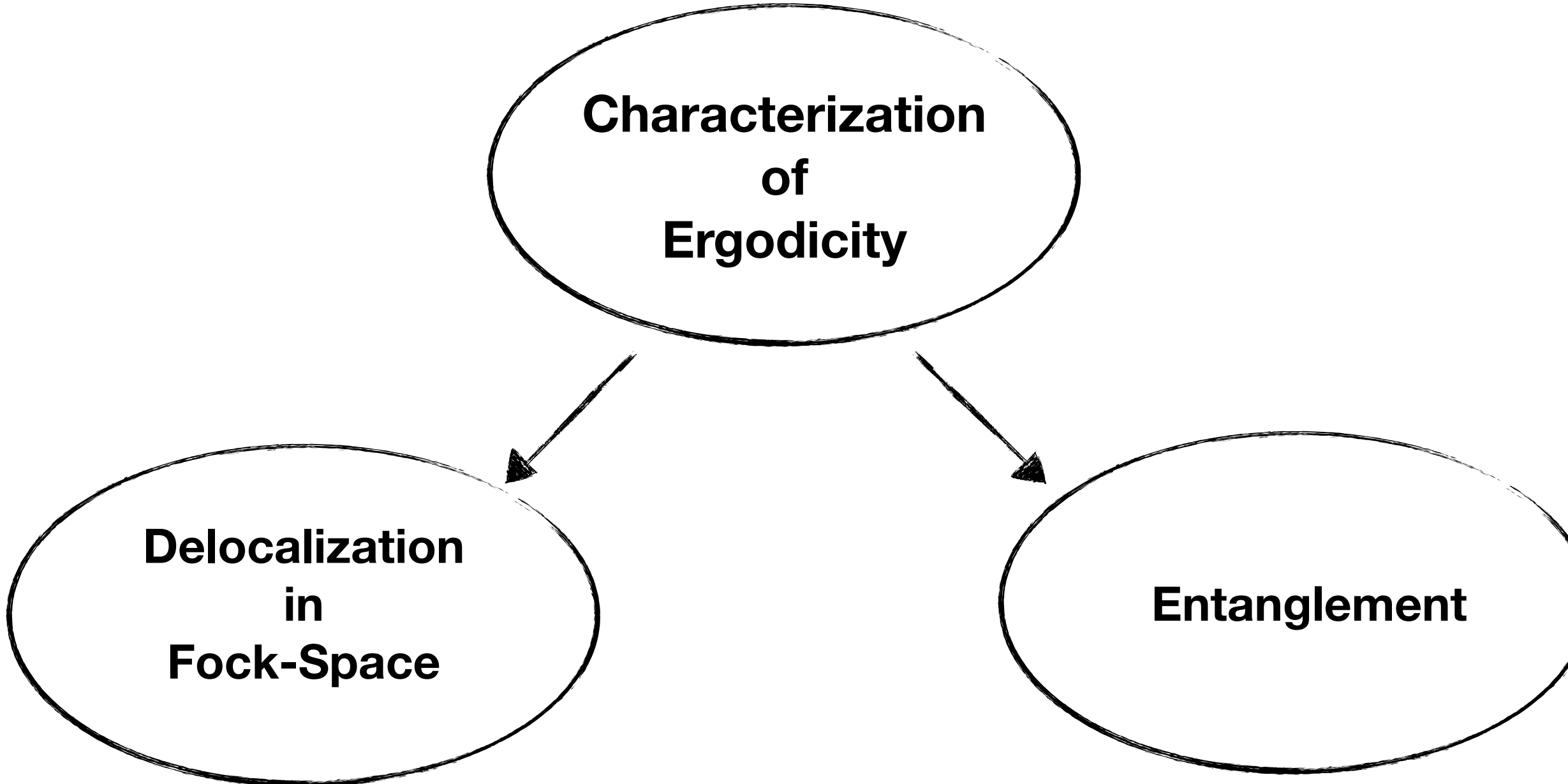
$$\overline{S_{1,Page}} = \log N_A - \frac{N_A}{2N_B}$$



$$N = 2^L, \quad N_A = 2^{L/2}$$

$$\overline{S_{1,Page}} = \frac{L}{2} \log 2 - \frac{1}{2}$$

**Characterization
of
Ergodicity**



```
graph TD; A([Characterization of Ergodicity]) --> B([Delocalization in Fock-Space]); A --> C([Entanglement]);
```

**Delocalization
in
Fock-Space**

Entanglement

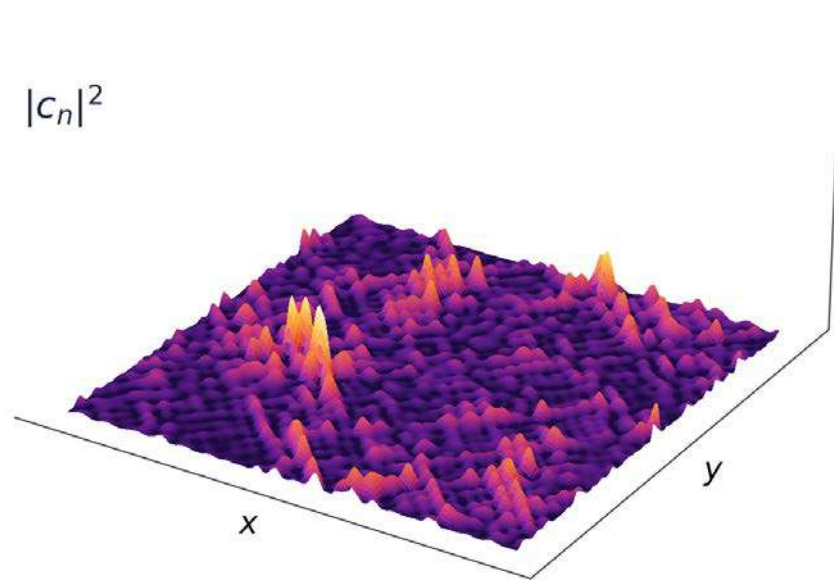
$$I_q = \sum_n |c_n|^{2q} \sim N^{(1-q)D}$$

Fractal Dimension

$$S_q(A) \sim D_{q,ent} S_{q,Page}(A)$$

Deviation from Page

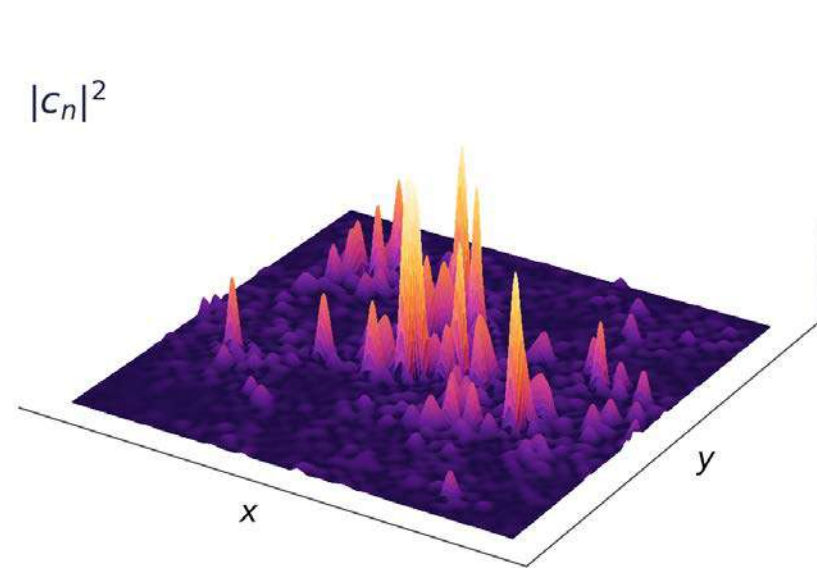
Ergodic



$$D_q = 1$$

$$D_{q,ent} = 1$$

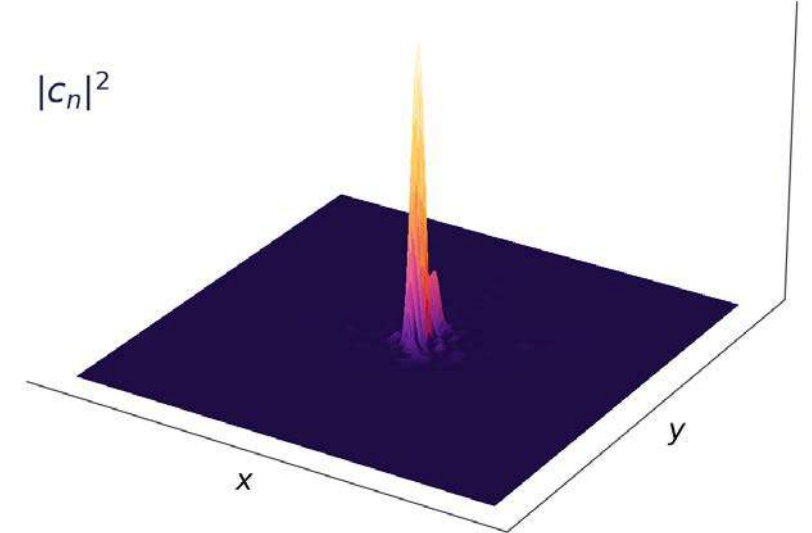
Multifractal



$$0 < D_q < 1$$

$$D_{q,ent} = ??$$

Localized



$$D_q = 0$$

$$D_{q,ent} = 0$$

Sparse Random Pure States

$$\#\{|c_n|^2 \neq 0\} \sim N^D, \quad \frac{\#\{|c_n|^2 \neq 0\}}{N} \rightarrow 0$$

Support of the wave-function

$$|\psi_{S-RPS}\rangle \sim$$

$$\begin{pmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_{37} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_{54} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

Sparse Random Pure States

$$\#\{|c_n|^2 \neq 0\} \sim N^D, \quad \frac{\#\{|c_n|^2 \neq 0\}}{N} \rightarrow 0$$

Support of the wave-function

$$c_n = g_n r_n$$

$$g_n \sim \mathcal{N}_{\mathbb{C}}(0,1)$$

$$P(r_n = 1) = N^{D-1}, \quad P(r_n = 0) = 1 - N^{D-1}$$

$$I_q \sim N^{(1-q)D}, \quad D_q = D$$

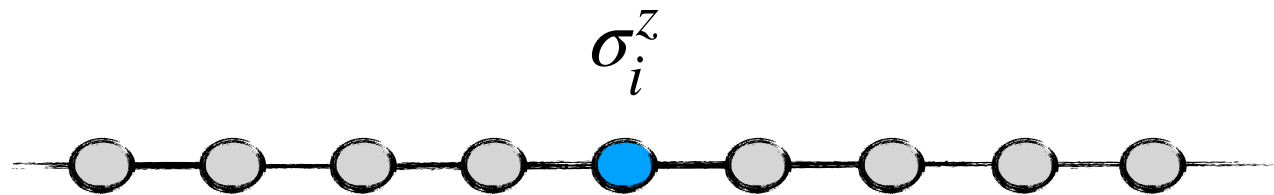
$$|\psi_{S-RPS}\rangle \sim$$

$$\begin{pmatrix} c_1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_{37} \\ 0 \\ \vdots \\ \vdots \\ 0 \\ c_{54} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

Warm-up: Local Observable

$$N = 2^L$$

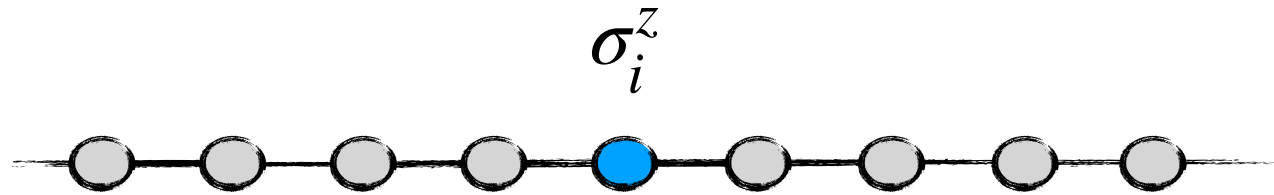
$$|\psi_{S-RPS}\rangle = \sum_{\sigma} a_{\sigma} |\sigma\rangle_z$$



Warm-up: Local Observable

$$N = 2^L$$

$$|\psi_{S-RPS}\rangle = \sum_{\sigma} a_{\sigma} |\sigma\rangle_z$$



$$\langle \psi_{S-RPS} | \sigma_i^z | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^z] + \mathcal{O}(N^{-D/2})$$

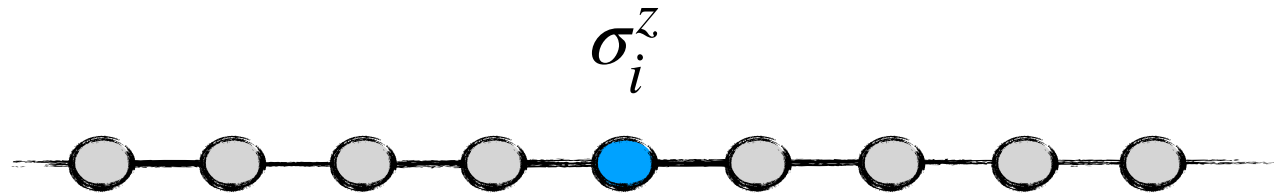
“Fading ergodicity”

M. Kliczkowski et al. PRB **110**, 134206 (2024)

Local Observables know about fractal dimension.

Warm-up: Local Observable

$$N = 2^L$$



$$|\psi_{S-RPS}\rangle = \sum_{\sigma} a_{\sigma} |\sigma\rangle_z$$

$$\langle \psi_{S-RPS} | \sigma_i^z | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^z] + \mathcal{O}(N^{-D/2})$$

“Fading ergodicity”

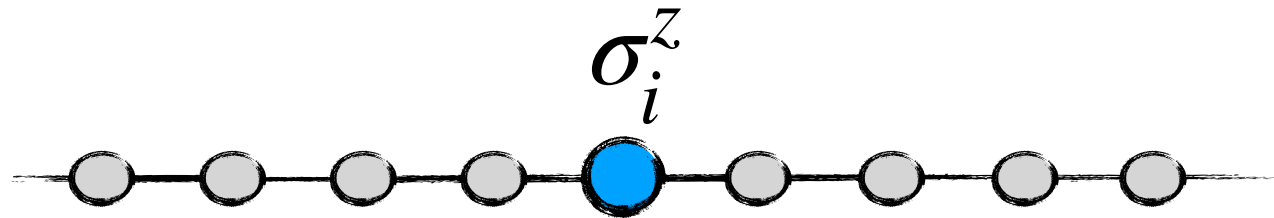
M. Kliczkowski et al. PRB **110**, 134206 (2024)

Local Observables know about fractal dimension.

Q: Do all observables escape ergodicity?

Warm-up: Local Observable

$$N = 2^L$$



$$|\psi_{S-RPS}\rangle = \sum_{\sigma} a_{\sigma} |\sigma\rangle_z$$

$$\langle \psi_{S-RPS} | \sigma_i^z | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^z] + \mathcal{O}(N^{-D/2})$$

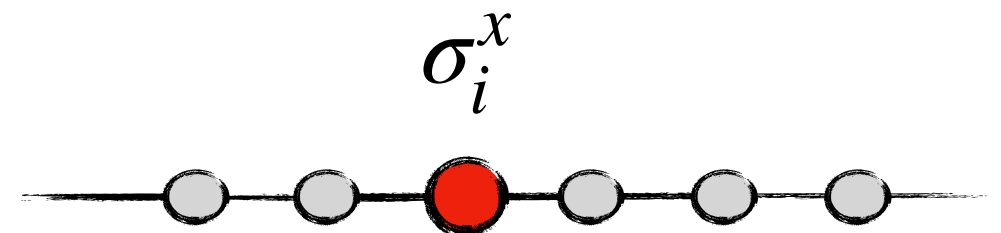
“Fading ergodicity”

M. Kliczkowski et al. PRB **110**, 134206 (2024)

Local Observables know about fractal dimension.

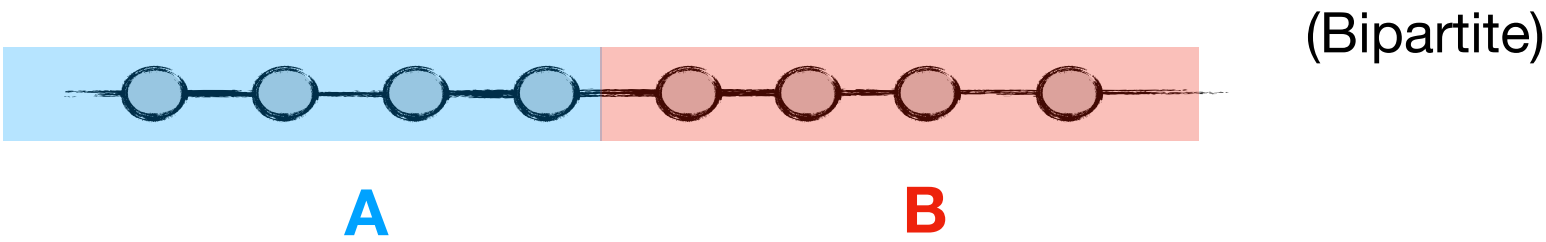
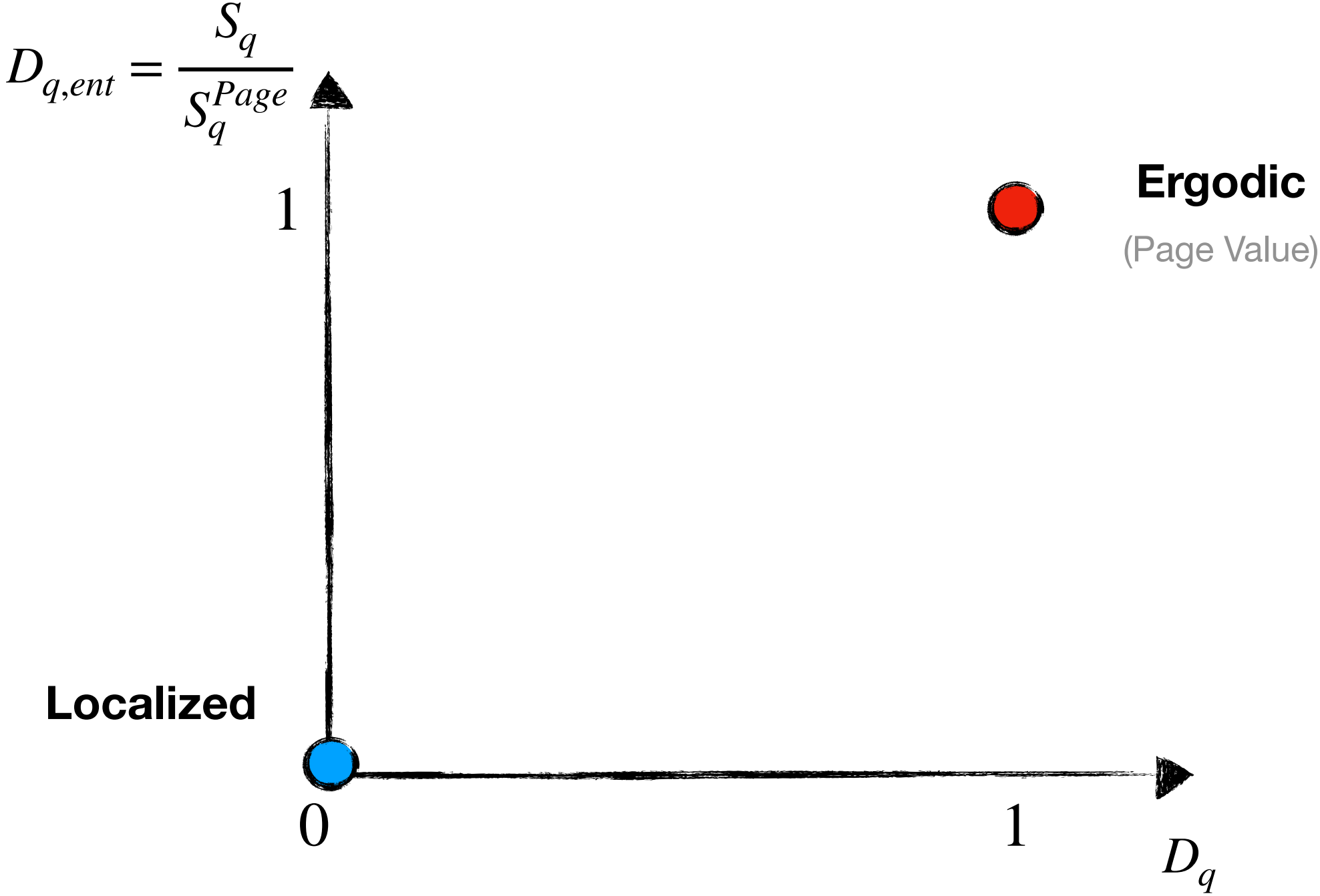
Q: Do all observables escape ergodicity? No!

$$\langle \psi_{S-RPS} | \sigma_i^x | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^x] + \mathcal{O}(N^{-1/2})$$



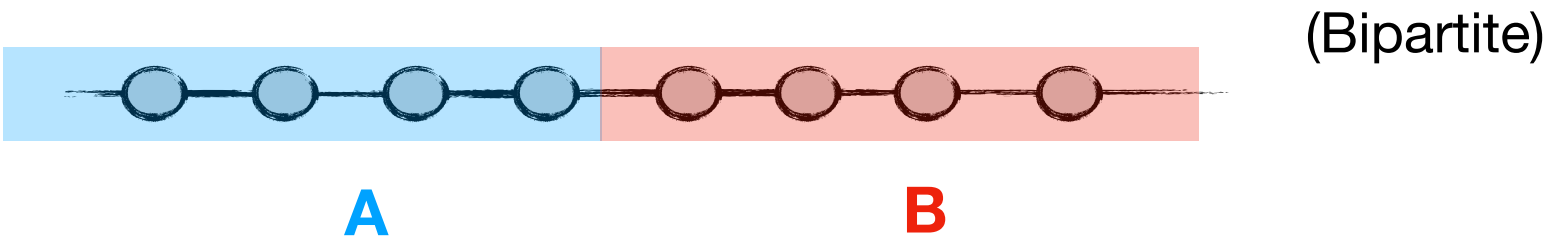
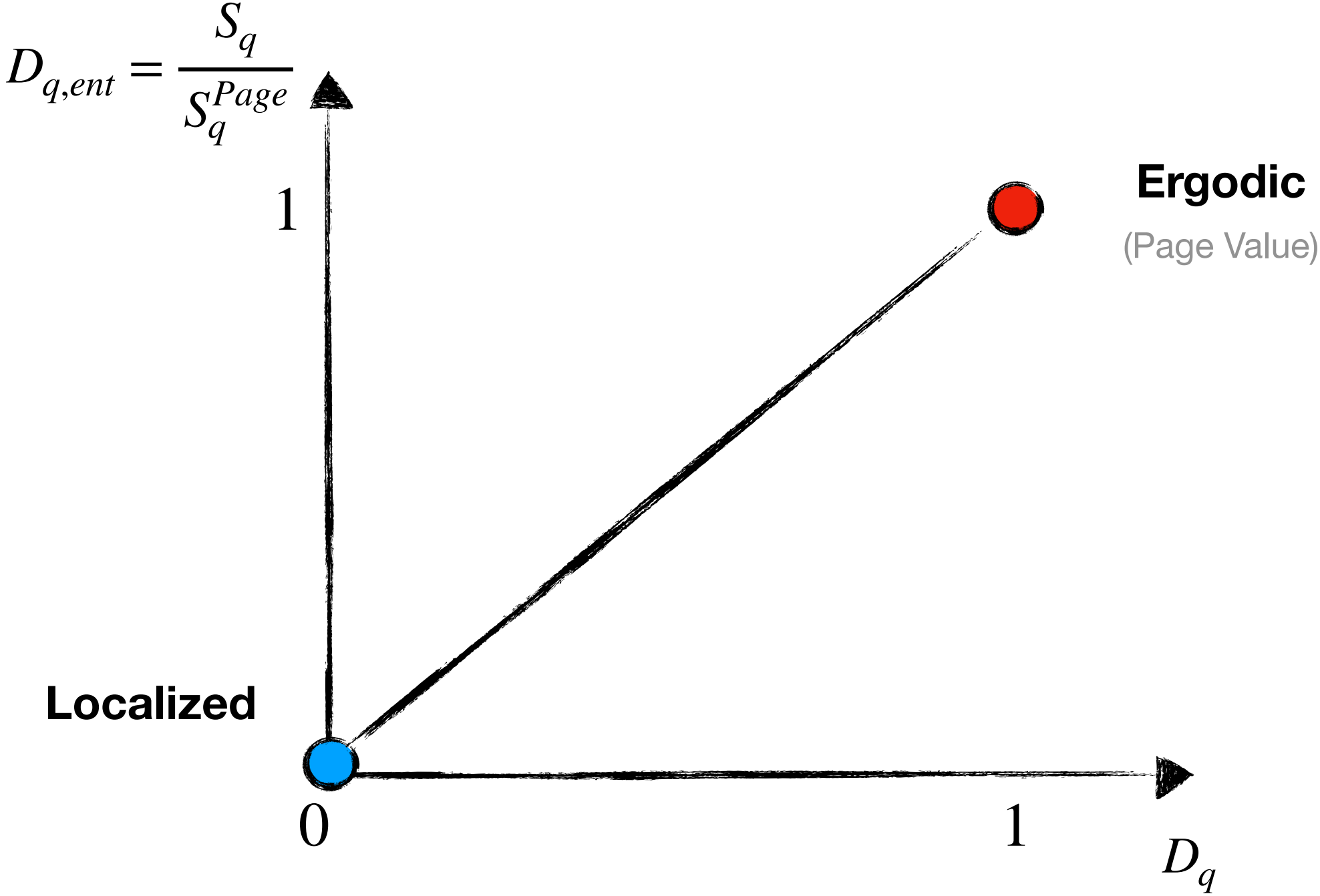
Back to Entanglement:

$$N = 2^L \quad N_A = N_B = 2^{L/2}$$



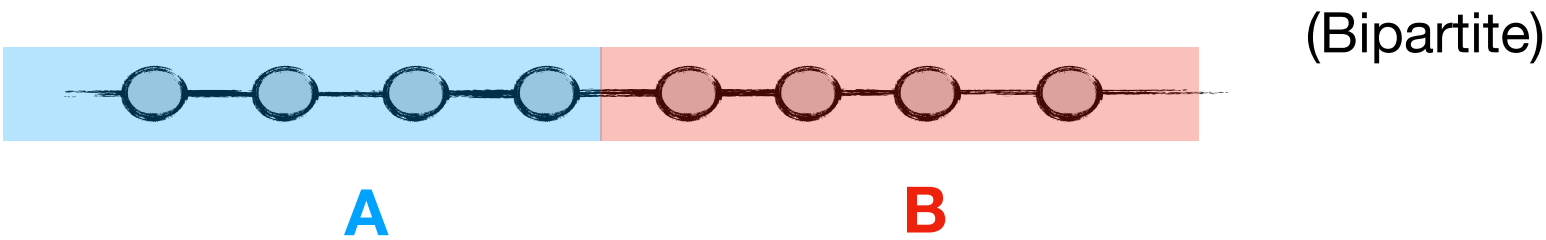
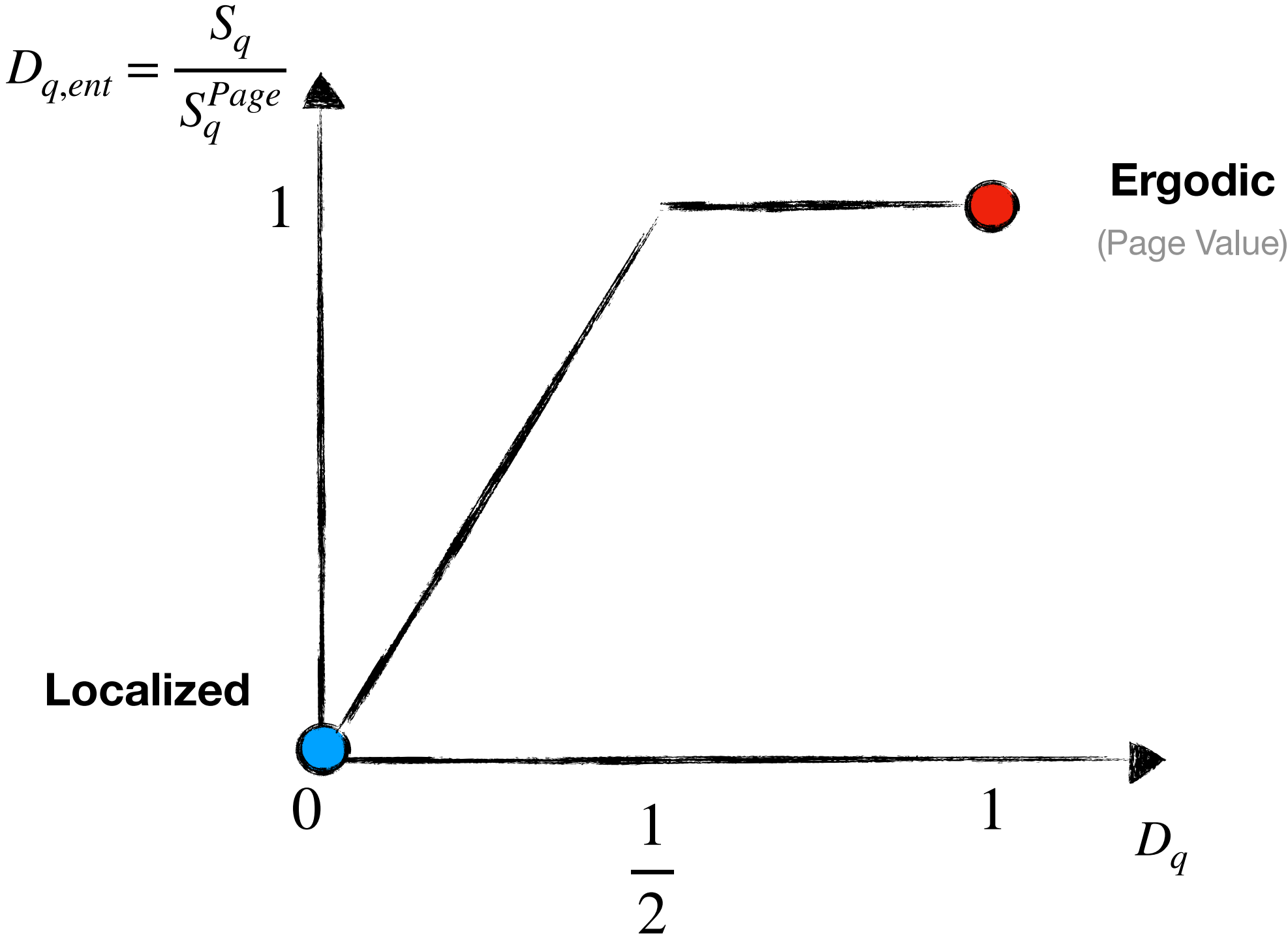
Back to Entanglement:

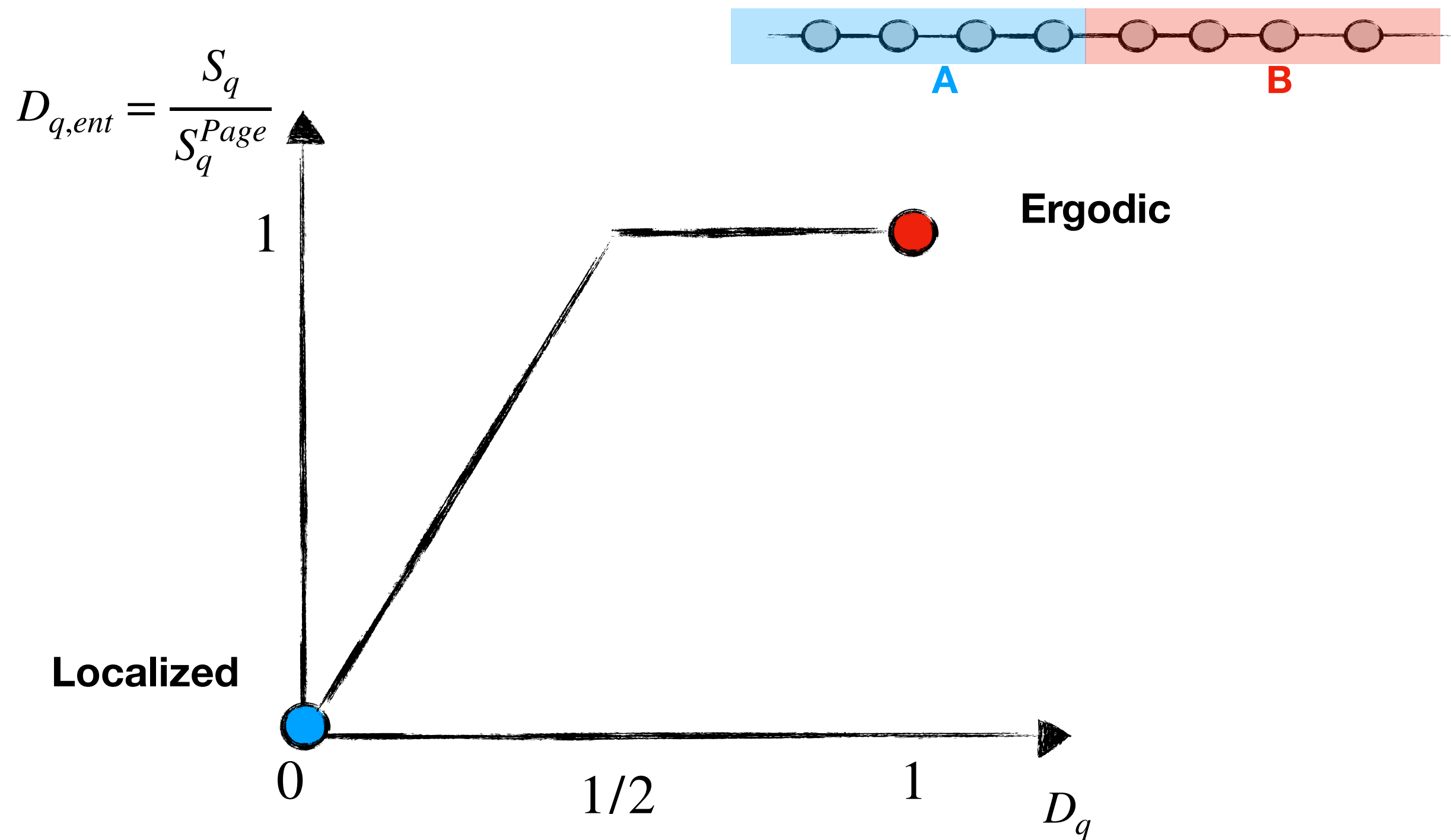
$$N = 2^L \quad N_A = N_B = 2^{L/2}$$



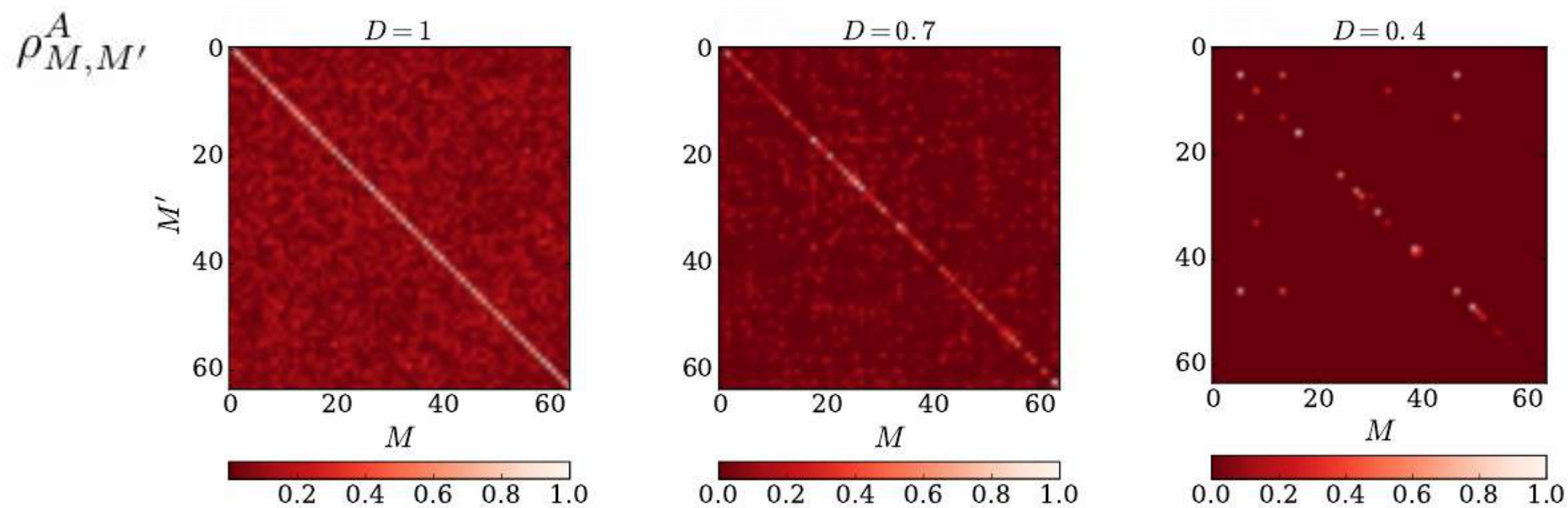
Back to Entanglement:

$$N = 2^L \quad N_A = N_B = 2^{L/2}$$





Entanglement Entropy can have an ergodic behavior
even though
the wave- function is highly non-ergodic

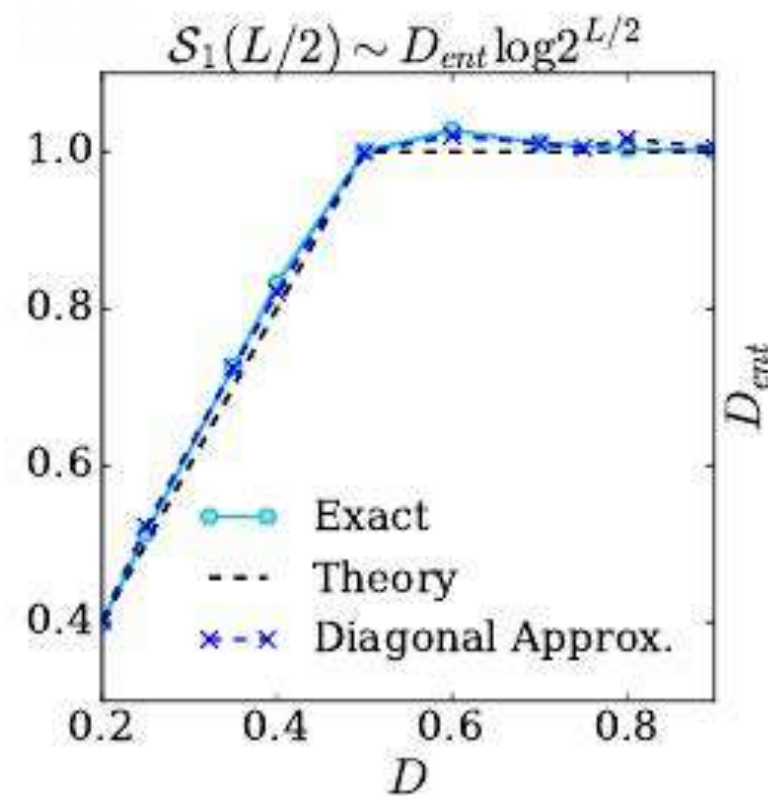


$$\overline{S_1}(A) \sim - \sum_M \overline{\rho_{M,M}^A \ln \rho_{M,M}^A} \sim \overline{\ln N_0}$$

$$N_0 = \#\{\rho_{M,M} \neq 0\}$$

Diagonal Approximation

$$\overline{S_1}(A) \simeq \begin{cases} D \ln N, & D < 1/2 \\ S_{1,Page}, & D > 1/2 \end{cases}$$



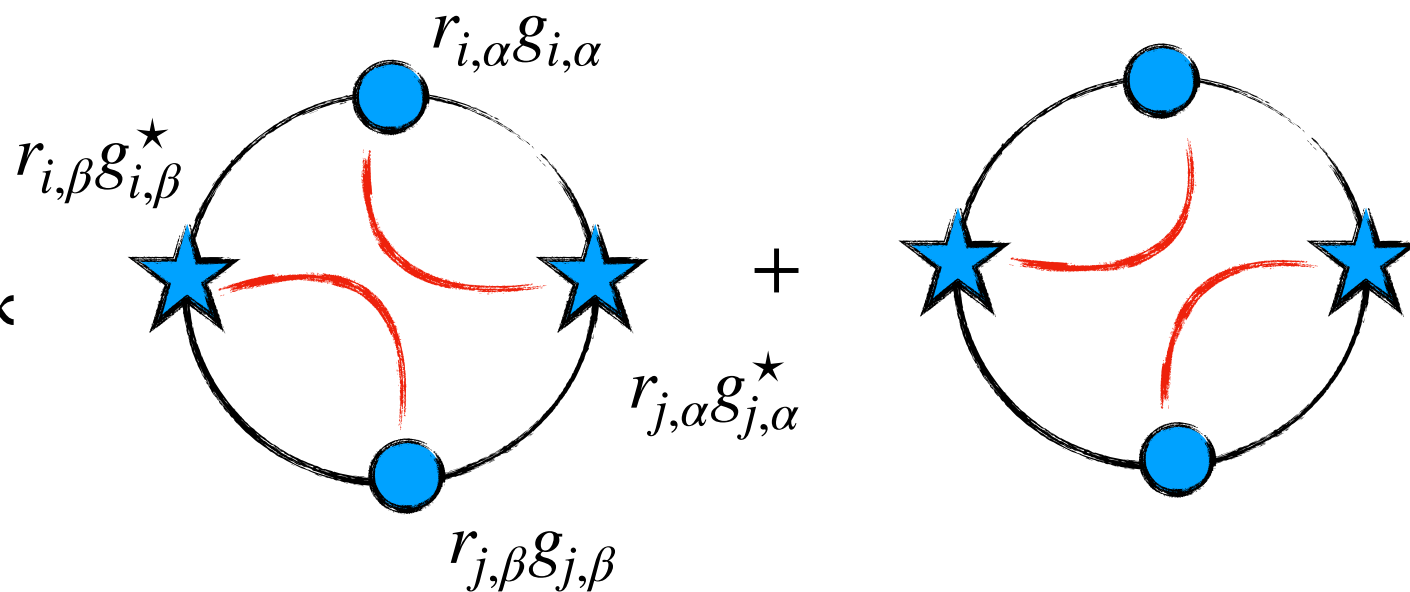
Purity

$$P_2 = \text{Tr}[\rho_A^2]$$

$$S_2 \leq -\log \bar{P}_2$$

$$|\psi_{S-RPS}\rangle = \sum_{i,\alpha} a_{i,\alpha} |i\rangle_A |\alpha\rangle_B \approx \frac{\sum_{i,\alpha} r_{i\alpha} g_{i\alpha} |i\rangle_A |\alpha\rangle_B}{\|\psi_{S-RPS}\rangle\|}$$

$$P_2 = \sum_{i,j,\alpha,\beta} a_{i,\alpha} a_{j,\alpha}^* a_{j,\beta} a_{i,\beta}^* \propto$$

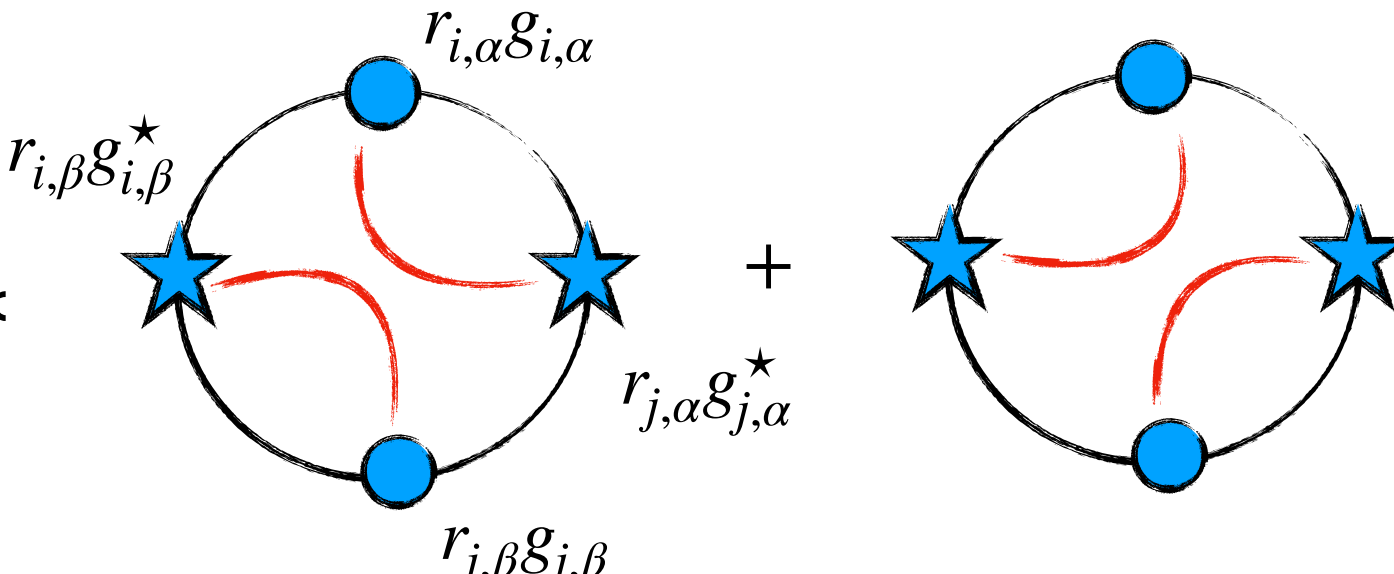


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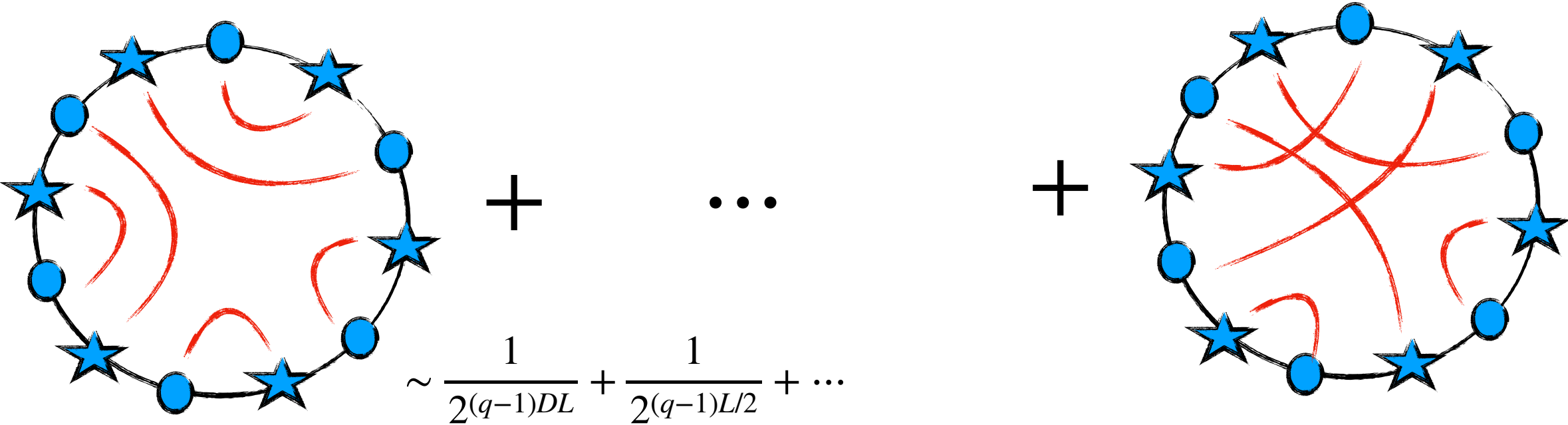
$$P_2 = \sum_{i,j,\alpha,\beta} a_{i,\alpha} a_{j,\alpha}^* a_{j,\beta} a_{i,\beta}^* \propto$$


$$= \frac{2}{2^{DL}} + \frac{2}{2^{L/2}} \begin{cases} \sim \frac{1}{2^{DL}} & D < 1/2 \\ \sim \frac{1}{2^{L/2}} & D > 1/2 \end{cases}$$

$$P_q = \text{Tr}[\rho_A^q]$$

$$S_q \leq -\frac{\log \overline{P}_q}{q-1}$$

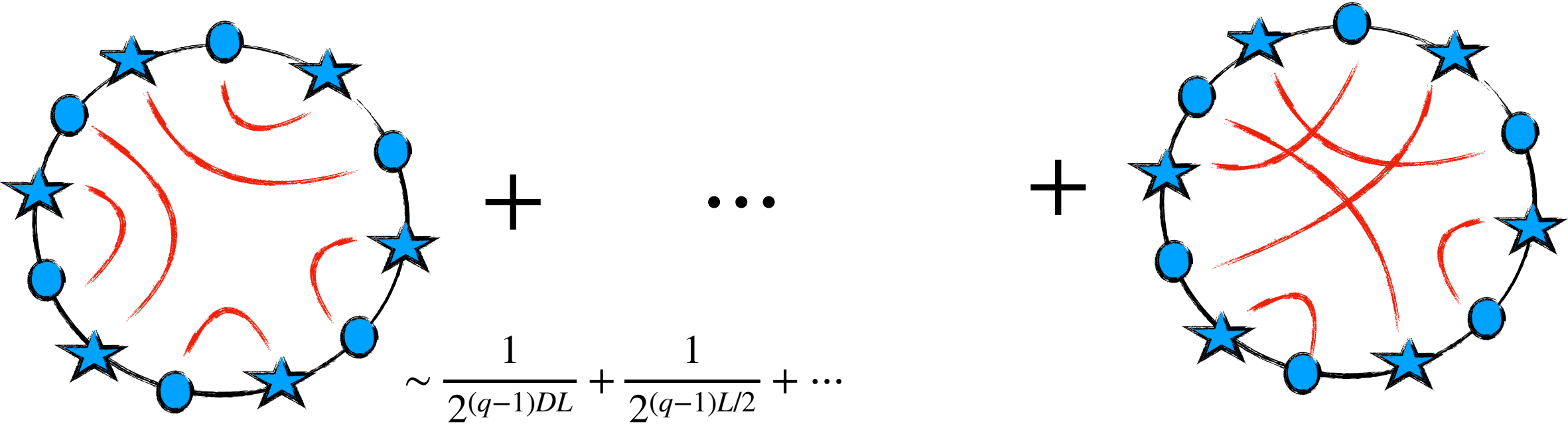
$$P_q = \sum_{i_1, \dots, i_q} \sum_{\alpha_1, \dots, \alpha_q} \prod a_{i_k, \alpha_k} a_{i_{k+1}, \alpha_k}^\star \propto$$



$$P_q = \text{Tr}[\rho_A^q]$$

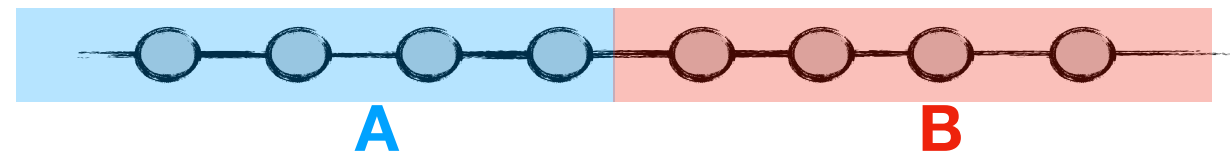
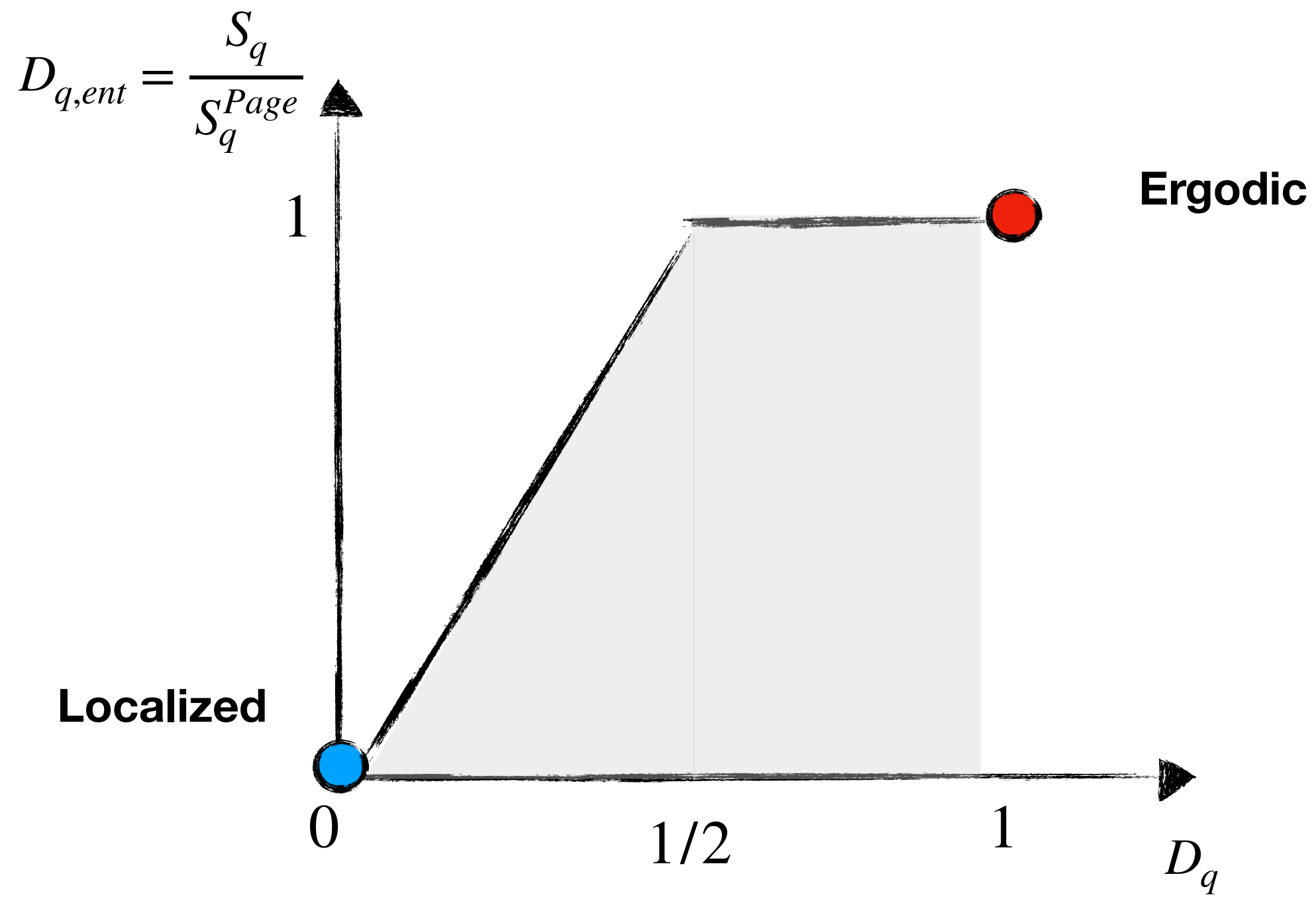
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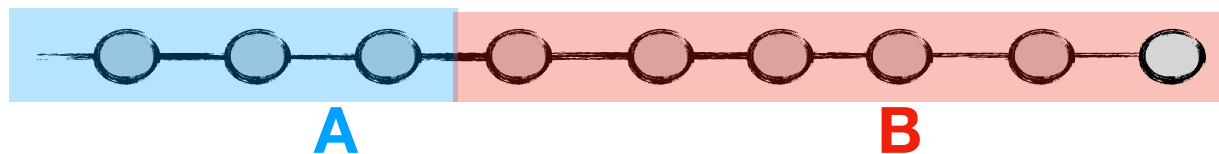
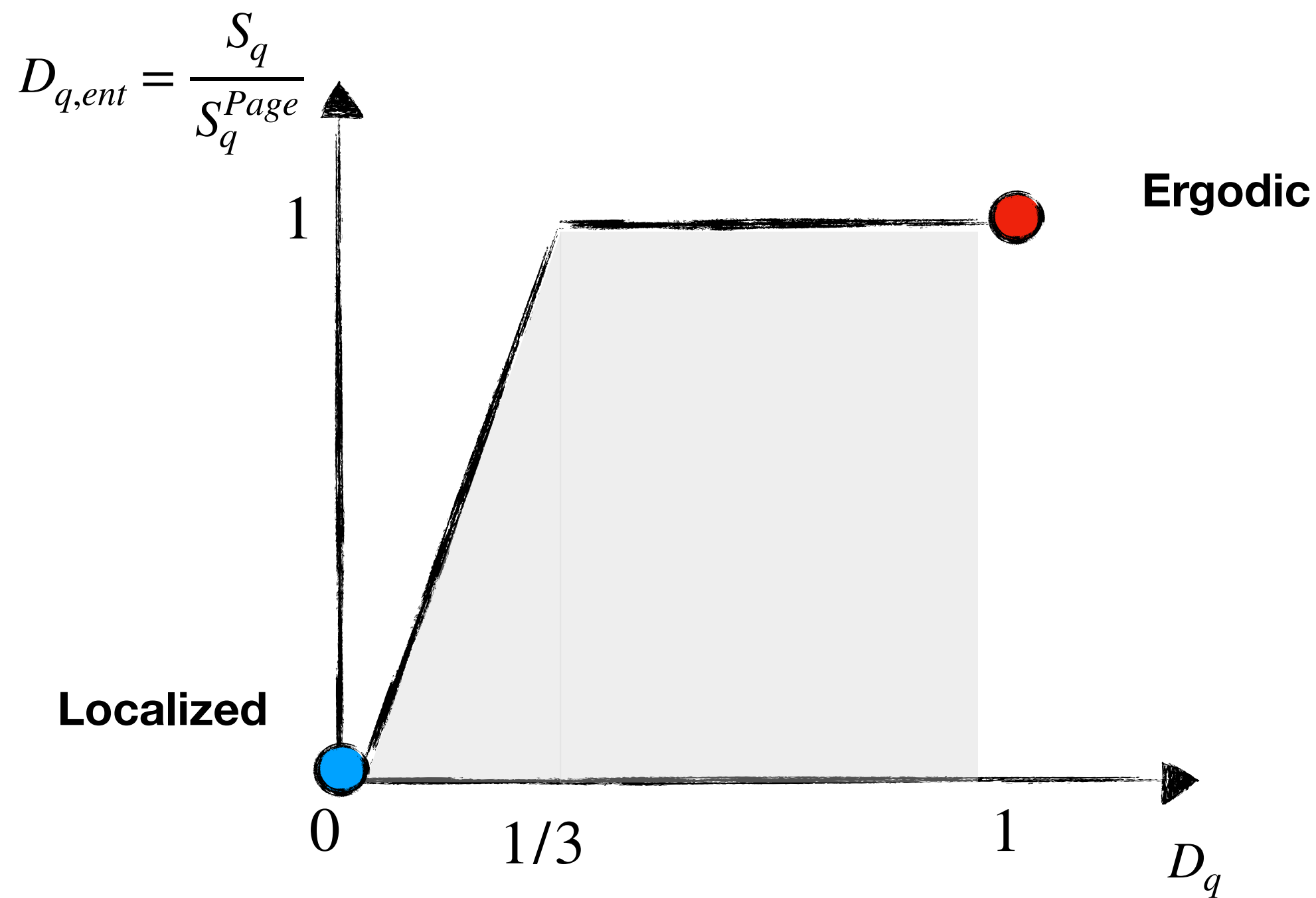
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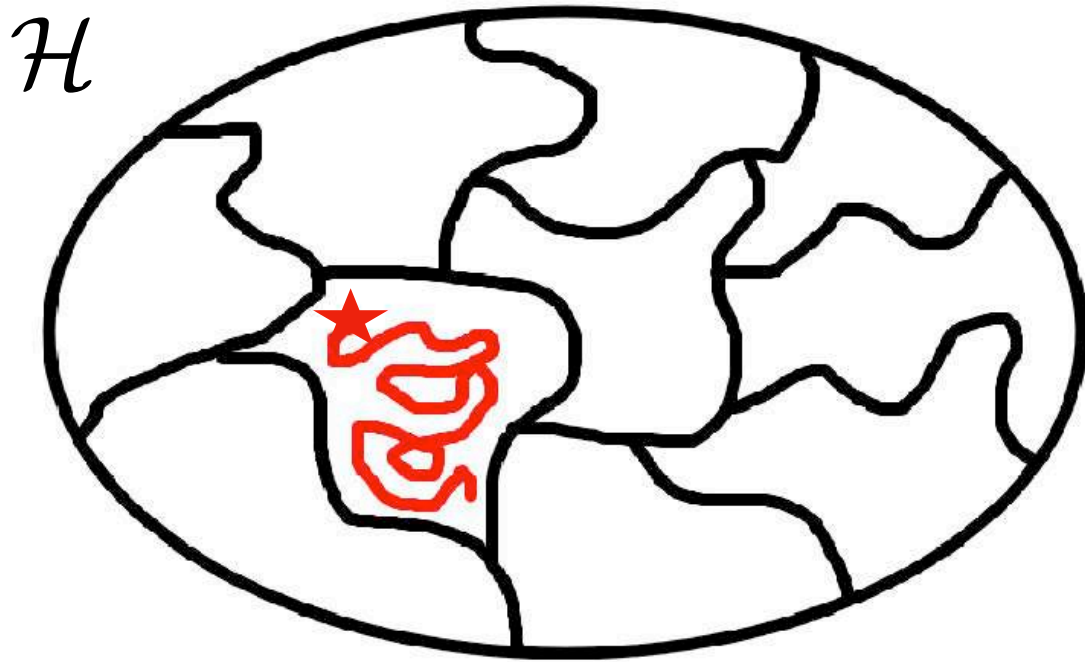
$$\sim \frac{1}{2^{(q-1)DL}} + \frac{1}{2^{(q-1)L/2}} + \dots$$

$$\approx \begin{cases} \frac{q!}{2^{(q-1)DL}} & D < 1/2 \\ \frac{C_q}{2^{(q-1)L/2}} & D > 1/2 \end{cases}$$





Application: Strong-Fragmentation

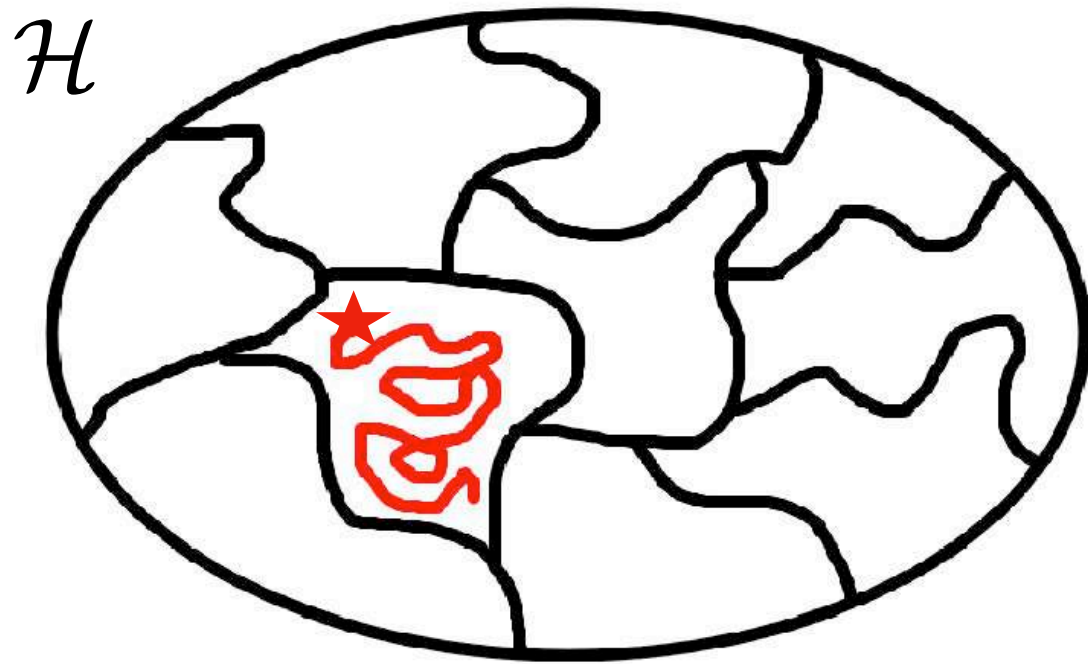


$$\hat{H}_\infty = \sum_x \hat{P}_x (\hat{c}_{x+1} \hat{c}_x + h.c.) \hat{P}_x + W \sum_x h_x \hat{n}_x$$

Kinematically Constrained Model

$$\hat{P}_x = 1 - (\hat{n}_{x+2} - \hat{n}_{x+1})^2$$

Application: Strong-Fragmentation



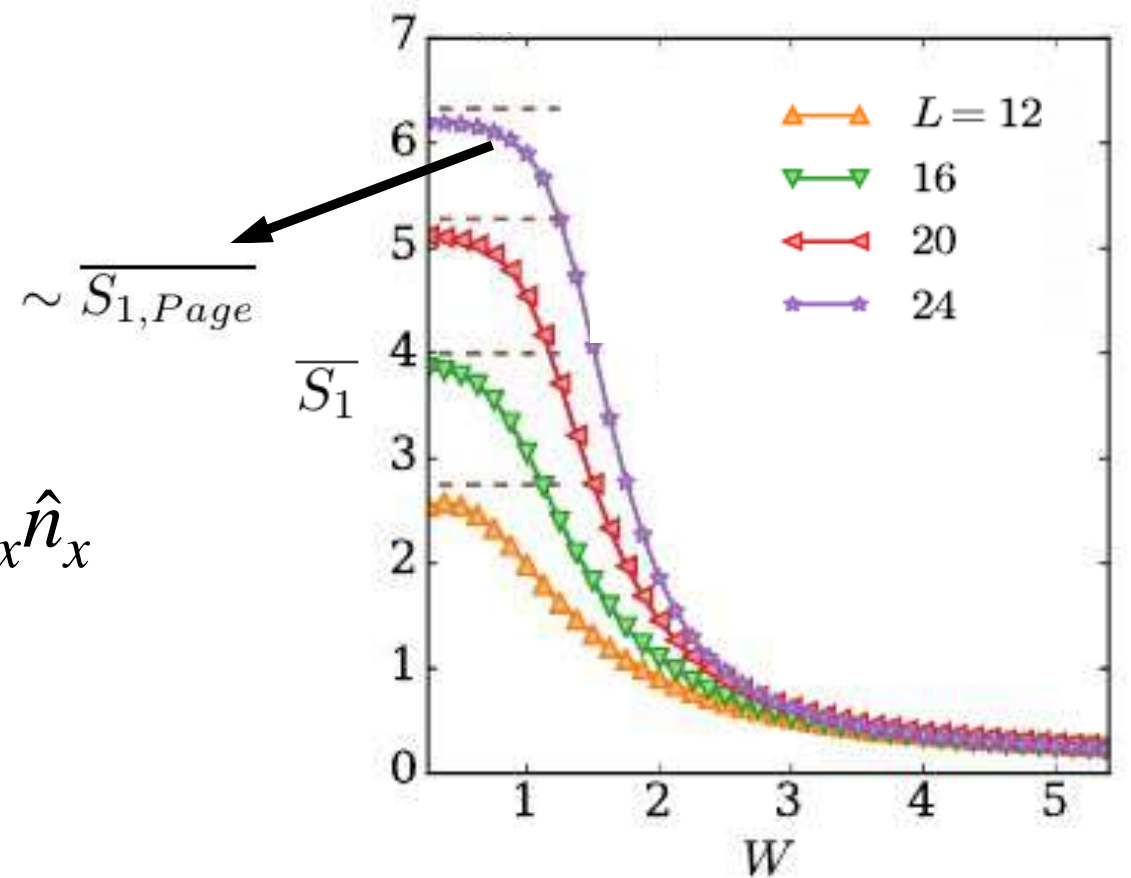
$$\dim \mathcal{H}_{frag} \sim 2^{DL}$$

$$D = \frac{3}{4} \log_2(3) - \frac{1}{2} \approx 0.7 > \frac{1}{2}$$

$$\hat{H}_\infty = \sum_x \hat{P}_x (\hat{c}_{x+1} \hat{c}_x + h.c.) \hat{P}_x + W \sum_x h_x \hat{n}_x$$

Kinematically Constrained Model

$$\hat{P}_x = 1 - (\hat{n}_{x+2} - \hat{n}_{x+1})^2$$



What about fluctuations?

Ergodic

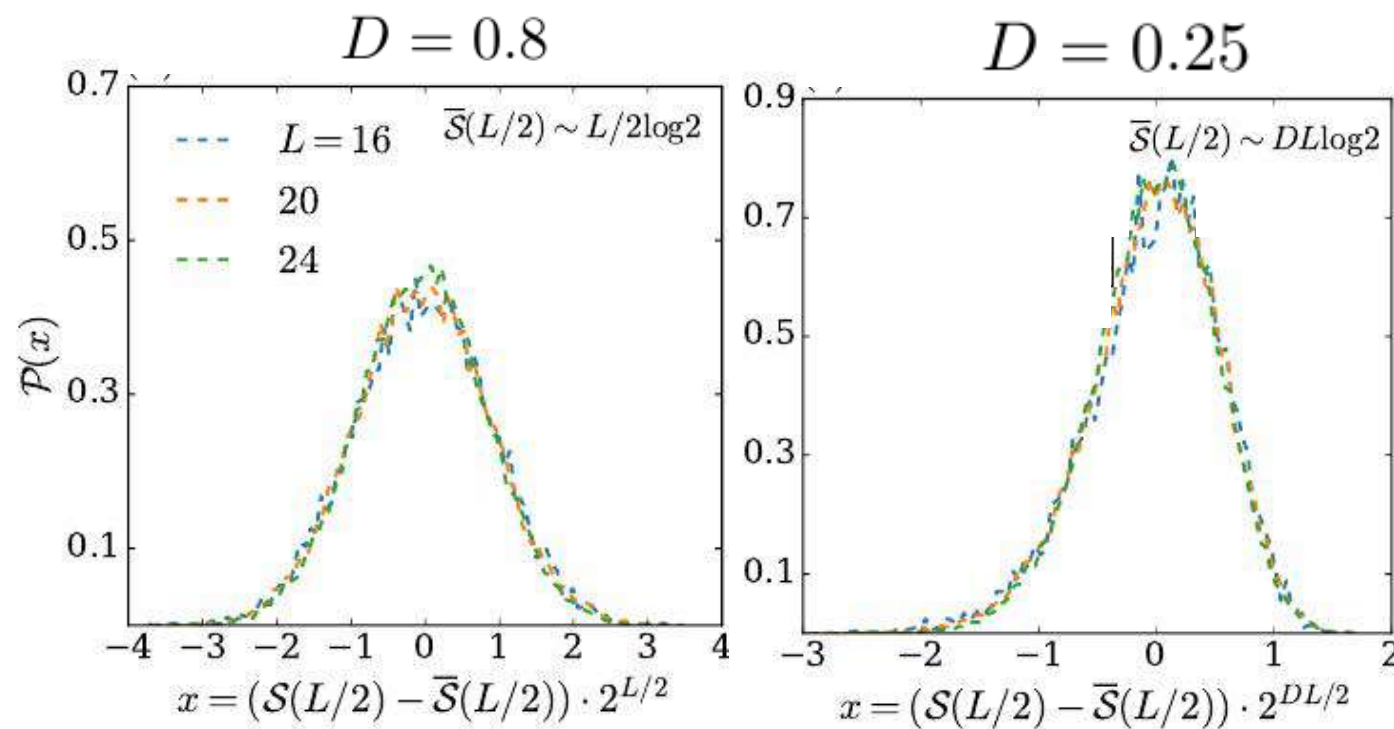


$$\delta(S_1^{Page})(A) = (\overline{S_1^2}(A) - \overline{S_1}(A)^2)^{1/2} \sim N_B^{-1}$$

Localized



$$\delta(S_1)(A) \sim \mathcal{O}(1)$$



D^{fluc}

1

Ergodic

1/2 3/4 1 D

$$\delta(S_1)(A) = (\overline{S_1^2}(A) - \overline{S_1}(A)^2)^{1/2} \sim N_B^{-D^{fluc}}$$

How random is a fractal state?

$$\langle \psi_{S-RPS} | \sigma_i^z | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^z] + \mathcal{O}(N^{-D/2})$$

“Fading Ergodicity”

How random is a fractal state?

$$\langle \psi_{S-RPS} | \sigma_i^z | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^z] + \mathcal{O}(N^{-D/2}) \quad \text{“Fading Ergodicity”}$$

$$\langle \psi_{S-RPS} | \sigma_i^x | \psi_{S-RPS} \rangle \approx \text{Tr}[\sigma_i^x] + \mathcal{O}(N^{-1/2}) \quad \text{Ergodic Behavior}$$

$$\langle \psi_{S-RPS} | \prod_i \sigma_i^{\alpha_i} | \psi_{S-RPS} \rangle \approx \mathcal{O}(N^{-1/2}), \quad \alpha_i \in \{x, y, z\}$$

(Random String)

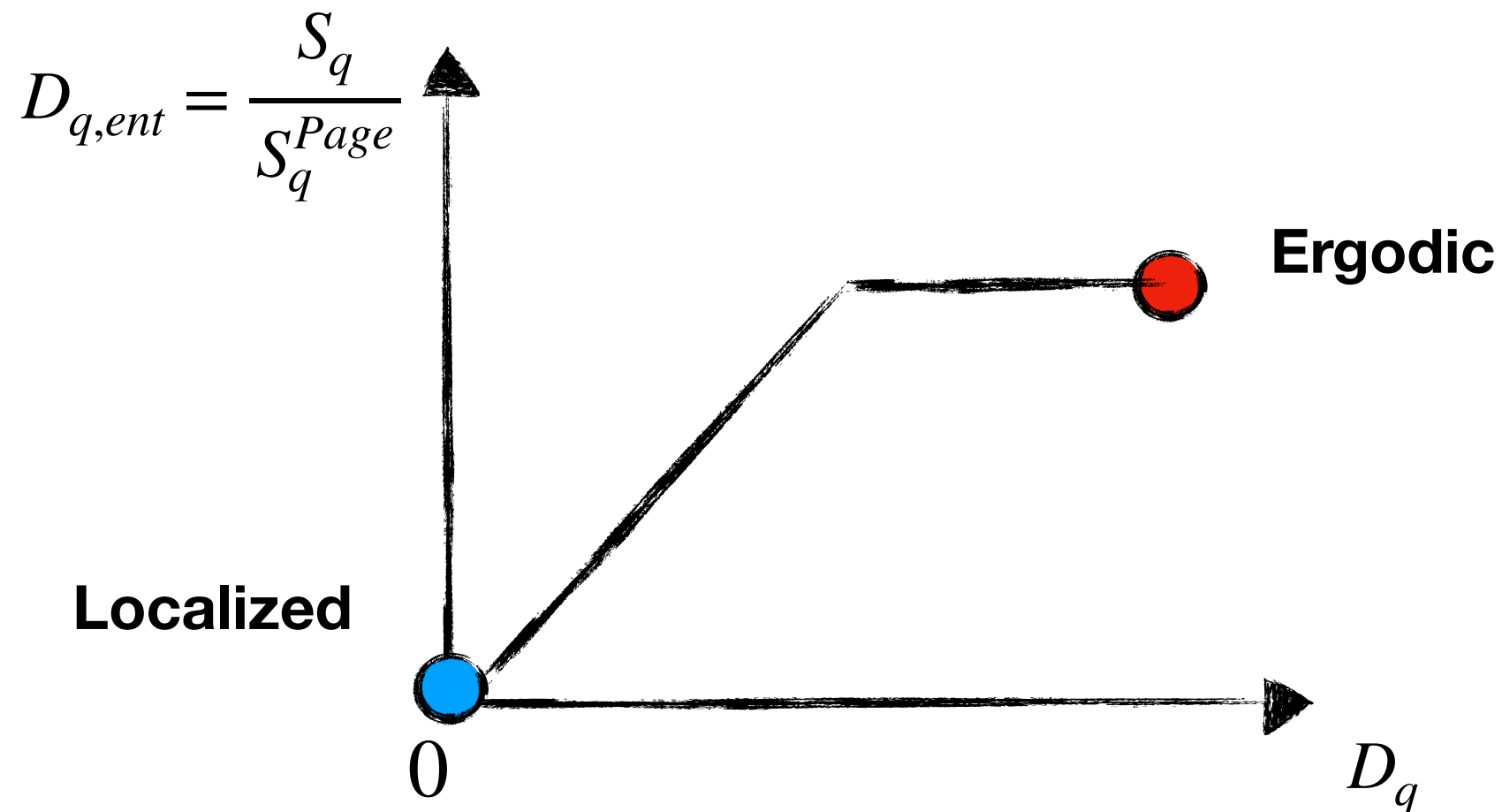
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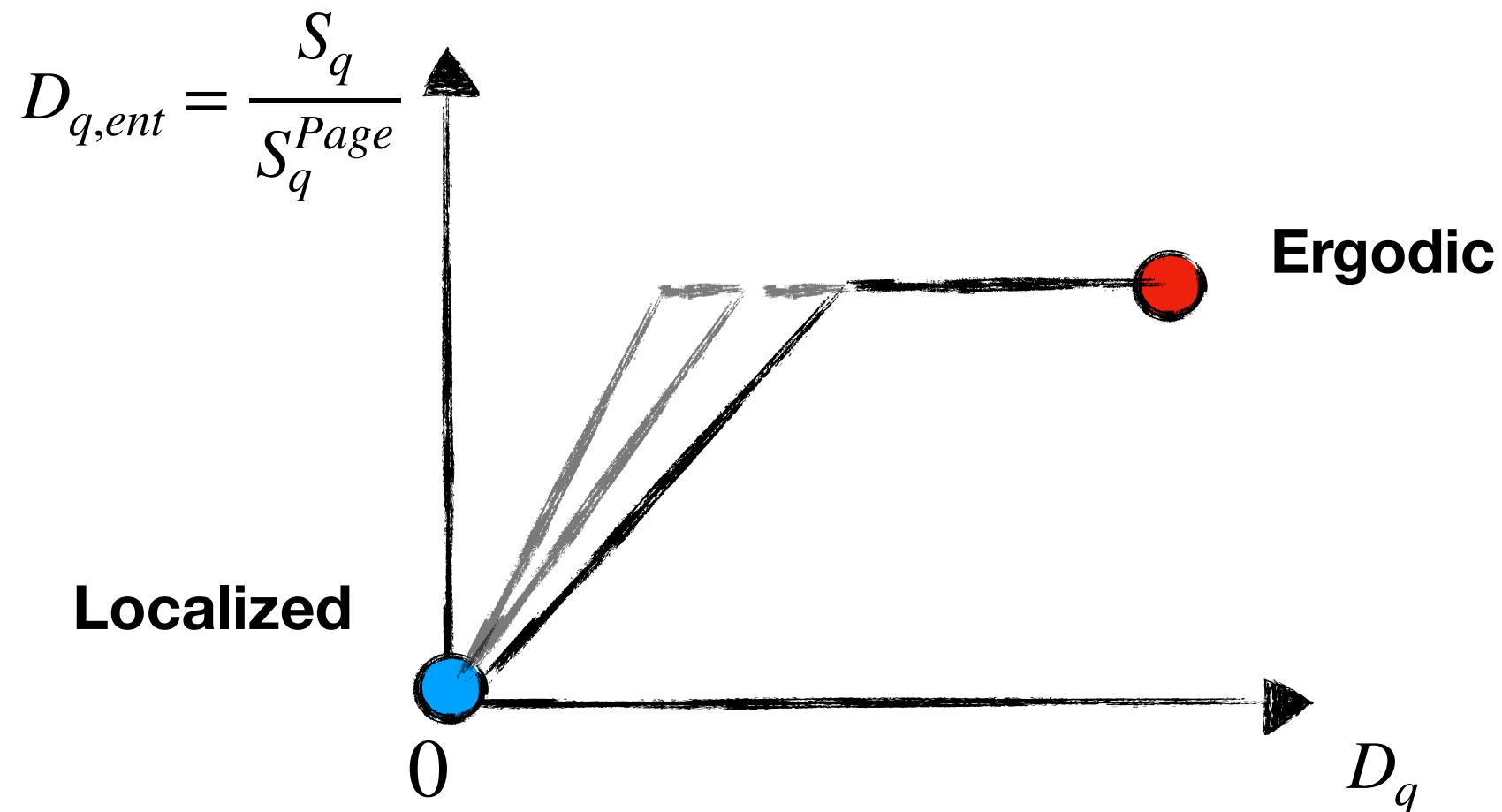
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(Random String)



Frame Potential

$$M_k^{Haar} = \mathbb{E}_{\text{Haar}} \left[|\psi\rangle\langle\psi|^{\otimes k} \right] = \frac{\Pi_{\text{sym}}^{(k)}}{\binom{N+k-1}{k}}$$

Frame Potential

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$$\begin{aligned} \mathcal{F}_{Haar}^{(k)} &= \mathbb{E} \left[|\langle\psi|\phi\rangle|^{2k} \right] \\ &= \binom{N+k-1}{k}^{-1} \end{aligned}$$

$$\langle\psi|\phi\rangle =$$

$$\begin{bmatrix} \psi_1^\star \\ \psi_2^\star \\ \psi_3^\star \\ \psi_4^\star \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}^\mathbf{T} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \\ \vdots \\ \vdots \end{bmatrix}$$

Frame Potential

$$M_k^{Haar} = \mathbb{E}_{Haar} \left[|\psi\rangle\langle\psi|^{\otimes k} \right] = \frac{\Pi_{\text{sym}}^{(k)}}{\binom{N+k-1}{k}}$$

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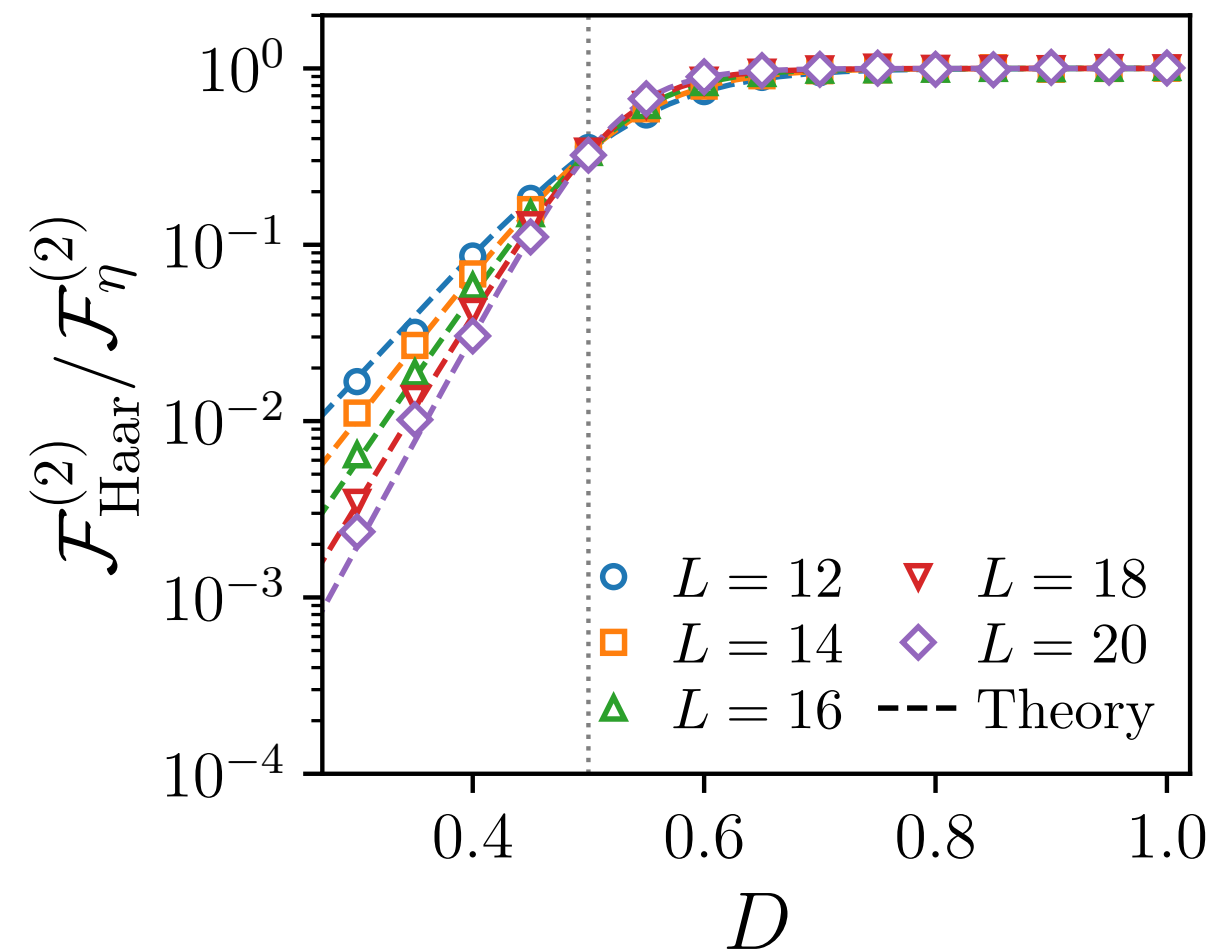
$$\begin{array}{c|c} \begin{array}{c} \psi_1^\star \\ \psi_2^\star \\ \psi_3^\star \\ \psi_4^\star \\ \vdots \end{array} & \begin{array}{c} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \vdots \end{array} \end{array}^T$$

$$\left\| M_k^\eta - M_k^{Haar} \right\|_1 \leq \sqrt{\frac{\mathcal{F}_\eta^{(k)}}{\mathcal{F}_{Haar}^{(k)}} - 1}$$

Bounds on the distance from Haar

$$\left\| M_k^\eta - M_k^{Haar} \right\|_2 = \sqrt{\mathcal{F}_\eta^{(k)} - \mathcal{F}_{Haar}^{(k)}}$$

Frame Potential for Fractal States

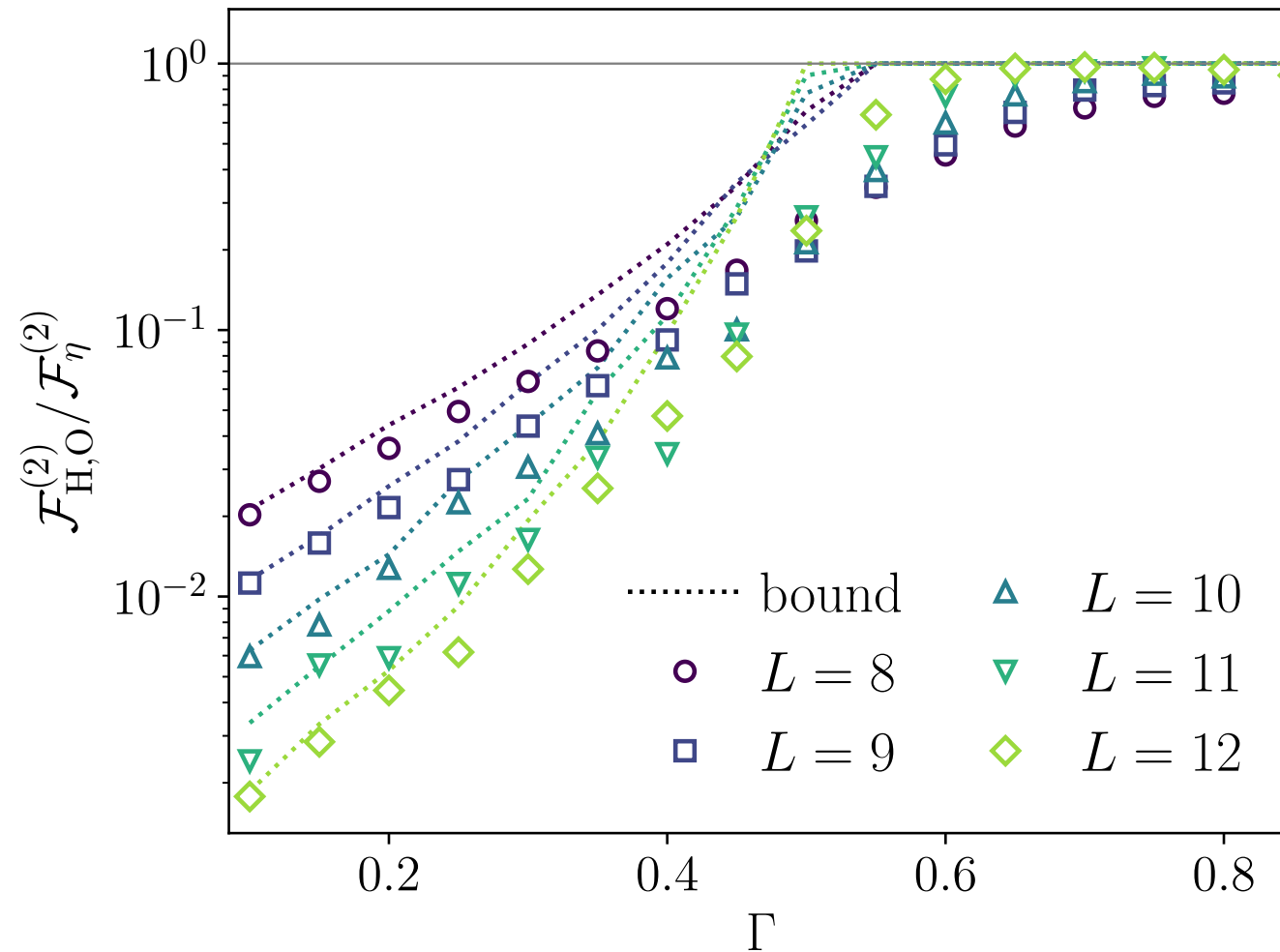


$$\langle \psi | \phi \rangle =$$

$$\begin{array}{c|c} & \mathbf{T} \\ \hline \psi_1^* & 0 \\ 0 & \phi_2 \\ \vdots & \phi_3 \\ \vdots & \vdots \\ \psi_{33}^* & \vdots \\ 0 & \vdots \\ \vdots & \phi_{47} \\ 0 & 0 \\ \psi_{156}^* & \vdots \\ & \vdots \end{array}$$

$$\frac{\mathcal{F}_{\text{Haar}}^{(k)}}{\mathcal{F}_{\eta}^{(k)}} \sim \begin{cases} 1 - \mathcal{O}(N^{-(2D-1)}), & D > 1/2 \\ \mathcal{O}(N^{-(k-1)(1-2D)}), & D < 1/2. \end{cases}$$

Frame Potential for **Multi-fractal** States



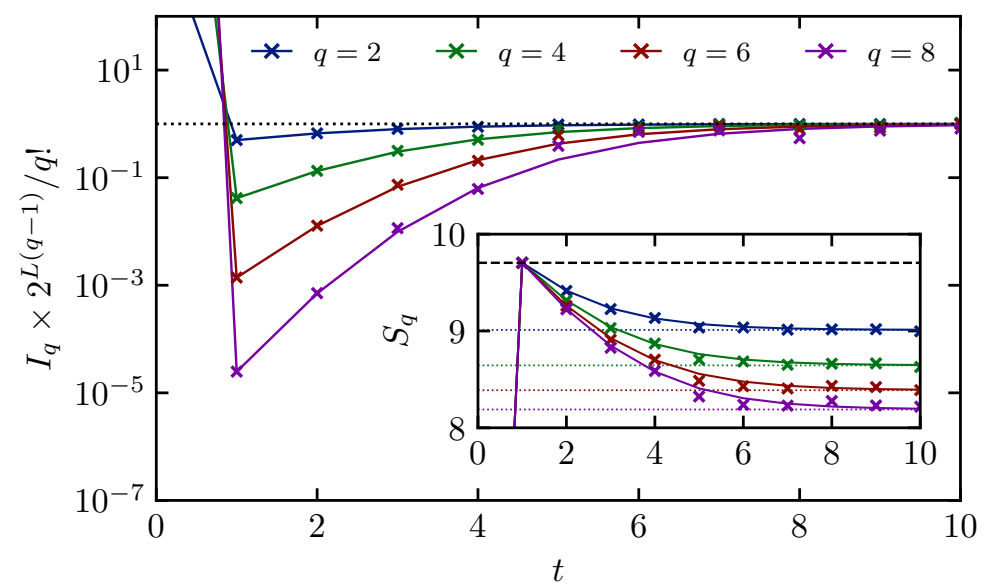
MBL Floquet Model

$$\frac{\mathcal{F}_{\text{Haar}}^{(k)}}{\mathcal{F}_{\eta}^{(k)}} \leq \min \left(1, \frac{N \mathcal{F}_{\text{Haar}}^{(k)}}{\max_{\{|x\rangle\}} (\mathbb{E}_{\eta} \text{IPR}_k)^2} \right) \sim \min \left(1, N^{(k-1)} (2D_k^{\min} - 1) \right)$$



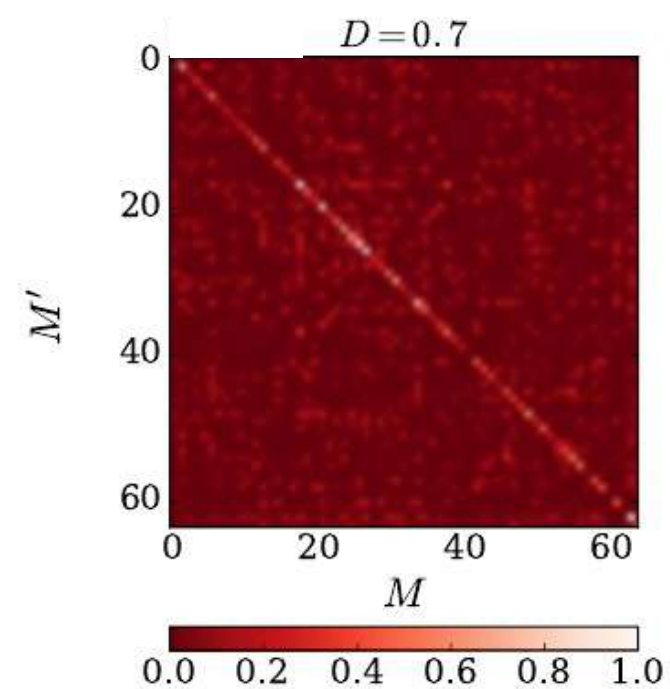
P. Claeys

Trinity College Dublin



Ivan

CNRS Toulouse



Conclusions:

● Emergence of the Porter-Thomas Distribution

Approach to PT distribution with finite-depth

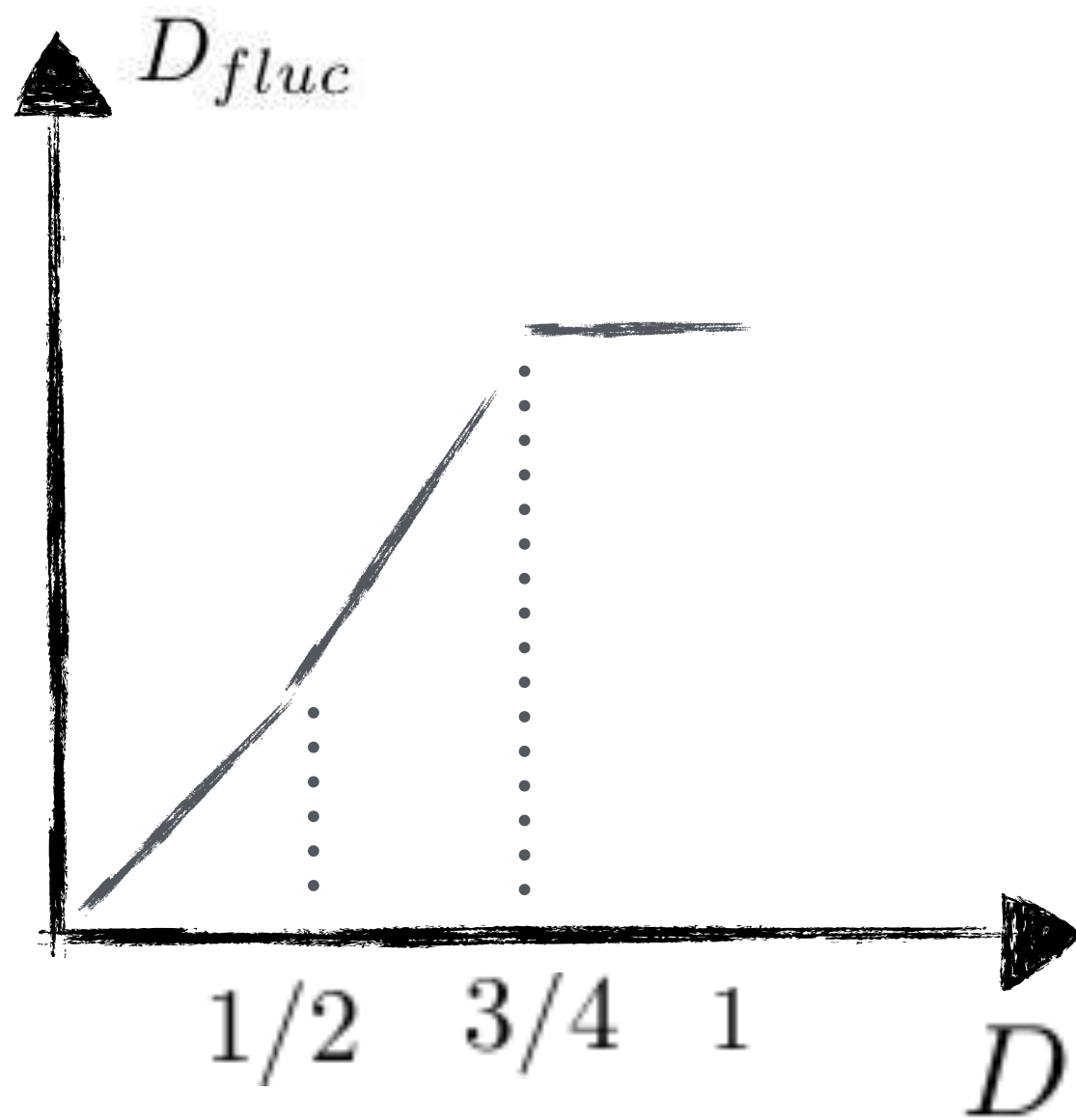
P.W .Claeys, [GDT](#), PRL 134 (5), 050405 (2025)

● Entanglement vs Inverse Participation Ratio

Entanglement Entropy can have an ergodic behavior even though
the wave- function is highly non-ergodic

[GDT](#), I.M. Khaymovich PRL 124 (20), 200602 (2020)

Thank you!



$$\overline{\delta S_1}(A) = (\overline{S_1^2}(A) - \overline{S_1}^2(A))^{1/2} \sim N^{-D_{fluc}/2}$$