



NORDITA 2026



Two-parameter scaling on the hyperbolic plane

From scale-dependent geometry to expanding graphs

Maksimilian Usoltcev · University of Cologne · Nordita 2026

with Alexander Altland, Tobias Micklitz, Devasheesh Sharma, and Carolin Wille

arXiv:2604.24917

When is one scaling variable enough?

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One-parameter scaling

One quantity tracks localization
across increasing scales.

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$\sigma(L)$
conductivity

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Avalanche / quantum sun^{3,4}

$\mathcal{D}_{\text{bath}} \uparrow$
bath size

$V_{\text{next}} \downarrow$
coupling

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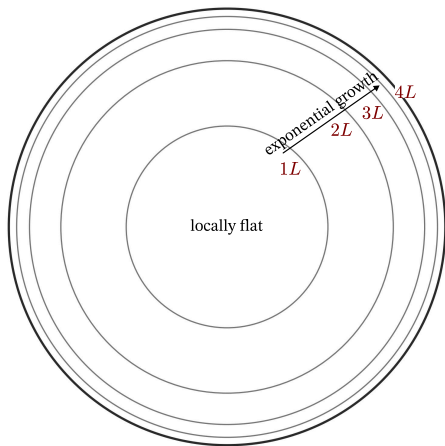
one-parameter scaling can break down⁵

*Goal: isolate the effect of scale dependence
in a simple localization problem.*

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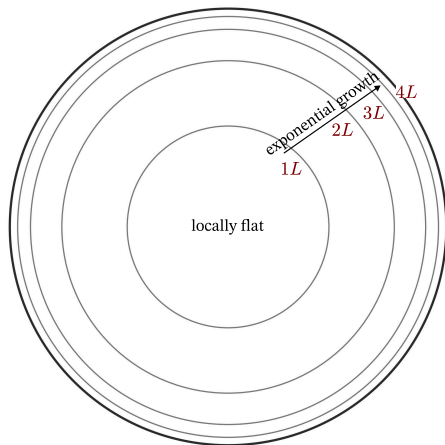
Scale-dependent geometry of $\mathbb{H}^2$



Poincaré disk

- $\mathbb{H}^2$: 2D manifold, constant negative curvature
- L : curvature radius

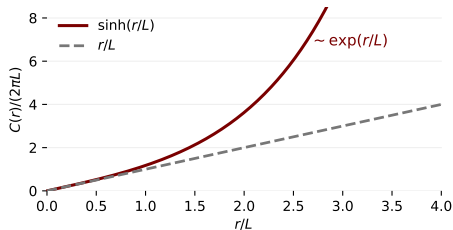
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- Shell circumference:

$$C(r) = 2\pi L \sinh(r/L)$$

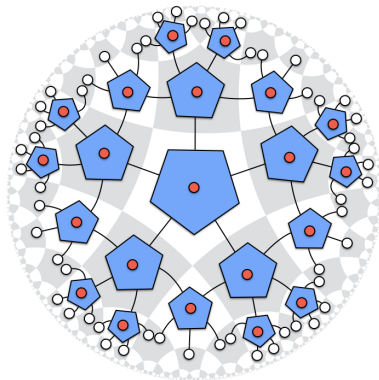


Hyperbolic lattices as a starting point

Hyperbolic tilings for quantum error correction.

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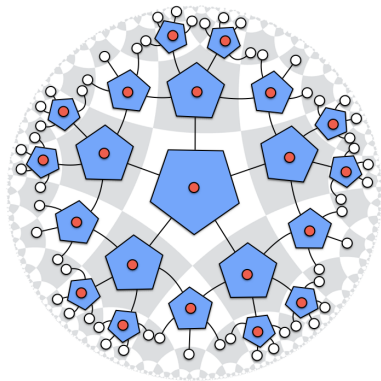
HaPPY code¹

¹ Harlow (2018); Pastawski et al. (2015)

Hyperbolic lattices as a starting point

Hyperbolic tilings for quantum error correction.

- ↪ Tensor networks on hyperbolic lattices
- ↪ Long-distance behavior^{2,3} on $\mathbb{H}^2$



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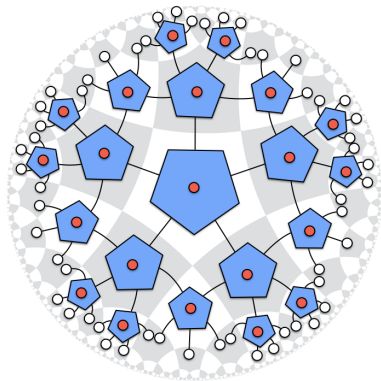
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*How does geometry modify
long-distance physics?*



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Negative curvature modifies diffusion

Geometric crossover:

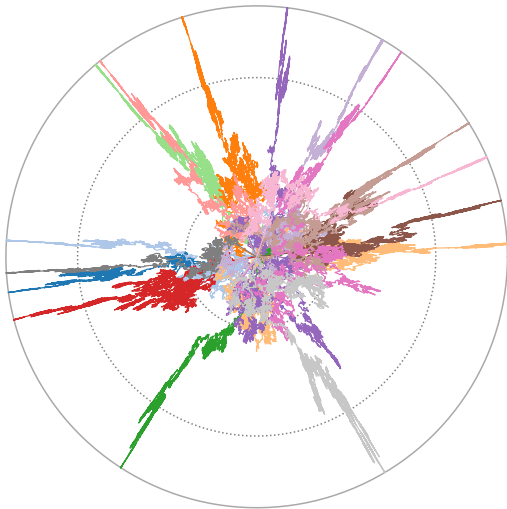
- $r \ll L$: locally 2D diffusion
- $r \gtrsim L$: radial escape
- Long returns suppressed

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Visible in classical Brownian motion.



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$$W_c = 0$$

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$$W_c = 0$$

$$W_c > 0$$

From local diffusion to expanding graphs

microscopic scale

Microscopic
model

disorder on $\mathbb{H}^2$

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**Microscopic
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$$r \ll L$$

Diffusion

continuum field theory

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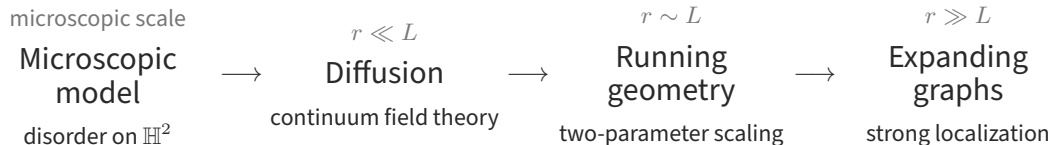


$$r \sim L$$

**Running
geometry**

two-parameter scaling

From local diffusion to expanding graphs



Disordered quantum particle on $\mathbb{H}^2$

- Continuum Hamiltonian:

$$\hat{H} = -\frac{1}{2m}\Delta_g + V(x)$$

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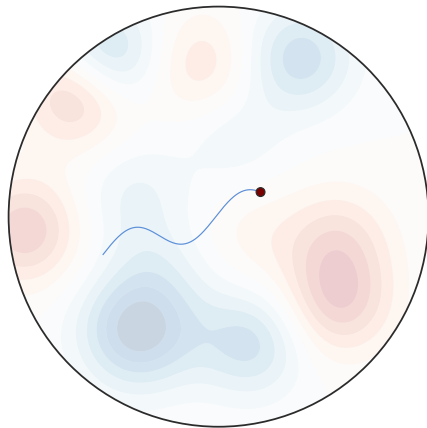
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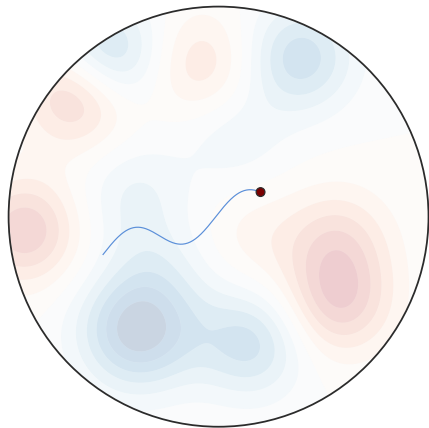
Schematic random potential on $\mathbb{H}^2$

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- Weak disorder: scattering is locally flat on scales $\ll L$



Schematic random potential on $\mathbb{H}^2$

Weak localization in the continuum theory

Disorder average $\rightarrow$ long-distance theory:

$$S[Q] = \frac{\sigma}{16} \int d^2x \sqrt{g} \operatorname{str}(Q \Delta_g Q)$$

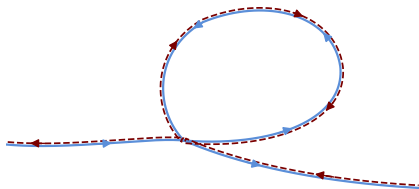
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time-reversed paths enhance return probability

$$\mathbb{R}^2 : \quad \frac{d\sigma}{d\ell} = -\frac{1}{\pi}$$

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Coarse graining increases running resolution:

$$a(\ell) = e^\ell / \Lambda$$

(Λ^{-1} : microscopic cutoff)

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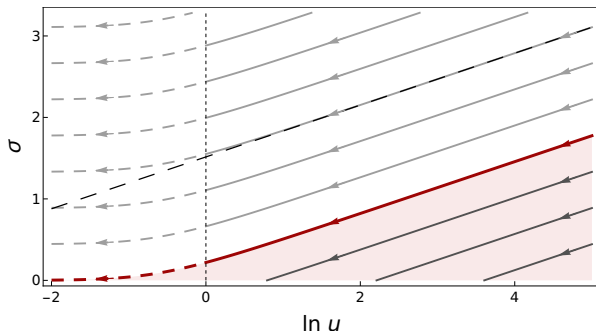
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$$\frac{d\sigma}{d\ell} = -\frac{1}{\pi} \frac{u^2 \tanh(\pi u)}{u^2 + 1/4}, \quad \frac{du}{d\ell} = -u.$$

(one-loop RG)

Localizing flow becomes nonperturbative at large scales

Perturbative control requires

$$\sigma \gg 1.$$

Along insulating flow,

$$\sigma \longrightarrow \mathcal{O}(1).$$

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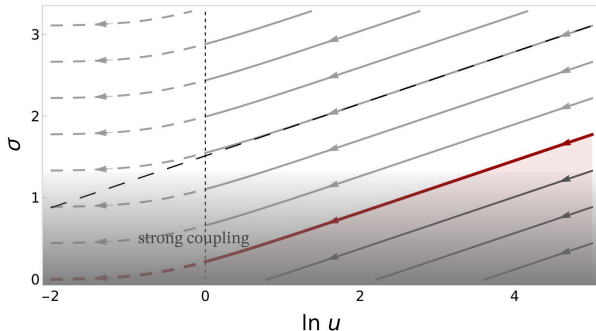
Perturbative control requires

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Along insulating flow,

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- one-loop RG becomes uncontrolled
- large-scale regime needs a nonperturbative description



From weak to strong localization

Discretizing the continuum theory

$$\int_{\mathbb{H}^2} d^2x \sqrt{g} \operatorname{str}(Q \Delta_g Q)$$

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$$\int_{\mathbb{H}^2} d^2x \sqrt{g} \operatorname{str}(Q \Delta_g Q) \longrightarrow - \sum_{\langle ij \rangle} \omega_{ij} \operatorname{str}[(Q_i - Q_j)^2]$$

ω_{ij} : geometric bond weights defining discrete Laplacian

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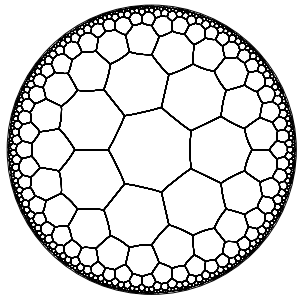
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which graph most faithfully discretizes $\mathbb{H}^2$?

Regular tessellation or irregular mesh?

Regular $\{p, q\}$ tessellation^{1,2}

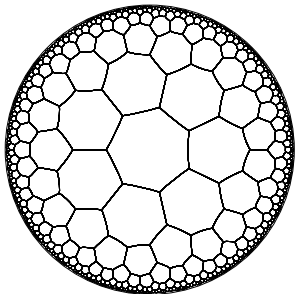


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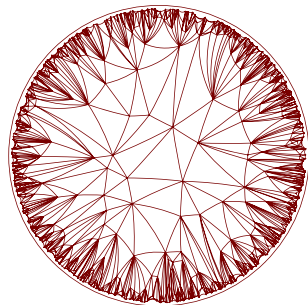
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Poisson–Delaunay (PD) mesh

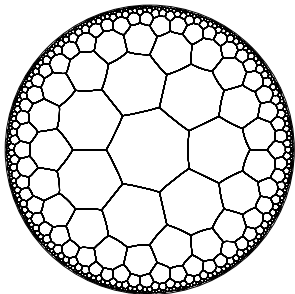


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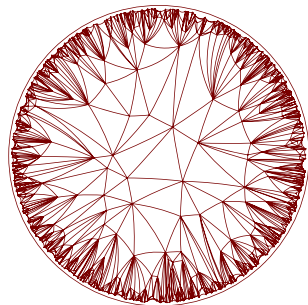


$$a_{\text{reg}}/L = \text{fixed}$$

no dense-mesh limit at fixed curvature

¹ Chen et al., PRL 133, 066101 (2024)

Poisson–Delaunay (PD) mesh



$$\rho_{\text{PD}} L^2 \uparrow \Rightarrow a_{\text{mesh}}/L \downarrow$$

parametric continuum limit

² Li et al., Commun. Phys. 7, 371 (2024)

A hierarchy of graph proxies

PD is faithful, but its irregular structure combines several ingredients

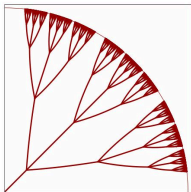
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Bethe tree



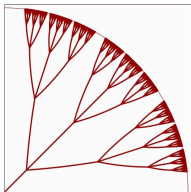
exponential growth

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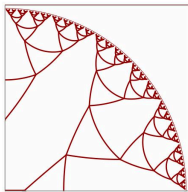
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exponential growth

Husimi cactus



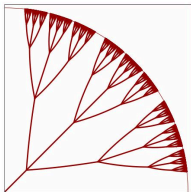
+ short loops

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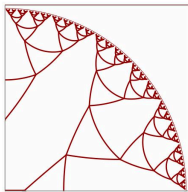
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PD bond disorder

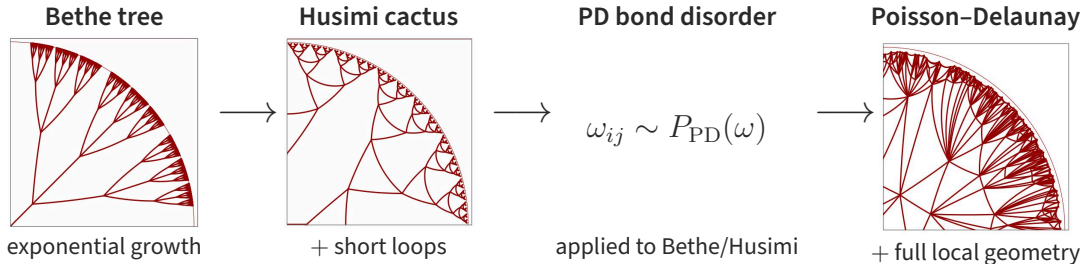
$$\omega_{ij} \sim P_{\text{PD}}(\omega)$$

applied to Bethe/Husimi

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Matching graph growth to $\mathbb{H}^2$

- Graph shells grow exponentially with generation n :

$$N_n \sim \gamma^n,$$

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$$C(r) \sim e^{r/L}$$

Matching graph growth to $\mathbb{H}^2$

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- Hyperbolic shells grow exponentially with radius r :

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- Match the two growth rates $\rightarrow$ assign radial length $a_{\mathcal{G}}$ to one graph generation:

$$\exp\left(\frac{\ln \gamma}{a_{\mathcal{G}}} r\right) \stackrel{!}{\sim} \exp\left(\frac{r}{L}\right)$$

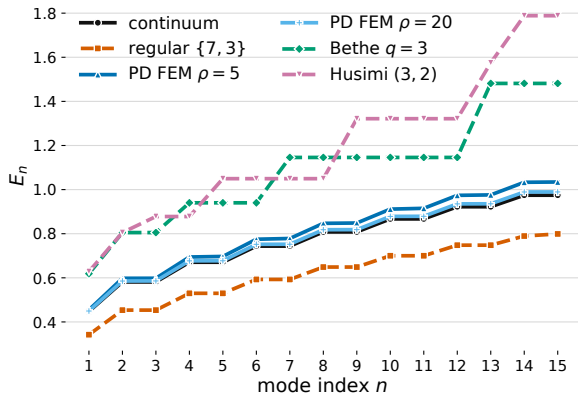
$$\implies \boxed{a_{\mathcal{G}} = L \ln \gamma}$$

Matching graph Laplacian spectra

- Solve Laplace equation on a finite $\mathbb{H}^2$ disc
- Fix the boundary $\rightarrow$ compare lowest eigenvalues
- Continuum-like graphs should match low-energy spectrum

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Disc of radius $R = 6L$

Strong localization on recursive graphs

- Cut parent link $\rightarrow$ forward branches decouple

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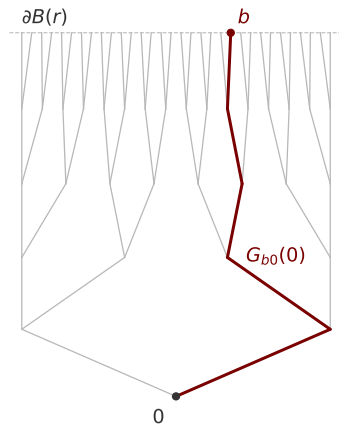
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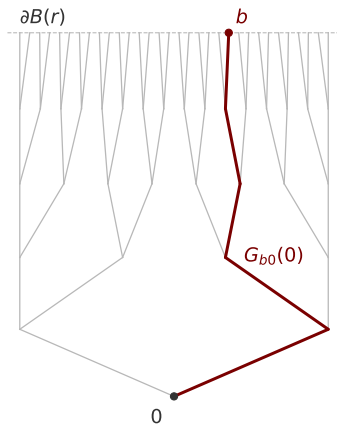
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- Bethe: branching only
Husimi: branching + short loops

$\hookrightarrow$ decay along a branch vs.
proliferation of branches



Attenuation versus proliferation

Along one path:

$$|G(r)|^2 \sim e^{-\alpha r/L}$$

Hyperbolic shell:

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Attenuation versus proliferation

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Total shell response:

$$N(r) |G(r)|^2 \sim e^{\kappa_\infty r/L}$$

with

$$\kappa_\infty \sim 1 - \alpha$$

(shell-response exponent)

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$$\kappa_\infty \begin{cases} > 0 & : \text{proliferation wins,} \\ < 0 & : \text{attenuation wins} \end{cases}$$



$$\kappa_\infty(W_c) = 0$$

transition

Localization from shell response

Shell-integrated response:

$$\chi(r) = \sum_{b \in \partial B(r)} |G_{b0}(0)|^2$$

Large-radius scaling:

$$\mathbb{E}[\ln \chi(r)] \sim \kappa_\infty \frac{r}{L}$$

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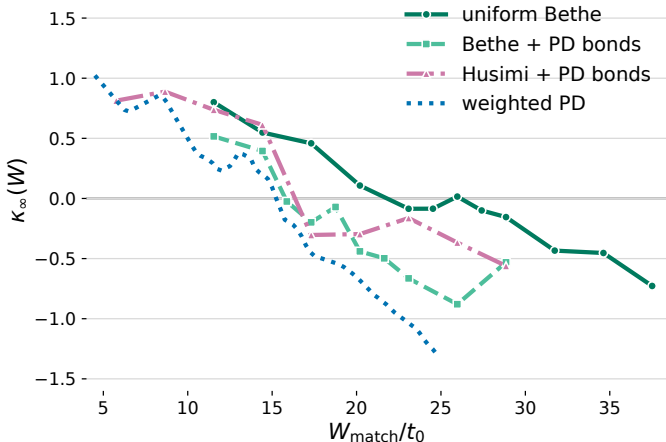
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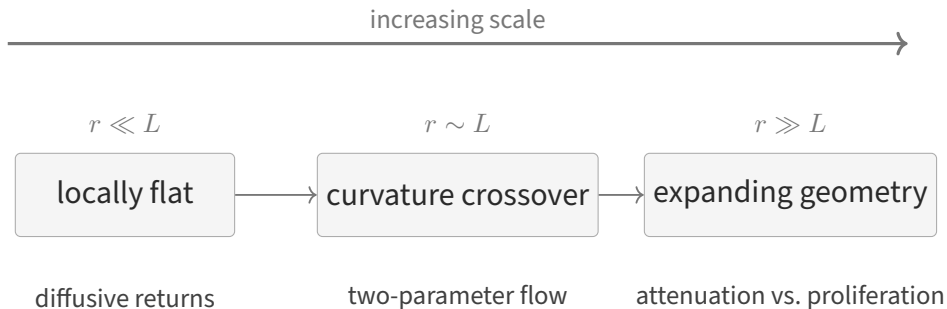
Localization criterion:

$$\boxed{\kappa_\infty = 0}$$

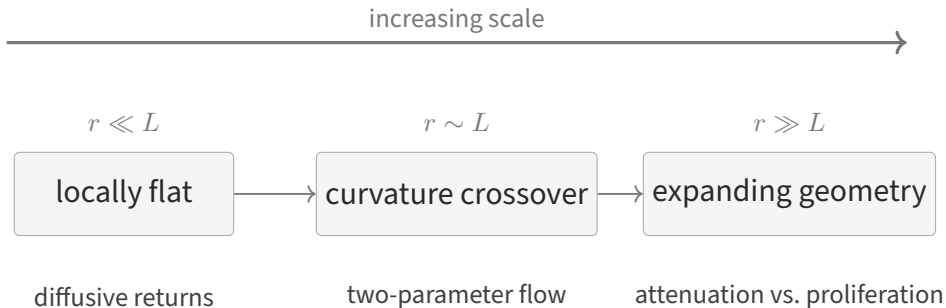


W_{match}/t_0 : growth-matched disorder strength in units of mean hopping t_0 .

Localization across increasing scales



Localization across increasing scales



*Curvature enables a genuine Anderson transition
in a setting that localizes in flat 2D.*

From geometry to many-body ergodicity breaking

Hyperbolic localization

$$\begin{array}{cc} \sigma(\ell) & u(\ell) = L/a(\ell) \\ \text{transport} & \text{geometry} \end{array}$$

scale dependence isolated geometrically



explicit two-parameter flow

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explicit two-parameter flow

Many-body ergodicity breaking

one-parameter scaling can break down

↪ accompanied by:

+ interactions

+ correlated disorder

+ bath-growth feedback

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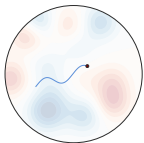
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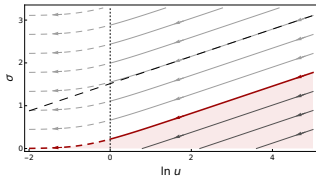
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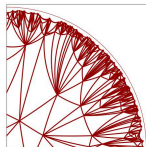
- How much of the many-body breakdown of one-parameter scaling comes from scale dependence alone?
- How is the geometric two-parameter flow changed by the many-body effects above?



random potential



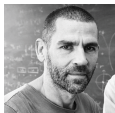
two-parameter flow



expanding regime

*Hyperbolic space as a controlled **testbed** for
scale dependence*

Thank you!



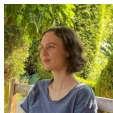
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Backup

Brownian motion on flat and hyperbolic space

same diffusion process, written in different coordinates/backgrounds

Flat space (Cartesian)

$$dx_t = dW_x, \quad dy_t = dW_y$$

no drift in Cartesian coordinates

Flat space (polar)

$$dr_t = \frac{1}{2r_t} dt + dW_r$$

$$d\theta_t = \frac{1}{r_t} dW_\theta$$

radial Itô drift from the Jacobian

Hyperbolic plane

$$dr_t = \frac{1}{2L} \coth(r_t/L) dt + dW_r$$

$$d\phi_t = \frac{1}{L \sinh(r_t/L)} dW_\phi$$

outward drift stays finite as

$$r \rightarrow \infty$$

key point: in flat space the radial drift decays as $1/r$, whereas on $\mathbb{H}^2$ it approaches $1/(2L)$

Bare scaling variables

Conductivity

$$\sigma_0 = 2\pi\nu D$$

$$D = \frac{v_0^2 \tau}{2} = \frac{v_0 \ell_{\text{mfp}}}{2}$$

$$\Rightarrow \sigma_0 = \pi\nu v_0 \ell_{\text{mfp}}$$

for a parabolic 2D band: $\sigma_0 \sim k_0 \ell_{\text{mfp}}$

Geometry

$$u_0 = \Lambda L$$

$$\Lambda^{-1} \sim \ell_{\text{mfp}}$$

$$\Rightarrow u_0 \sim \frac{L}{\ell_{\text{mfp}}} = \frac{L}{v_0 \tau}$$

curvature radius measured in microscopic units

$$\tau^{-1} \propto W^2 \quad \Rightarrow \quad \ell_{\text{mfp}}, D, \sigma_0 \propto W^{-2}, \quad u_0 \propto W^2$$

Born regime, with v_0 and density of states varying weakly with W

Three examples of two-parameter scaling

RRG¹

Physical variable:

$$D = \frac{dS}{d \ln K}$$

wave-function fractal dimension

Geometric variable:

$$K(L) = K_0^L$$

$$\frac{d \ln D}{d \ln K} = \beta_D(D, K)$$

growing block connectivity

Hyperbolic plane

Physical variable:

$$\sigma(\ell)$$

conductivity

Geometric variable:

$$u(\ell) = \frac{L}{a(\ell)}$$

$$\frac{d\sigma}{d\ell} = -\frac{1}{\pi} \frac{u^2 \tanh(\pi u)}{u^2 + 1/4}, \quad \frac{du}{d\ell} = -u$$

locally 2D $\rightarrow$ expanding geometry

Quantum sun²

Physical variable:

$$s = \frac{\bar{S}_A}{\ln 2}$$

single-site entanglement

Second parameter:

$$a = a(\alpha)$$

$$\beta_s = (1-s) \left[\ln a - \ln \frac{1-s}{s} \right]$$

coupling hierarchy / bath environment

¹ Vanoni et al., PNAS 121, e2401955121 (2024)

² Świątek et al., PRL 135, 170401 (2025)

Cavity method on recursive graphs

Cavity Green function on the directed edge $i \rightarrow j$:

$$G_{i \rightarrow j}(E) = \left[E + i\eta - \epsilon_i - \sum_{k \in \partial i \setminus j} t_{ik}^2 G_{k \rightarrow i}(E) \right]^{-1}$$

Uniform hopping

$$t_{ik} = t_0$$

$$G_{i \rightarrow j}^{-1} = E + i\eta - \epsilon_i - t_0^2 \sum_{k \in \partial i \setminus j} G_{k \rightarrow i}$$

Bethe / Husimi reference models

Random hopping

$$t_{ik} = t_0 \omega_{ik}, \quad \omega_{ik} \sim P_{\text{PD}}(\omega)$$

$$G_{i \rightarrow j}^{-1} = E + i\eta - \epsilon_i - t_0^2 \sum_{k \in \partial i \setminus j} \omega_{ik}^2 G_{k \rightarrow i}$$

PD bond-weight distribution on recursive graphs