Effects of topological heterogeneity on random matrix spectra

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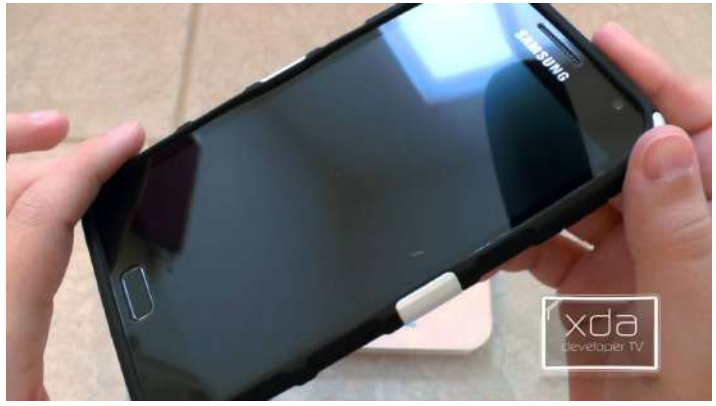
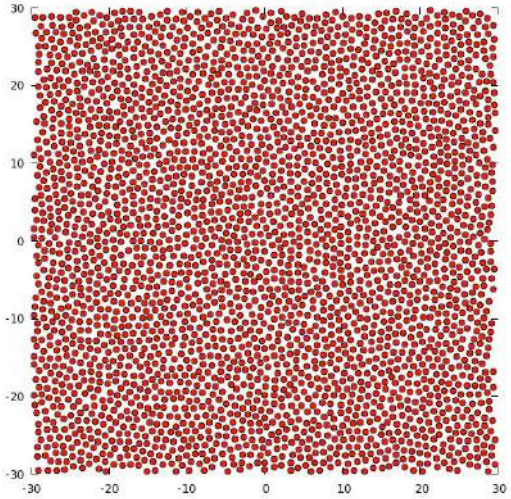
Glassy systems



Disordered solids: dense & amorphous

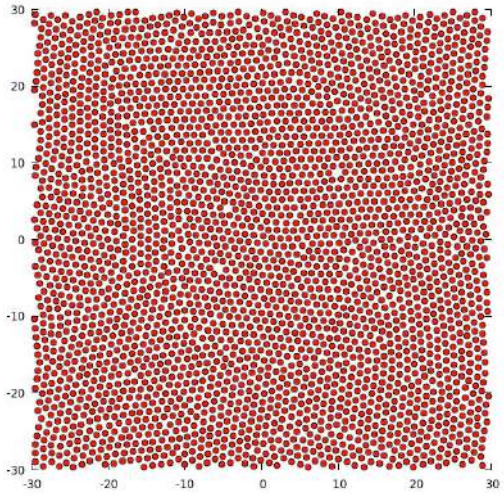
Making a glass

Monodisperse Liquid

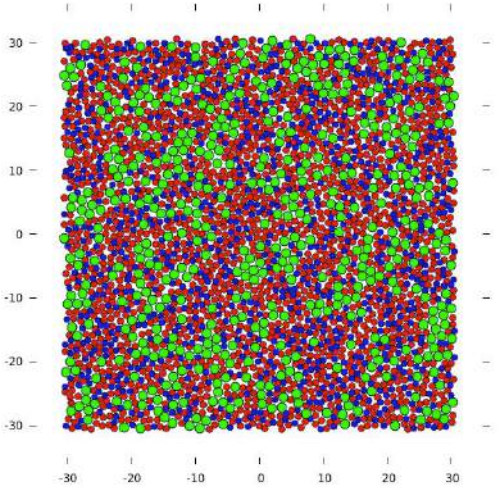


Temperature Reduction

Crystal

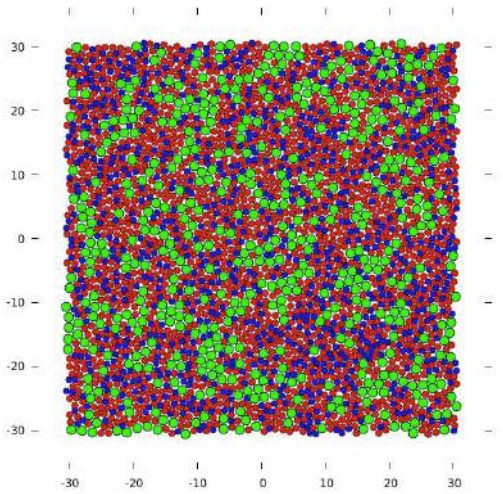


Polydisperse Liquid

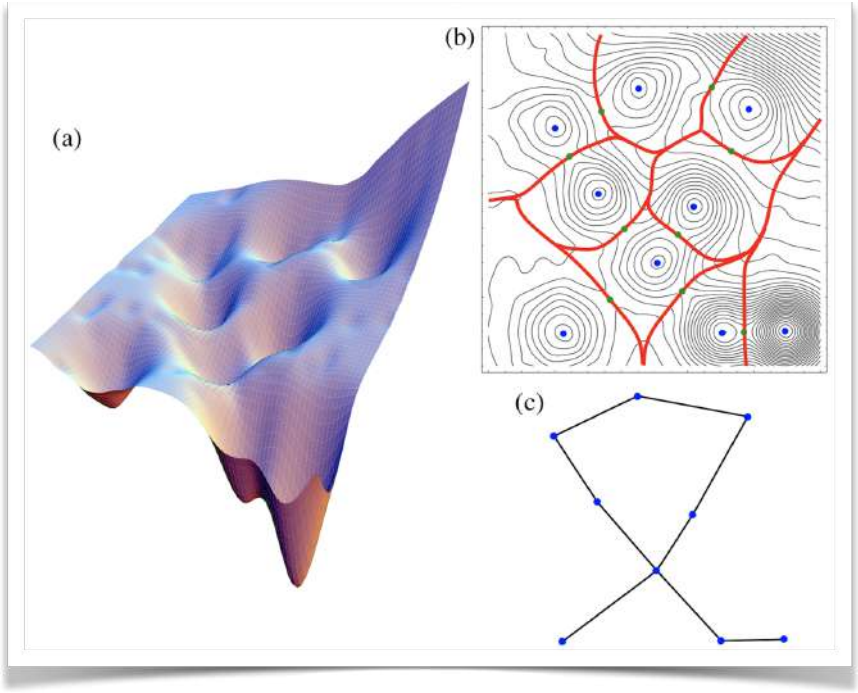


Temperature Reduction

Glass



Configuration space dynamics



Doye 2002

- Rough energy landscape, many minima
- Low temperature dynamics: activated hops
- Effective network of states $i = 1, \ldots, N$

Master operator

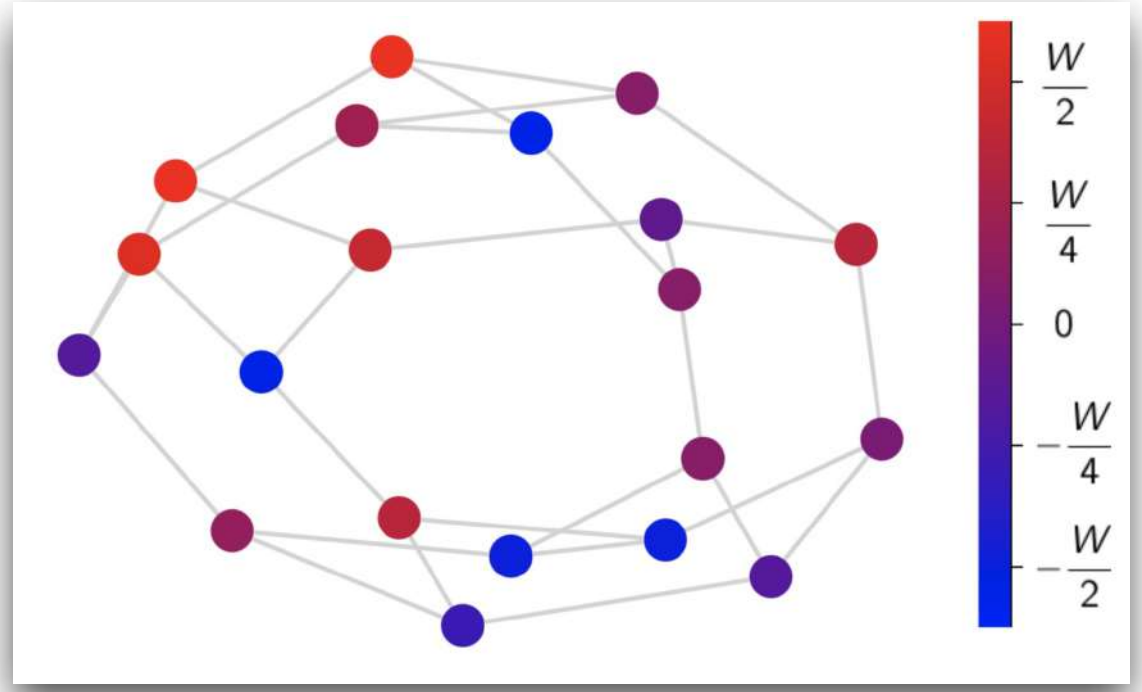
- Probability of being in state i : $p_i(t)$
- Hopping rate $i \to j$: W_{ji}
- Network structure: $a_{ij} = 1$ if edge present, else $= 0$
- Time evol'n: $\frac{d}{dt} p_i = \sum_j a_{ij}(W_{ij} p_j - W_{ji} p_i) = -\sum_j L_{ij} p_j$
- Master operator L is graph Laplacian $L = D - A$
- $A_{ij} = a_{ij} W_{ij}$: off-diagonal, (weighted) adjacency matrix
- $D_{ii} = \sum_j W_{ji}$: balances diagonal terms

Quantum analogue

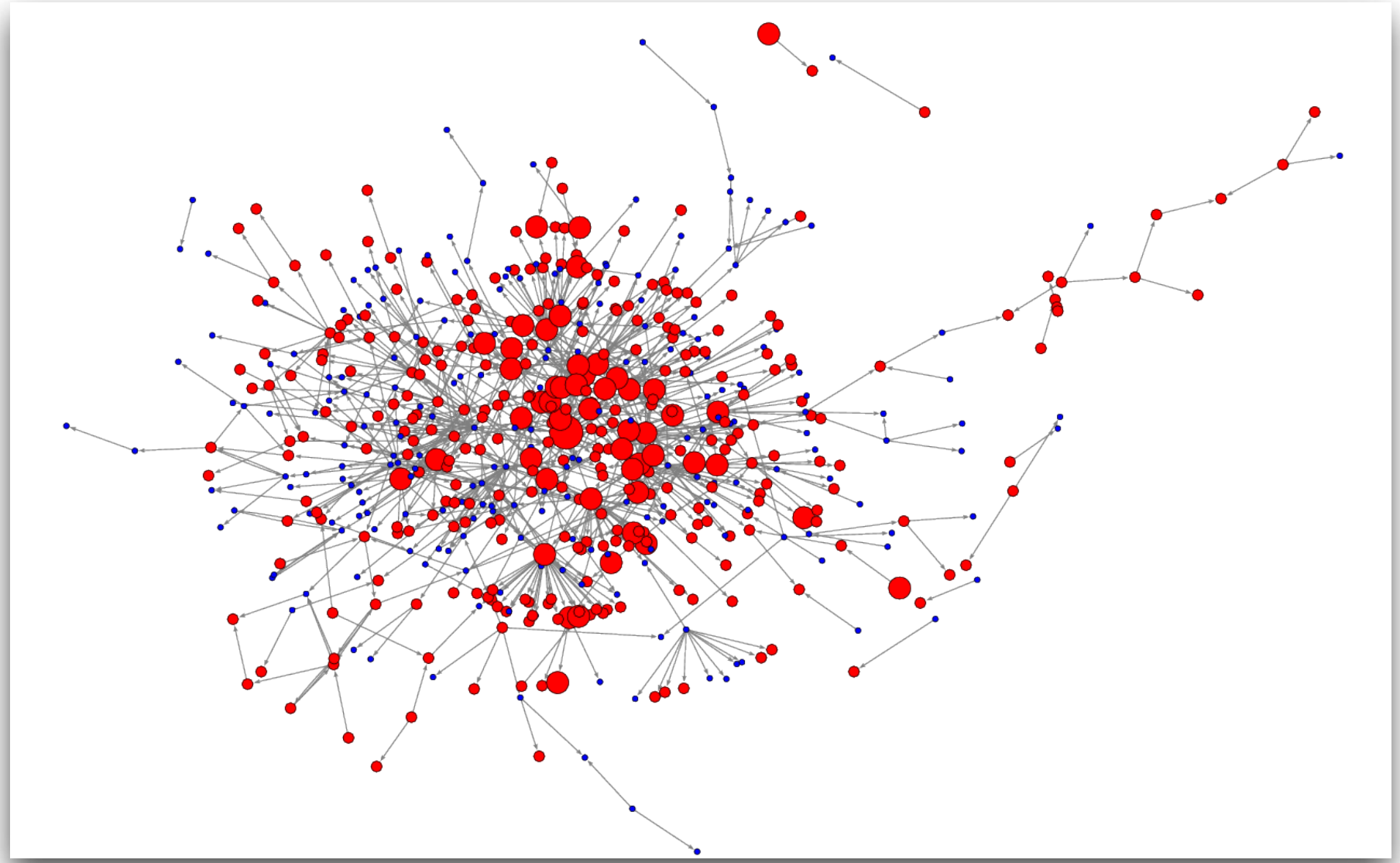
- Corresponding quantum Hamiltonian:

$$H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{(ij)} W_{ij}(c_i^\dagger c_j + \text{h . c.}), \quad \epsilon_i = -D_{ii}$$

- Similar to Anderson model on network, widely studied e.g. for Many-Body Localisation
- Disorder also in hopping terms, correlated with on-site terms
- For now, ignore correlations



Network model & disorder



Network model & disorder

- Focus: Effects of network topology
  Classically: e.g. from kinetic constraints in real space
  Quantum: Fock space connectivity

- Network model: fix distribution p_k of degrees $k_i = \sum_j a_{ij}$

- Network otherwise random: configuration model

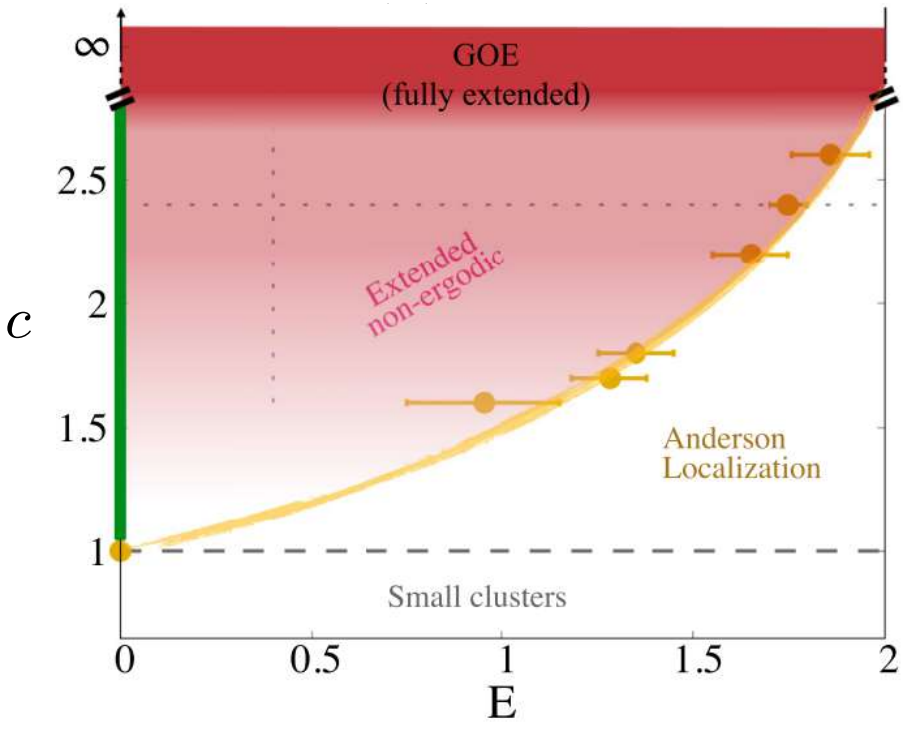
- Connectivity $c = \langle k \rangle =$ average degree

- Weights: Gaussian with zero mean and $\langle W_{ij}^2 \rangle = 1/c$

Topological heterogeneity

- Study limit of high connectivity $c = \langle k \rangle$
  Why? Physical glass in volume V has $N \sim e^V$, $c \sim V \sim \ln N$
- Degree distribution p_k? For Erdös-Renyí graphs, Poisson distribution, relative variance $c/c^2 \to 0$ for $c \to \infty$
- Heterogeneity effects disappear for $c \to \infty$, effectively random regular graph

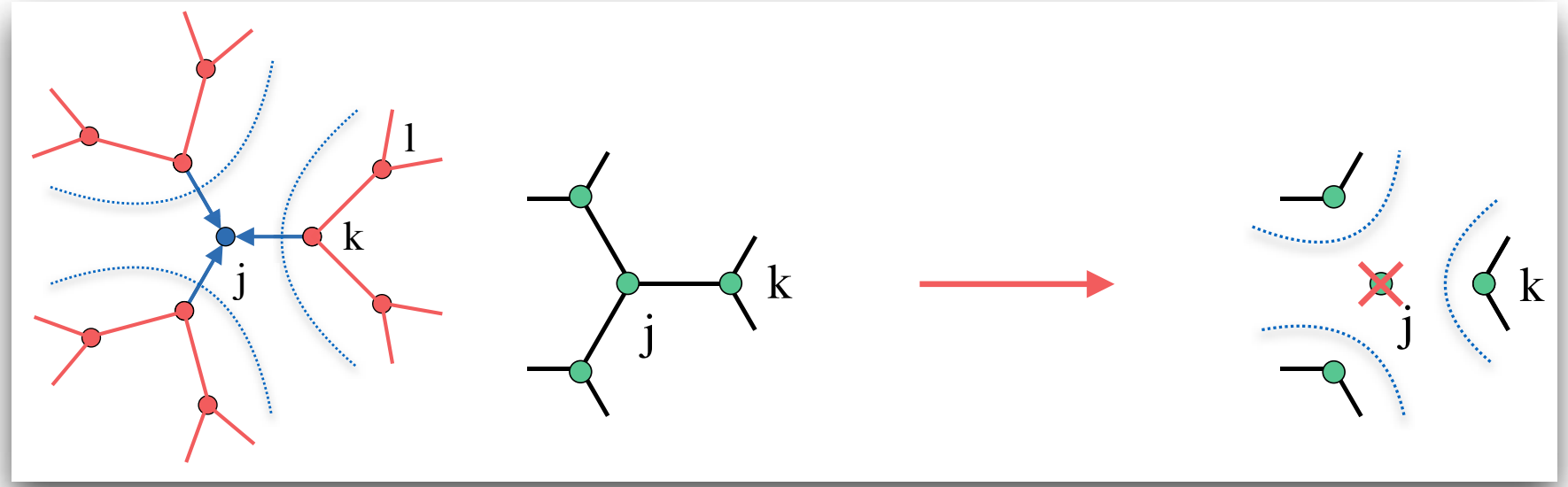
(Cugliandolo Schehr Tarzia Venturelli 2024)



Topological heterogeneity

- Alternative choices for degree distribution p_k?
- Negative binomial distribution:
  scaled degrees $\kappa = k/c$ have Gamma-distribution $\nu(\kappa) \propto \kappa^{\gamma-1} e^{-\gamma\kappa}$
- Variance $1/\gamma$ (relative to mean $\langle\kappa\rangle = 1$) does not vanish for $c \to \infty$
- Small $\gamma \Leftrightarrow$ heterogeneous networks
- Study spectral density $\rho(E) = \langle \mathrm{Im}\, G_{ii} \rangle / \pi$ of eigenvalues E of H
- Local density of states $\mathrm{Im}\, G_{ii}$ from resolvent $G = (z - H)^{-1}$,
  with $z = E - i\eta$

Cavity method

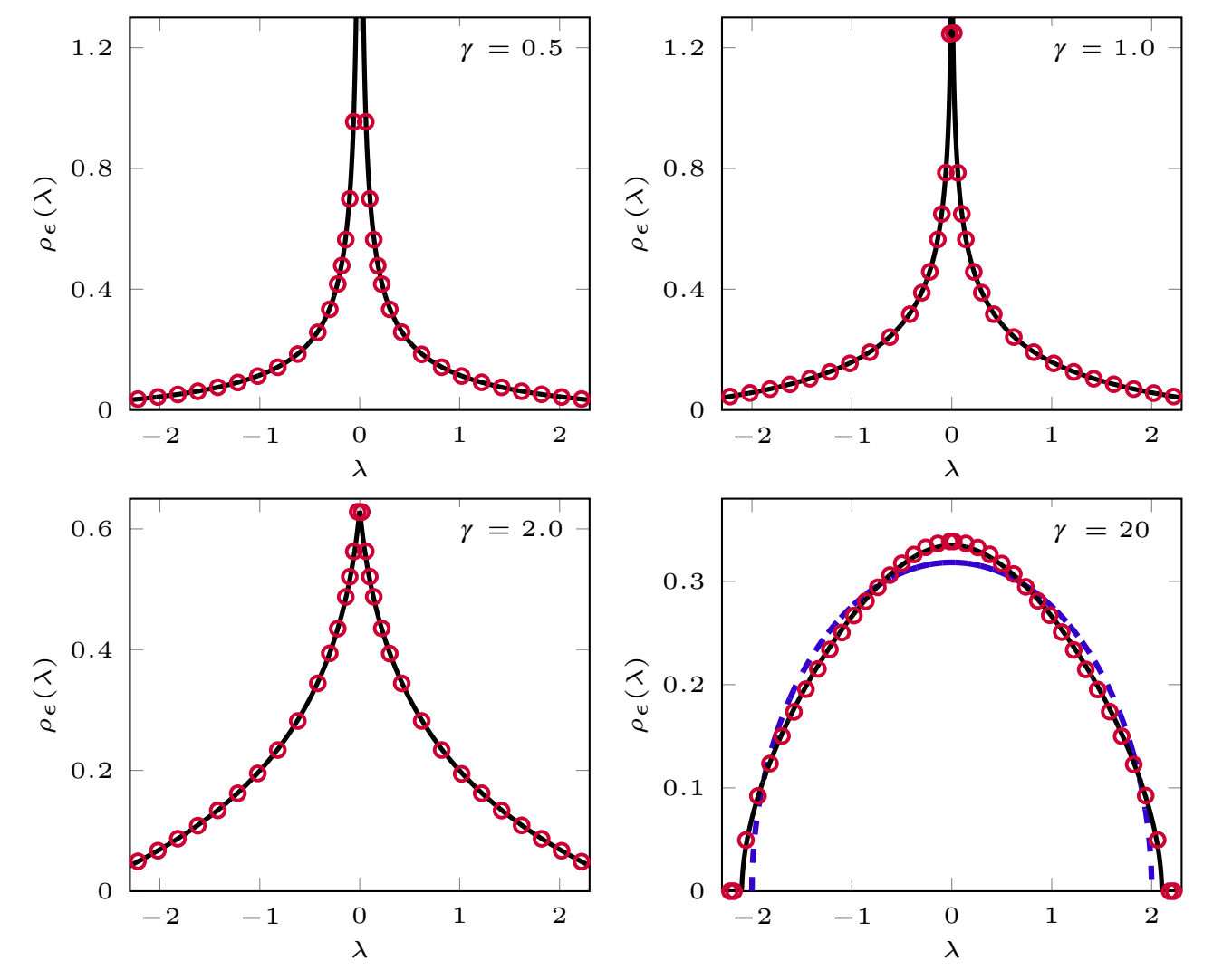


- Network becomes locally tree-like for $N \to \infty$, can use cavity method to get recursive equations for cavity resolvent entries $G_{ii}^{(j)}$
- Large c: self-consistent equation for average of $G_{ii}^{(j)}$
- Limits: $\lim_{\eta \to 0} \lim_{c \to \infty} \lim_{N \to \infty} \dots$

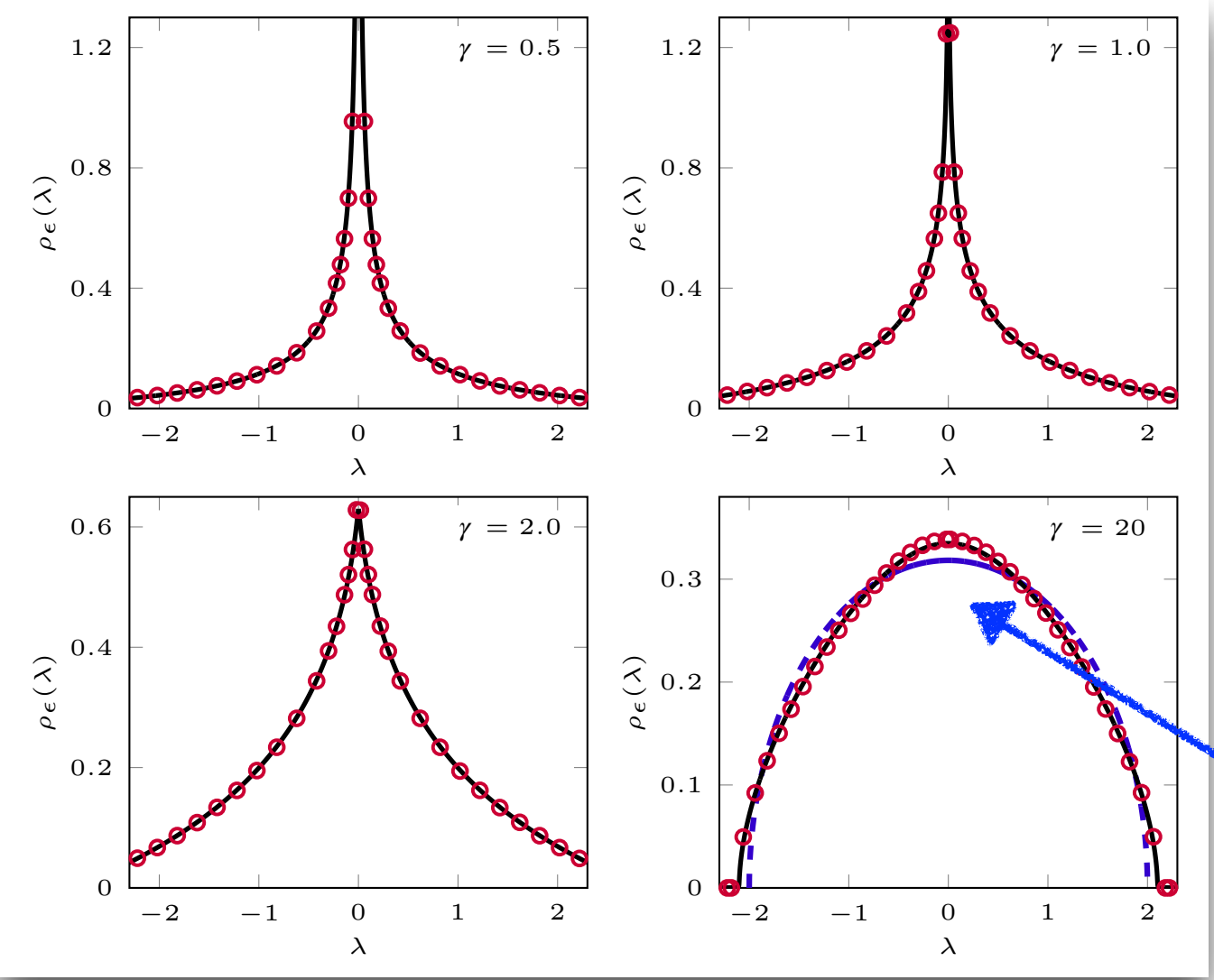
Part 1: No on-site disorder

$$H = \sum_{(ij)} W_{ij}(c_i^\dagger c_j + \text{h . c.})$$

Spectral density

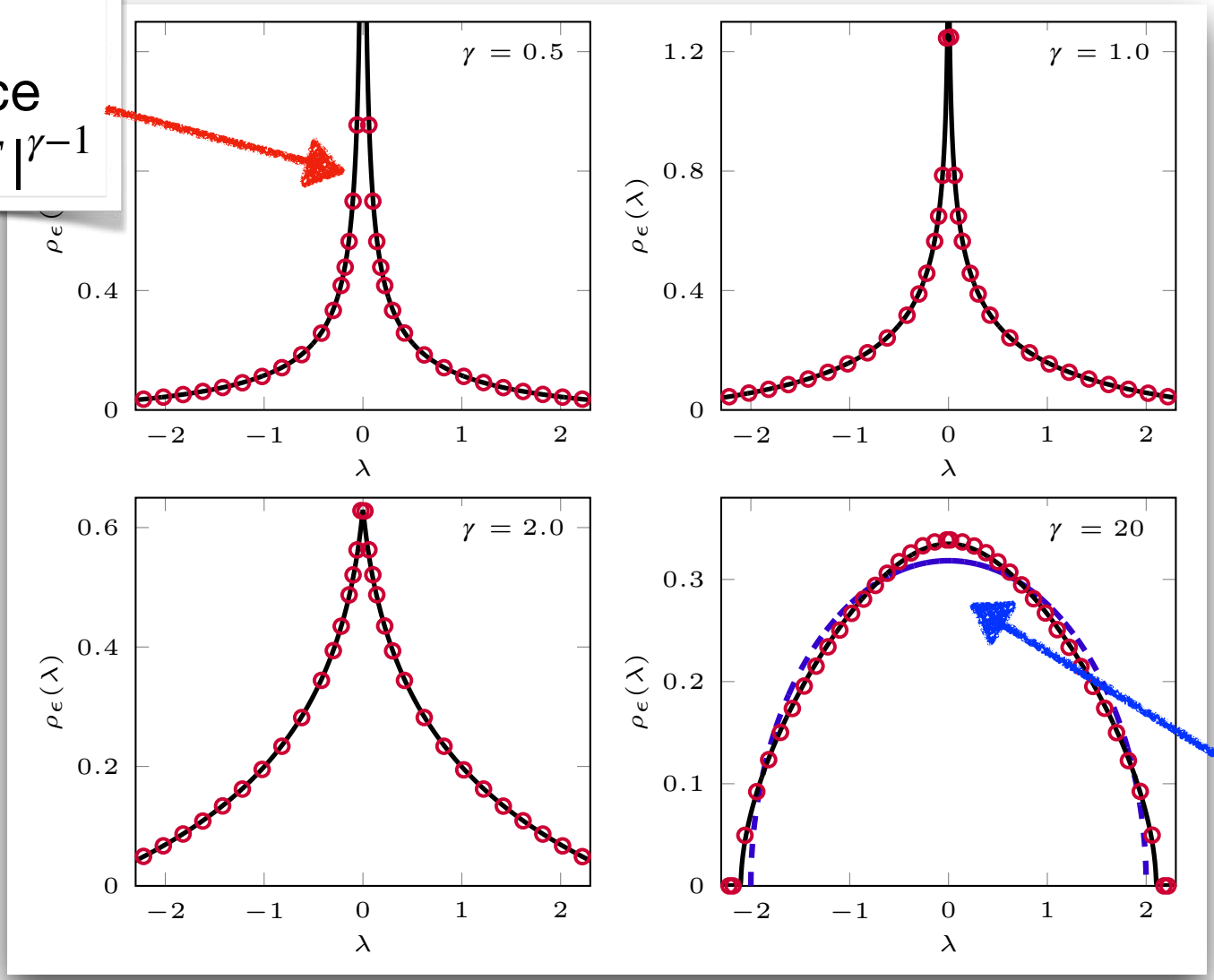


Spectral density



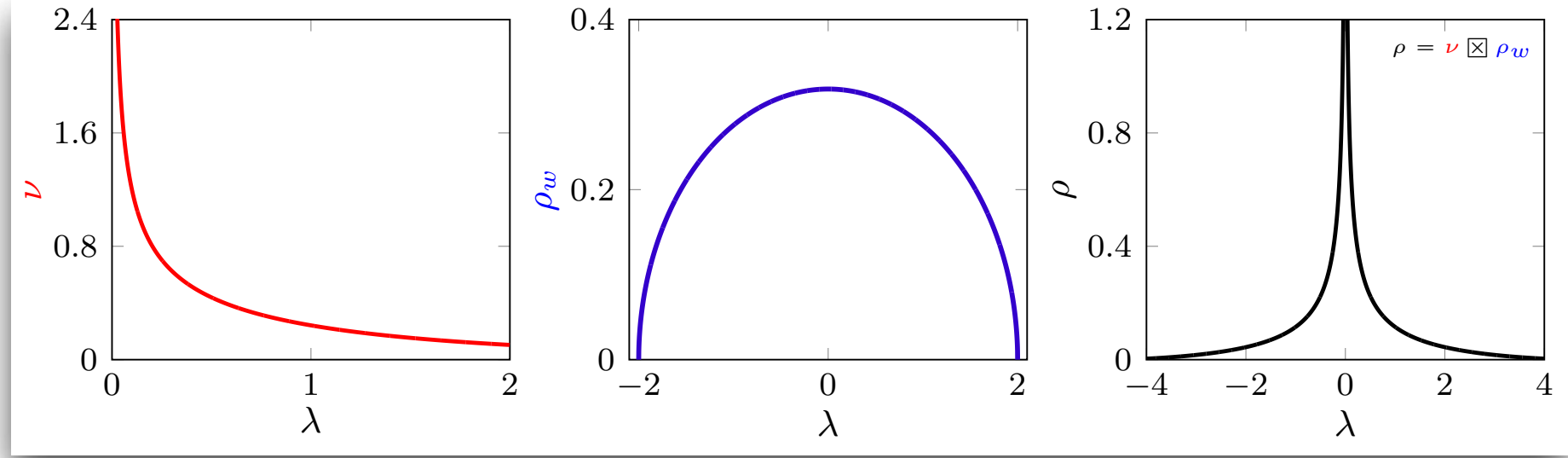
Spectral density

Spectral divergence
$\rho(E) \sim |E|^{\gamma-1}$



Wigner

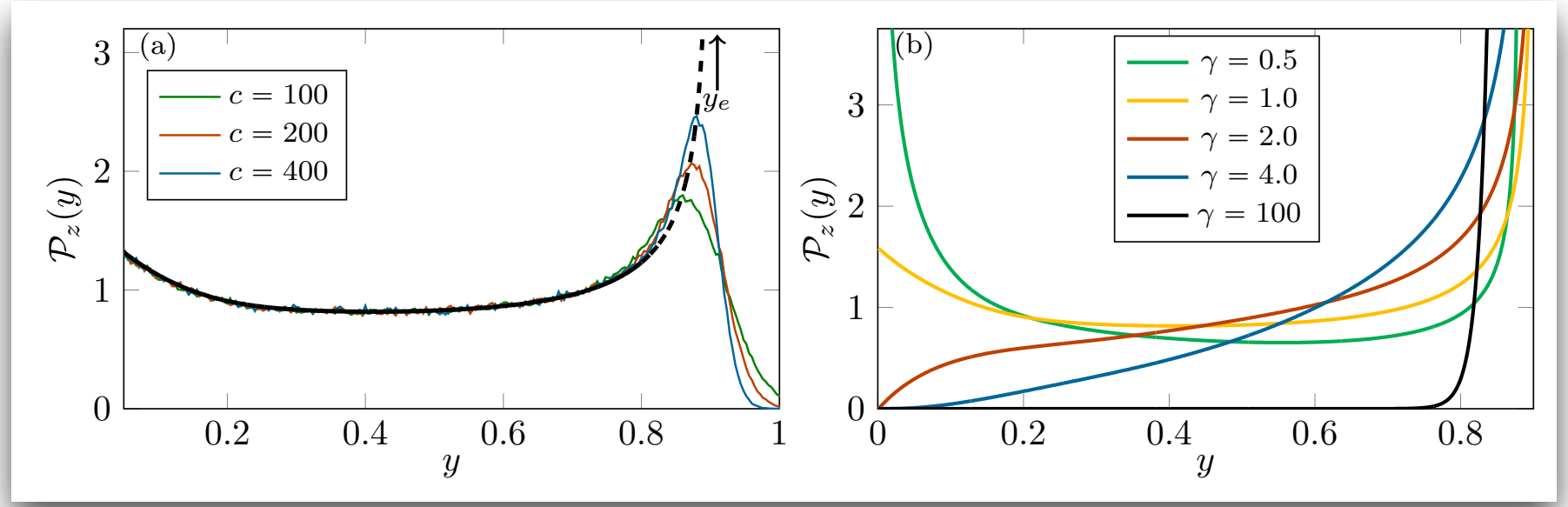
Free multiplication



- Can show: spectral density = free multiplicative convolution (Dembo+ 2021)
- Effective decomposition: $H = D^{1/2} W D^{1/2}, D_{ii} = \kappa_i$

Local density of states

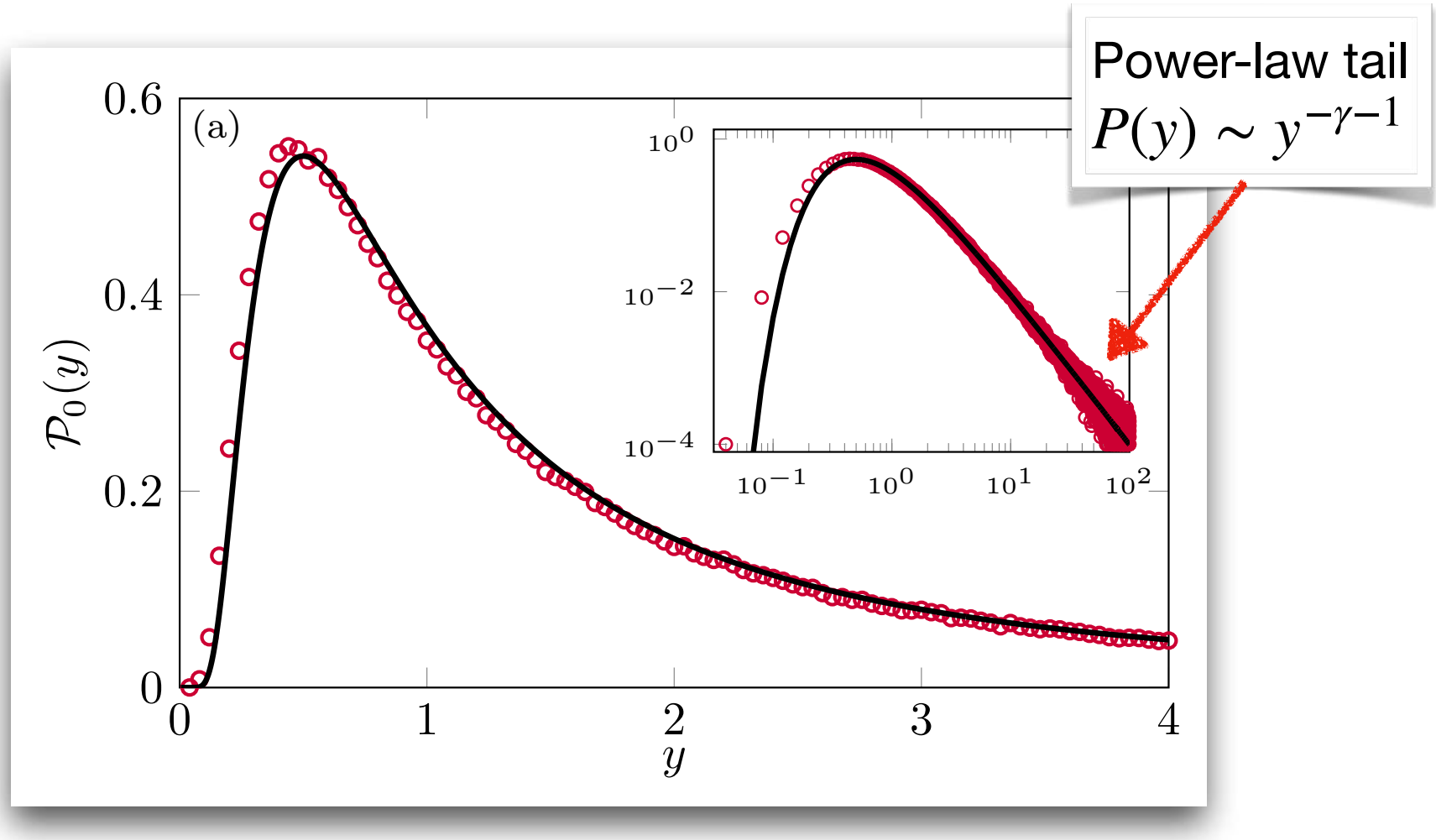
Distribution of $y_i = \mathrm{Im}\, G_{ii}$ at $E \neq 0$



- Distribution of y has finite support
- Eigenvectors are extended

Local density of states

Distribution of $y_i = \mathrm{Im}\, G_{ii}$ at $E = 0$



Multifractality

- Characterise structure of eigenvectors
- Generalised inverse participation ratio (IPR):

$$I_q(N) = \sum_i |\psi_i|^{2q} \sim N^{-\tau(q)}$$

- Multifractal exponents $\tau(q)$:
  Extended state $|\psi_i|^2 \sim 1/N$ gives $\tau(q) = q - 1$
  Localised state $|\psi_i|^2 = O(1)$ (for $O(1)$ nodes): $\tau(q) = 0$
  Single fractal dimension: $\tau(q) = d_f(q - 1)$

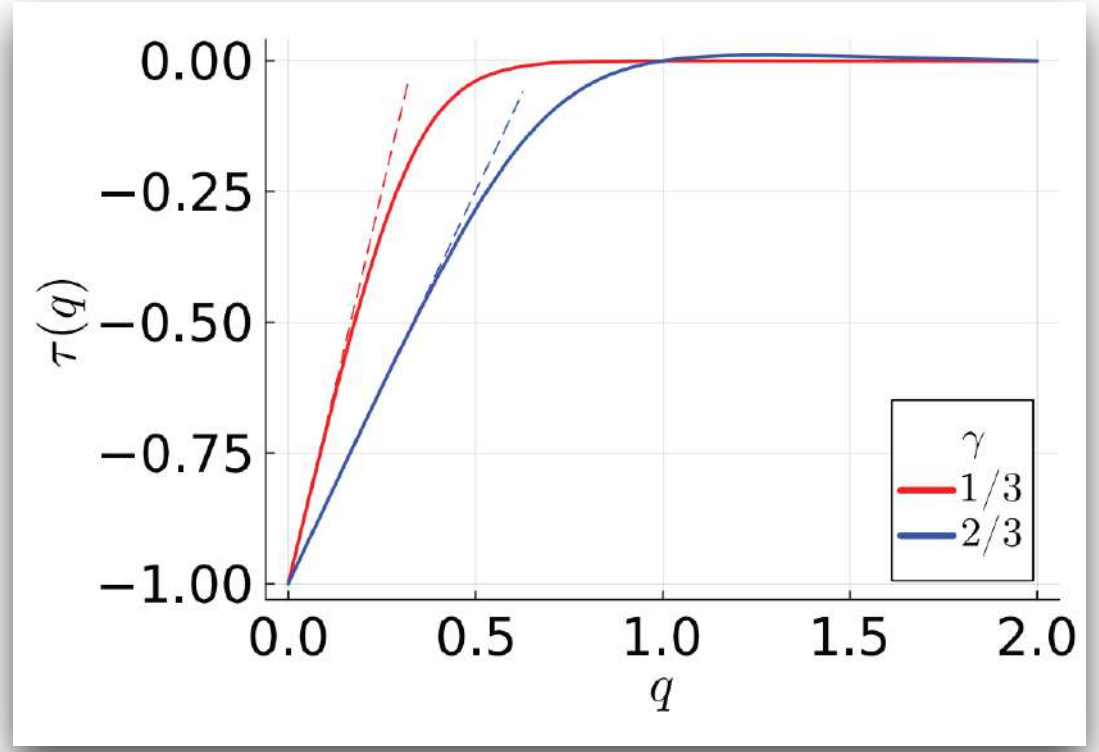
- Evaluate I_q from scaling of LDoS distribution
  ("envelope distribution", De Luca+ 2014)

Multifractal exponents

$$E = 0, \gamma < 1$$

Eigenvectors localised with multifractal features:

$$\tau = \begin{cases} q/\gamma - 1 \\ 0 \end{cases}$$



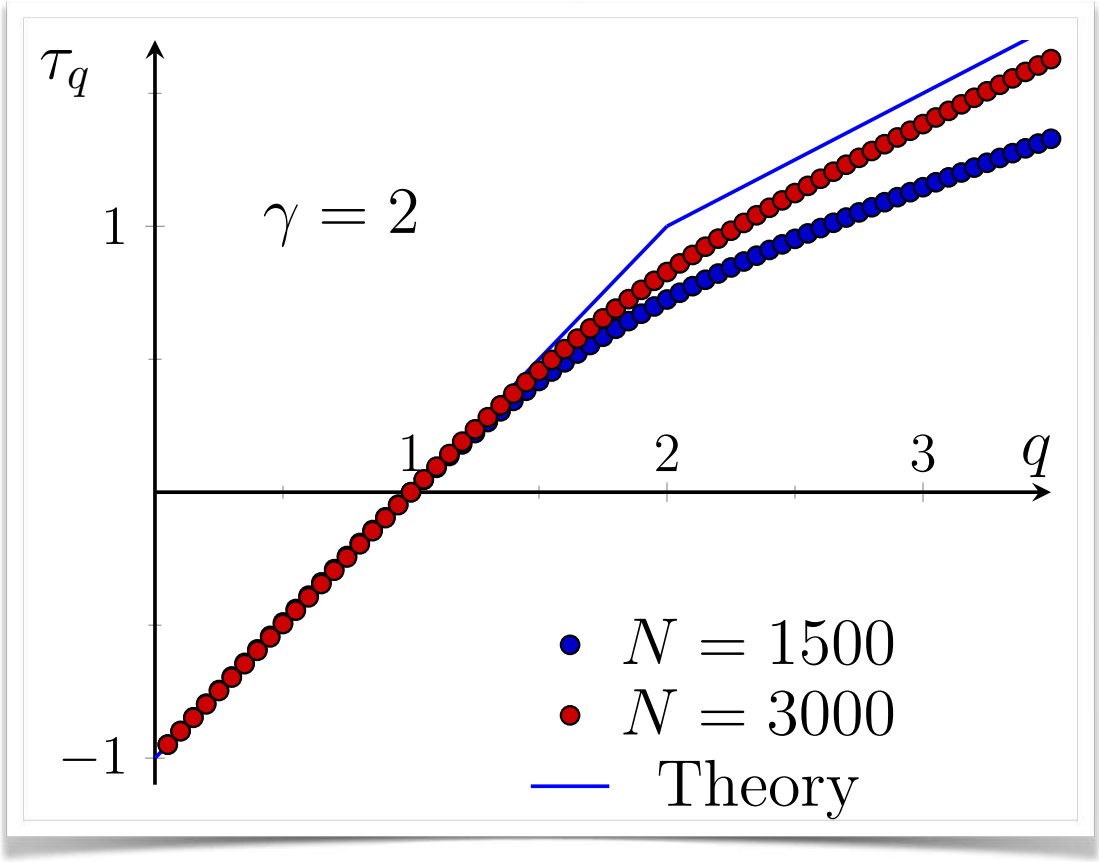
Multifractal exponents

$$E = 0, \gamma > 1$$

Eigenvectors extended non-ergodic:

$$\tau = \begin{cases} q - 1 \\ q\,(1 - 1/\gamma) \end{cases}$$

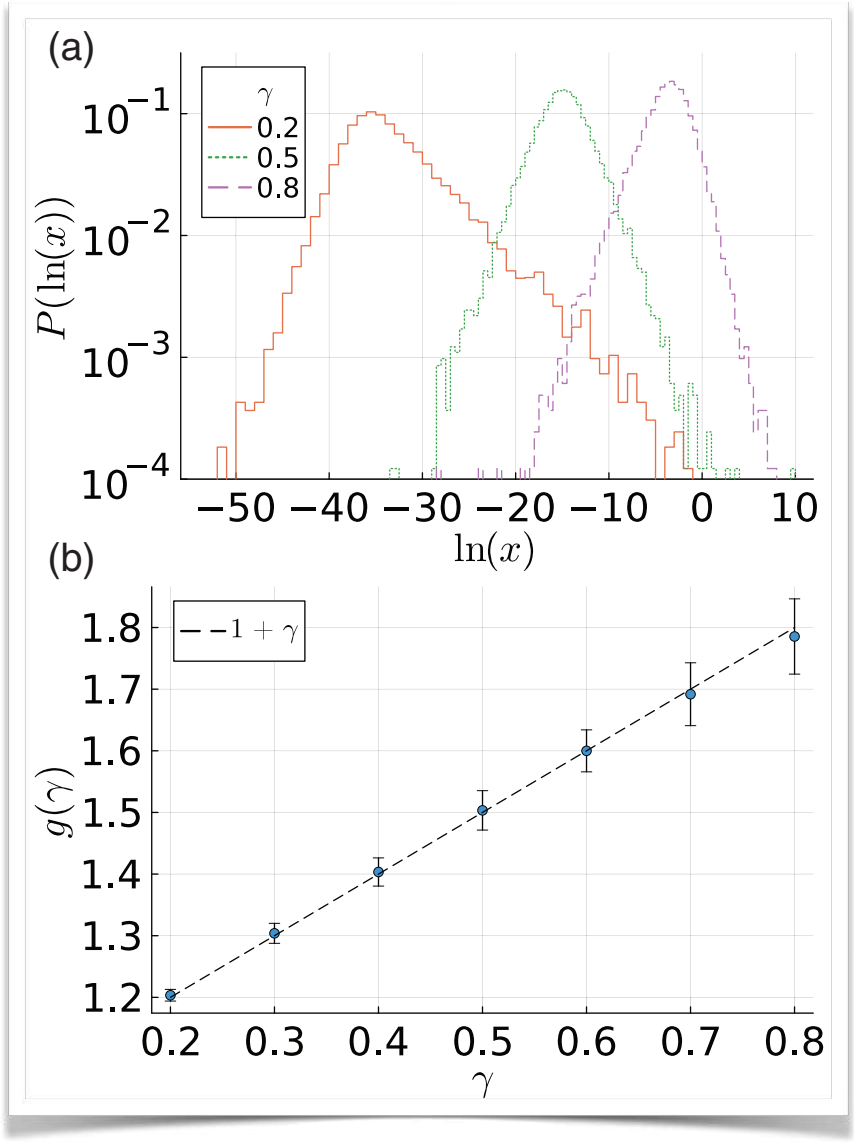
(for $q \lesssim \gamma$)



Wavefunction amplitudes

- Local density of states is
  $y_i \propto \sum_\mu |\psi_i^\mu|^2 \delta_\eta(E - E_\mu)$

- Expect similar features in distribution of squared wavefunction amplitudes
  $x_i = N|\psi_i|^2$
  (see e.g. Monthus 2017)



Statistical localisation

- What is mechanism of localisation at $E = 0$?
- Not power law in “space” on network, or competing correlation length scales
  (Kravtsov+ 2015, Monthus 2016; García-Mata+ 2020, 2022, Das+ 2025, …)
- Strong correlation between $|\psi_i|^2$ and degree k_i
- Localisation determined by statistical properties of nodes

Statistical localisation

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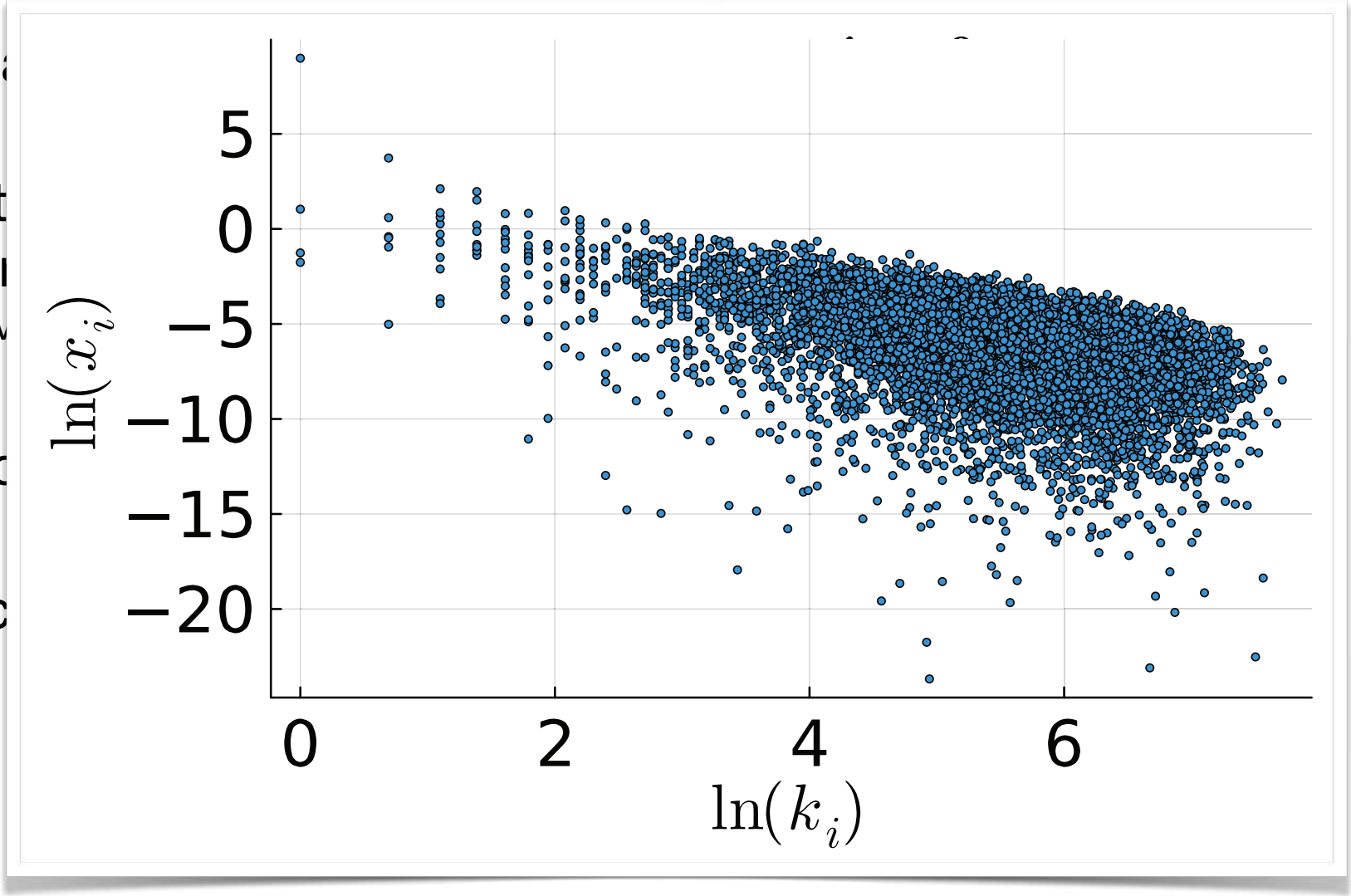
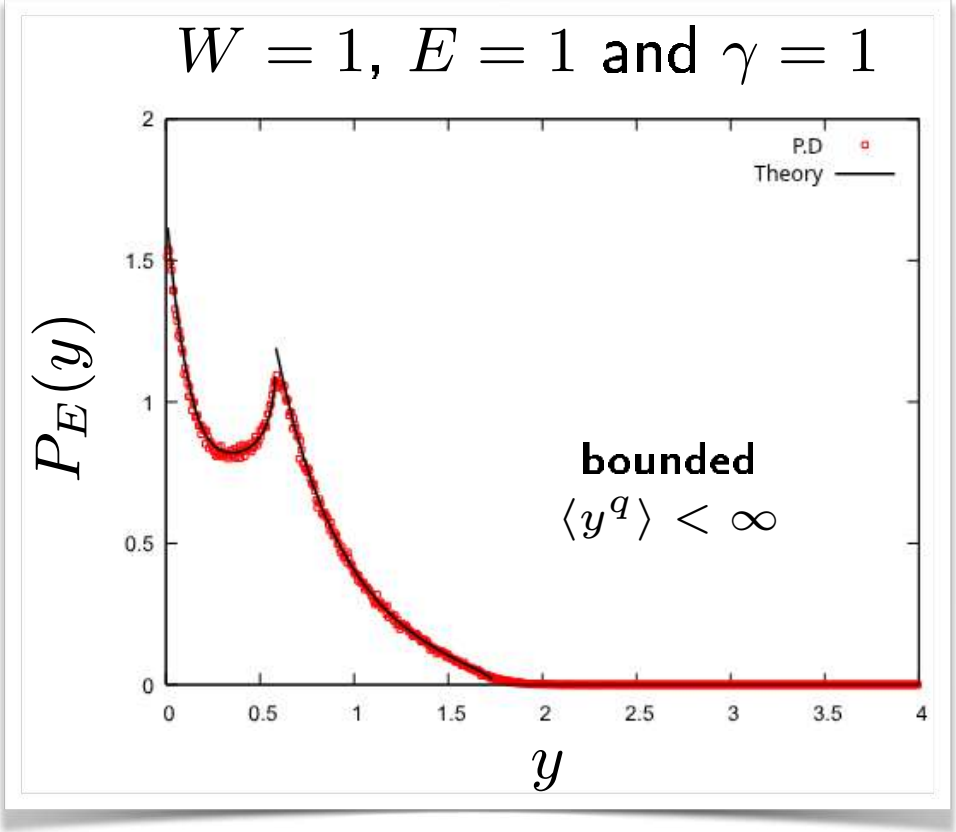


Part 2: Adding on-site disorder

$$H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{(ij)} W_{ij}(c_i^\dagger c_j + \text{h . c.}), \quad |\epsilon_i| \leq \frac{1}{2}W$$

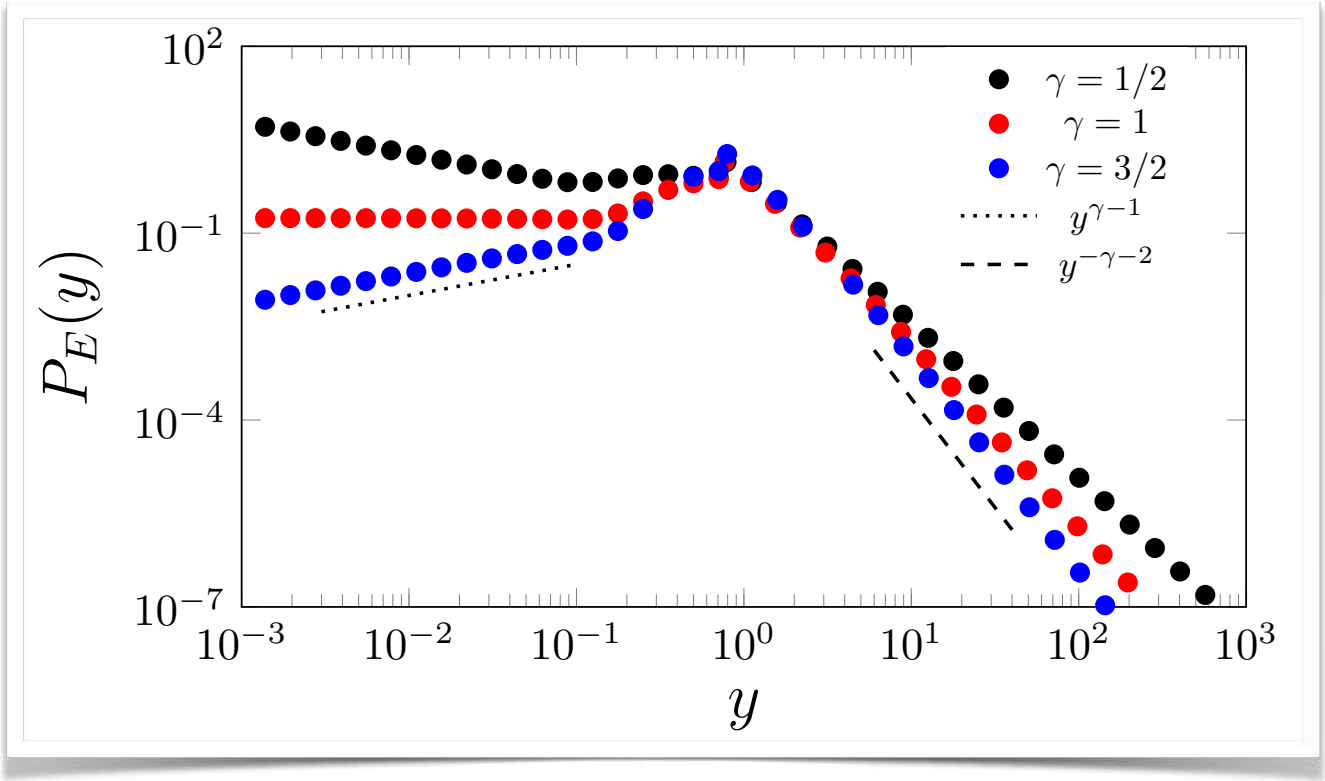
What changes?

- Spectral density $\rho(E)$: no divergences, still has unbounded support
- LDoS distribution now has large-y power law tail
  $P(y) \sim y^{-\gamma-2}$
  for all $|E| < \frac{1}{2}W$
- For $|E| > W/2$, LDoS has compact support again

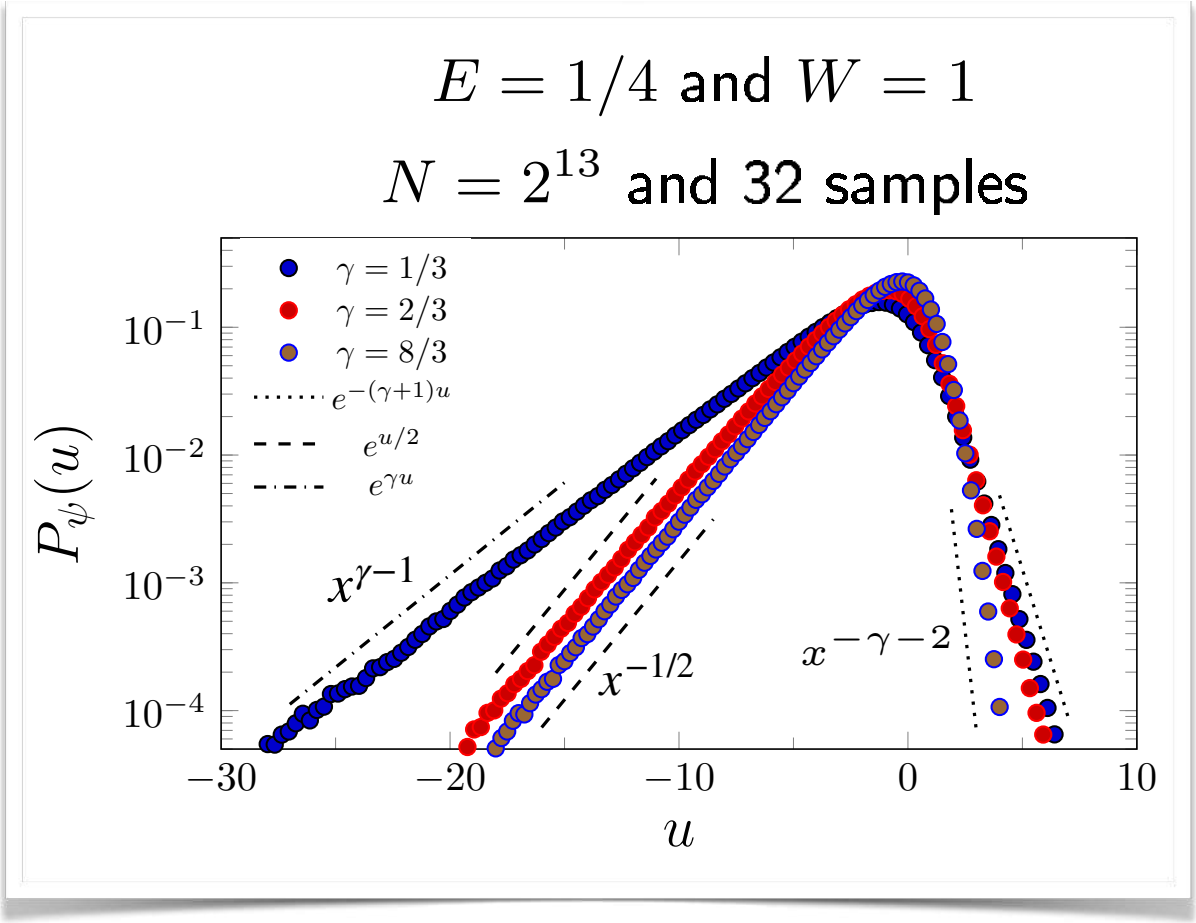


What changes?

- LDoS distribution acquires additional small-y power law tail $P(y) \sim \nu(y) \sim y^{\gamma-1}$ (for all E)

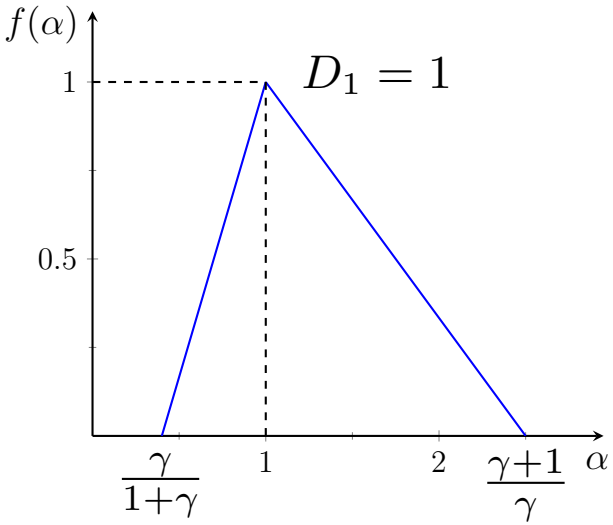


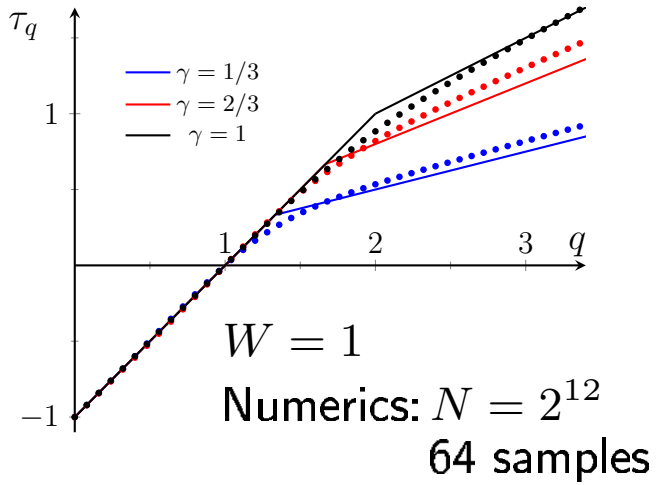
Wavefunction amplitudes



Spectrum of fractal dimensions

$P(y) \sim \nu(y)$ vs $P(y) \sim y^{-3}\nu(1/y)$: Mirlin-Fyodorov symmetry

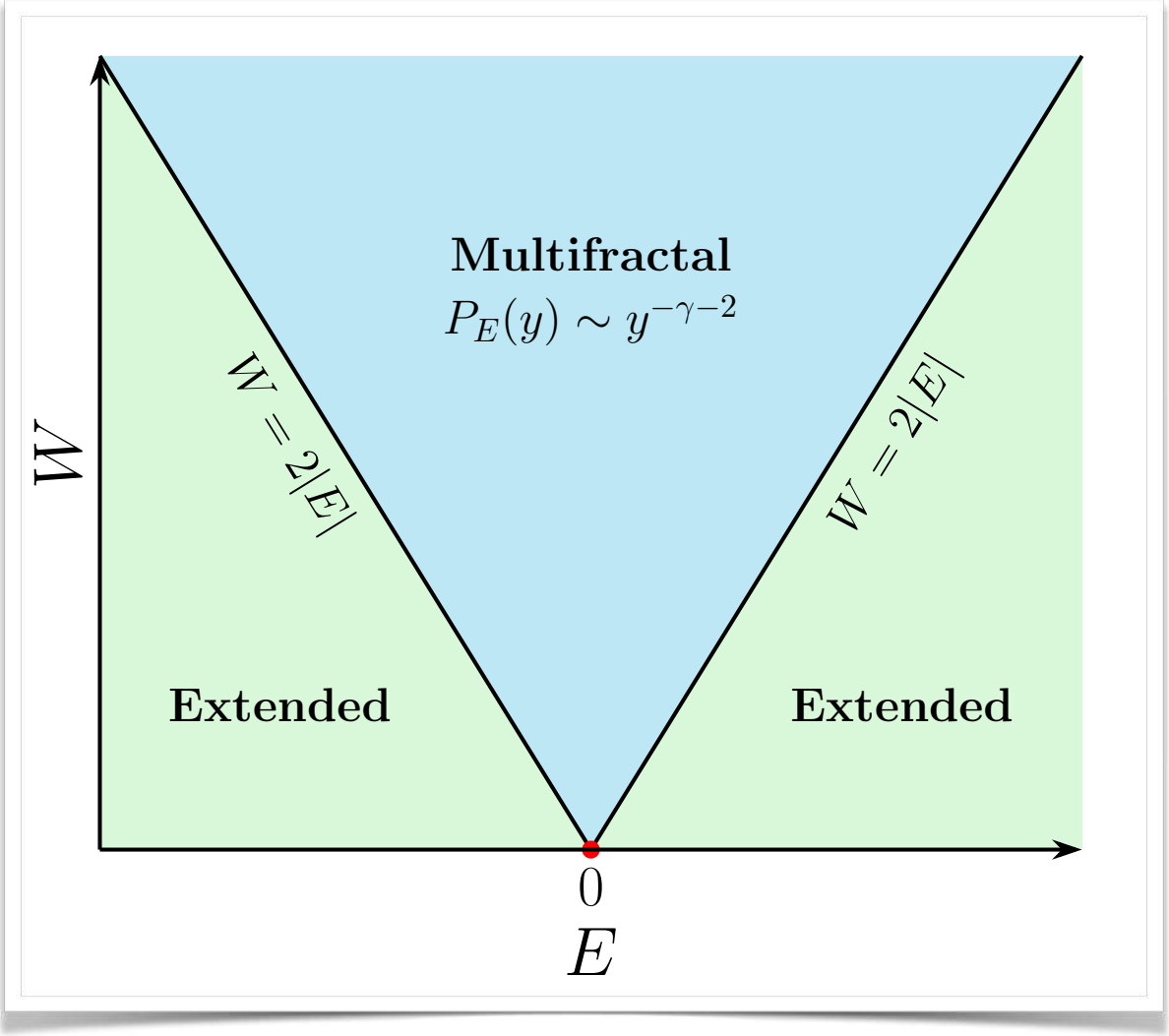




$$f(\alpha) = \begin{cases} -\gamma + (1+\gamma)\alpha & \frac{\gamma}{1+\gamma} \le \alpha \le 1 \\ \gamma + 1 - \gamma\alpha & 1 \le \alpha \le \frac{\gamma+1}{\gamma} \end{cases}$$

$$\tau_q = \begin{cases} q - 1 & q < 1+\gamma \\ \left(\frac{\gamma}{1+\gamma}\right) q & q > 1+\gamma \end{cases}$$

Phase diagram



Part 2b: Power-law on-site disorder

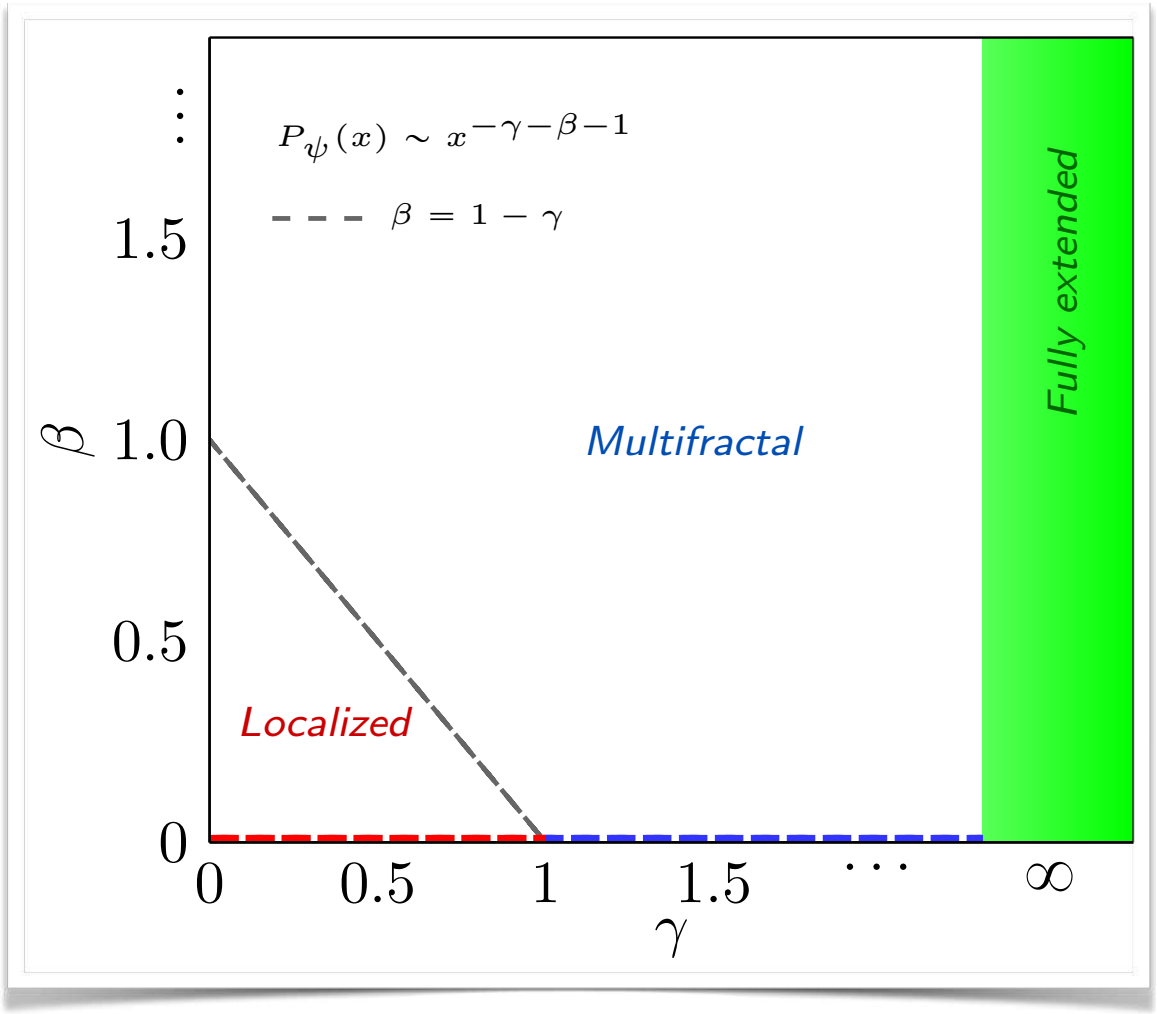
$$p_\epsilon(\epsilon) \sim |\epsilon|^{\beta-1}$$

What changes?

LDoS distribution acquires distinct large-y power-law tail for $E = 0$:

$$P(y) \sim \frac{\nu(1/y)}{y^3} p_\epsilon(1/y)$$

$$P(y) \sim y^{-(\beta+\gamma+1)}$$

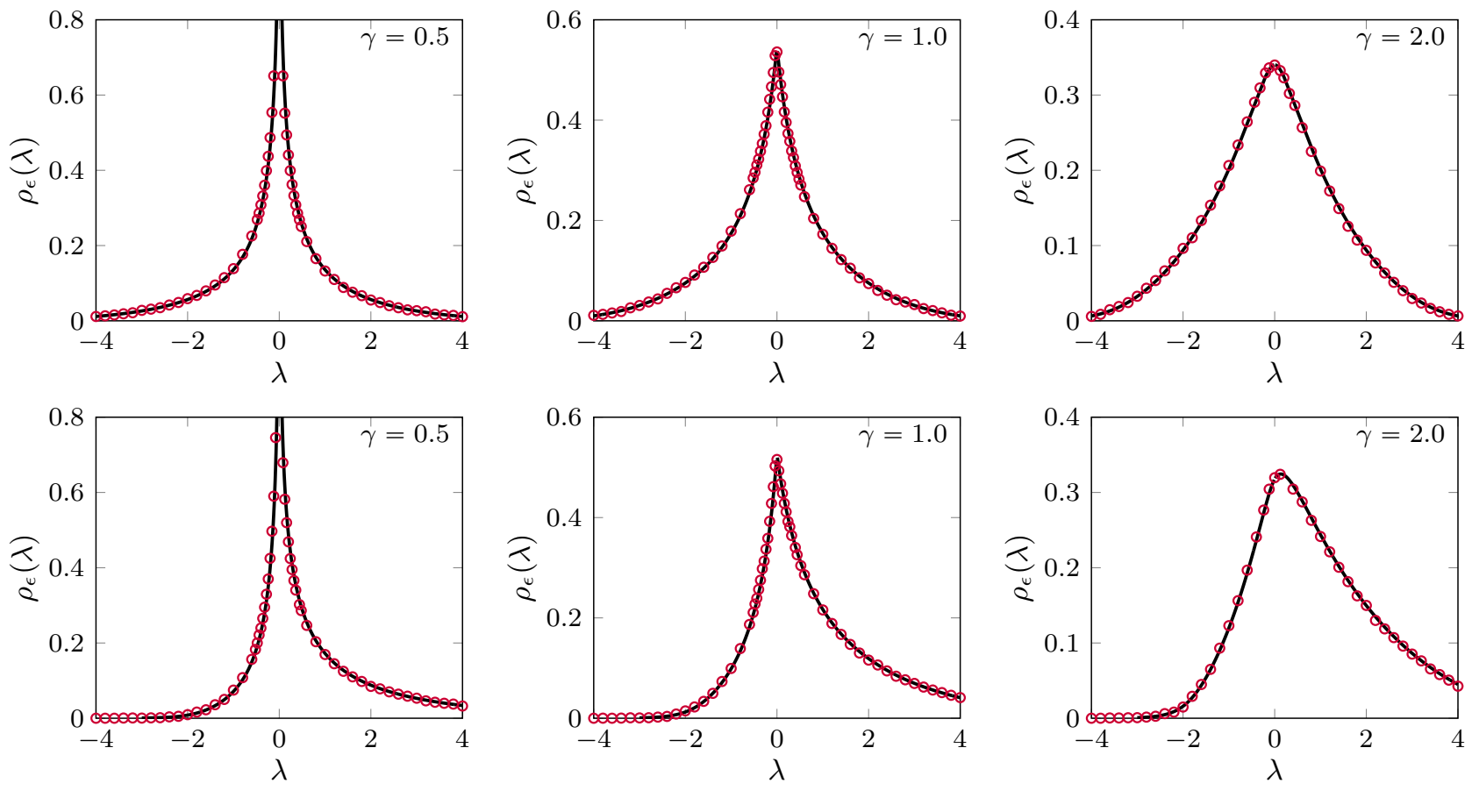


Part 3:
Graph Laplacians

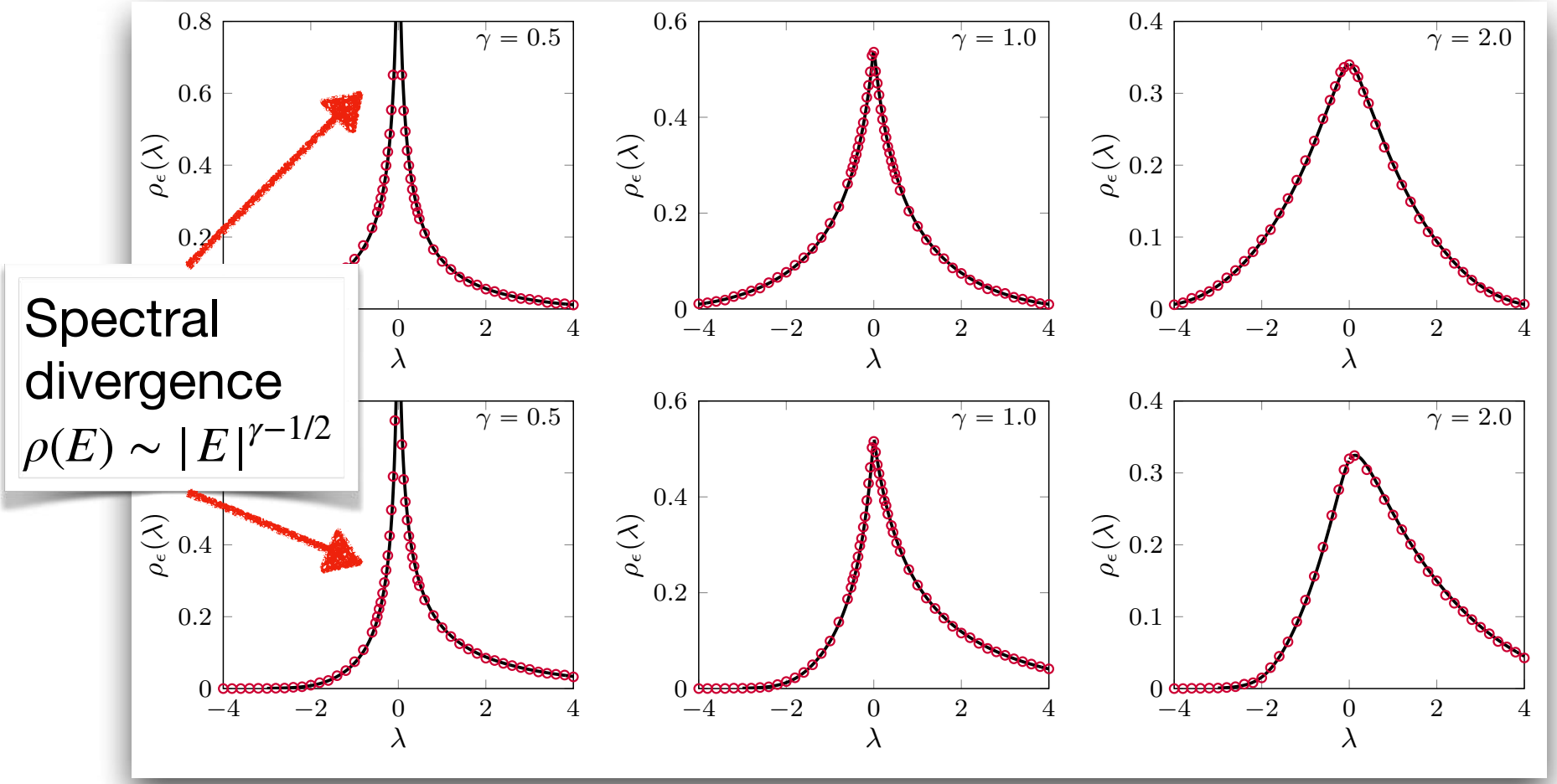
$$H = \sum_i \epsilon_i c_i^\dagger c_i + \sum_{(ij)} W_{ij}(c_i^\dagger c_j + \text{h . c.})$$

$$\epsilon_i = -D_{ii} = -\sum_j W_{ji}$$

Spectral density



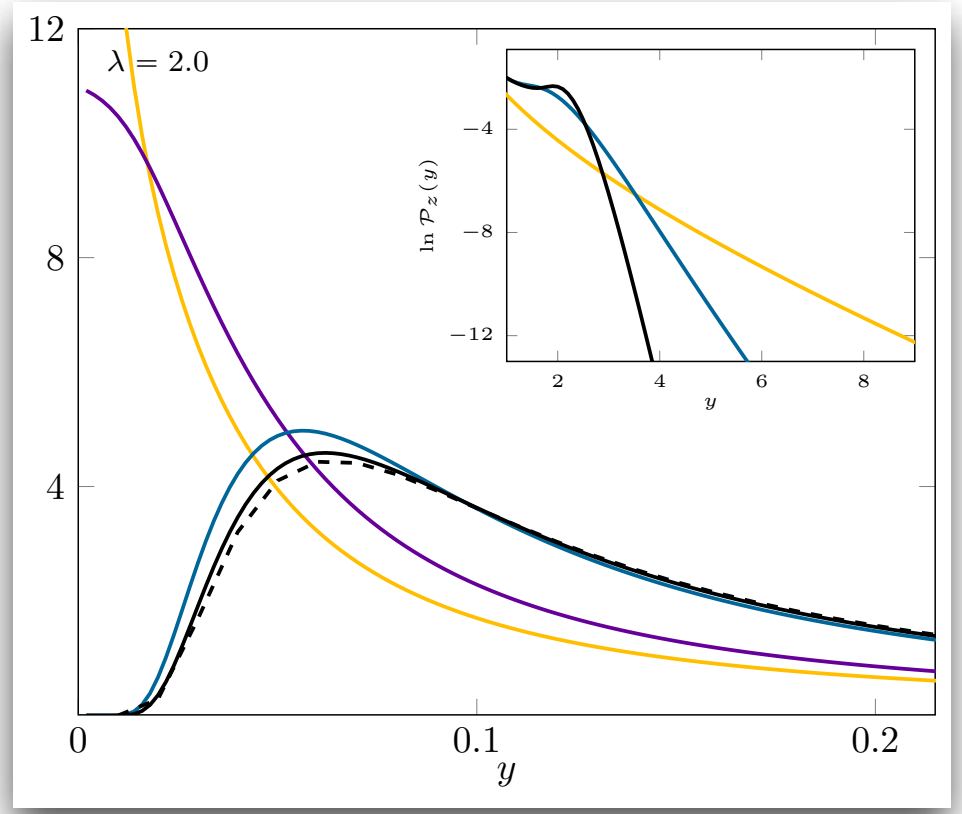
Spectral density



Local density of states

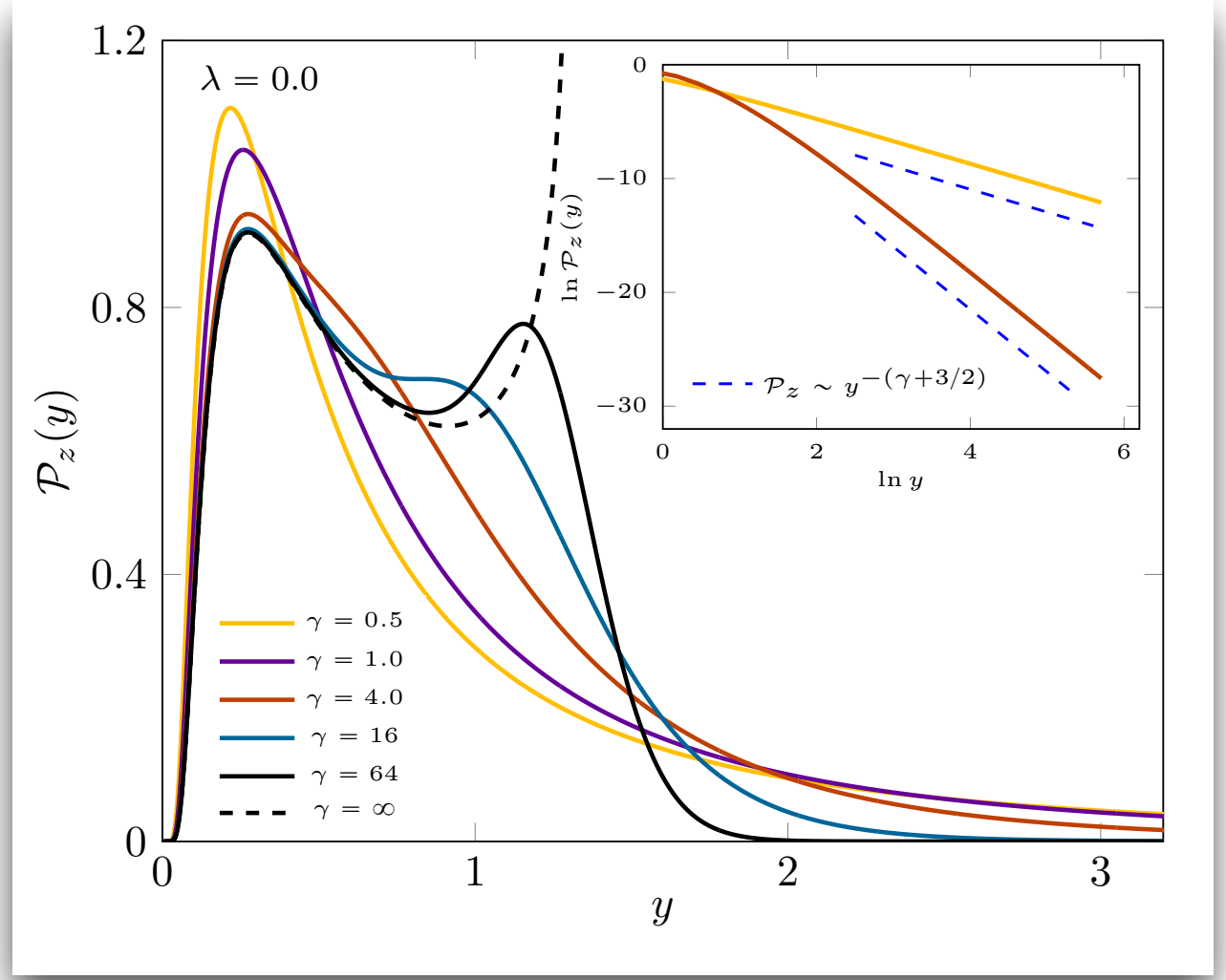
Distribution of $y_i = \operatorname{Im} G_{ii}$ at $E \neq 0$

- Distribution of y has exponential tail
- Eigenvectors are extended



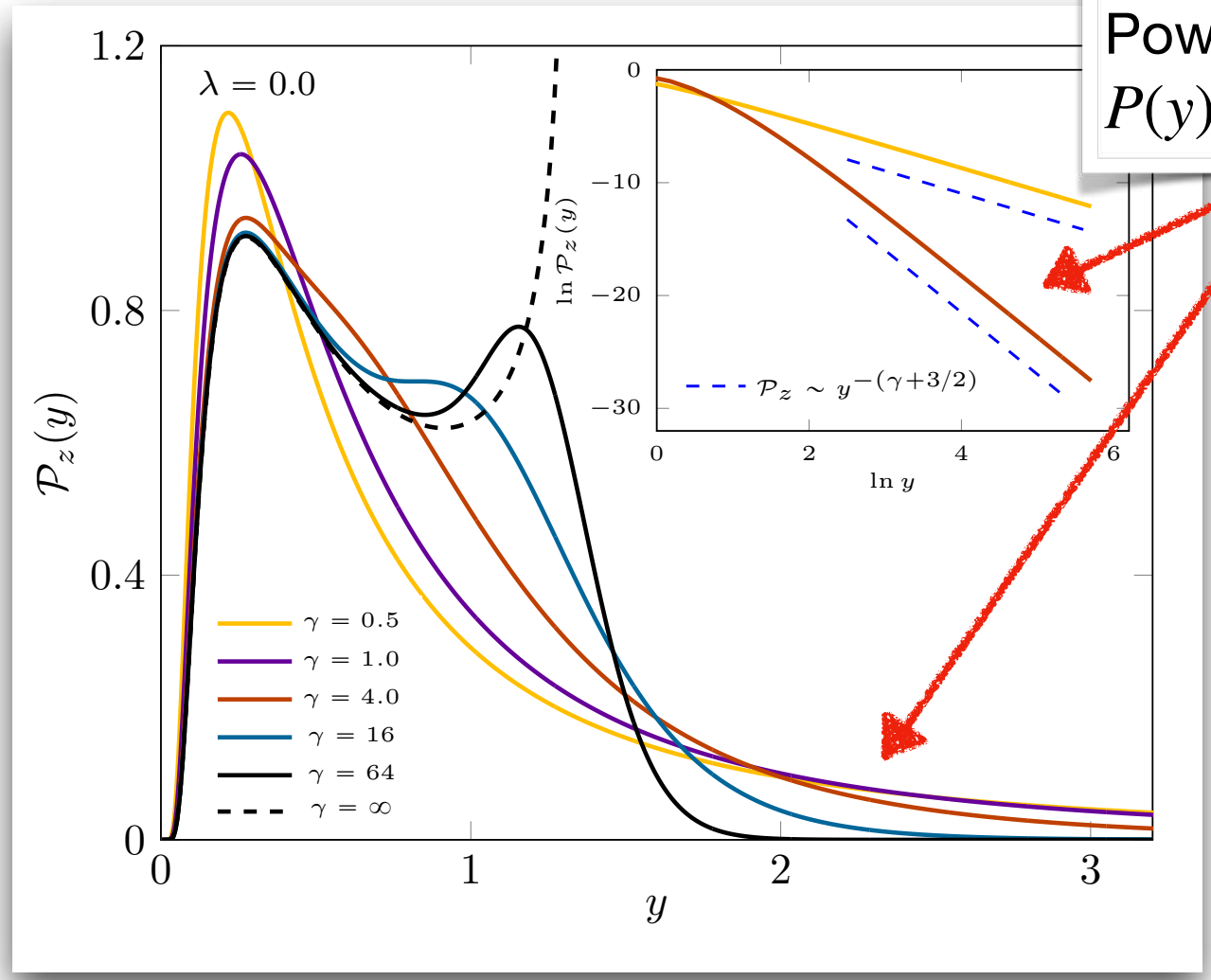
Local density of states

Distribution of $y_i = \mathrm{Im}\, G_{ii}$ at $E = 0$



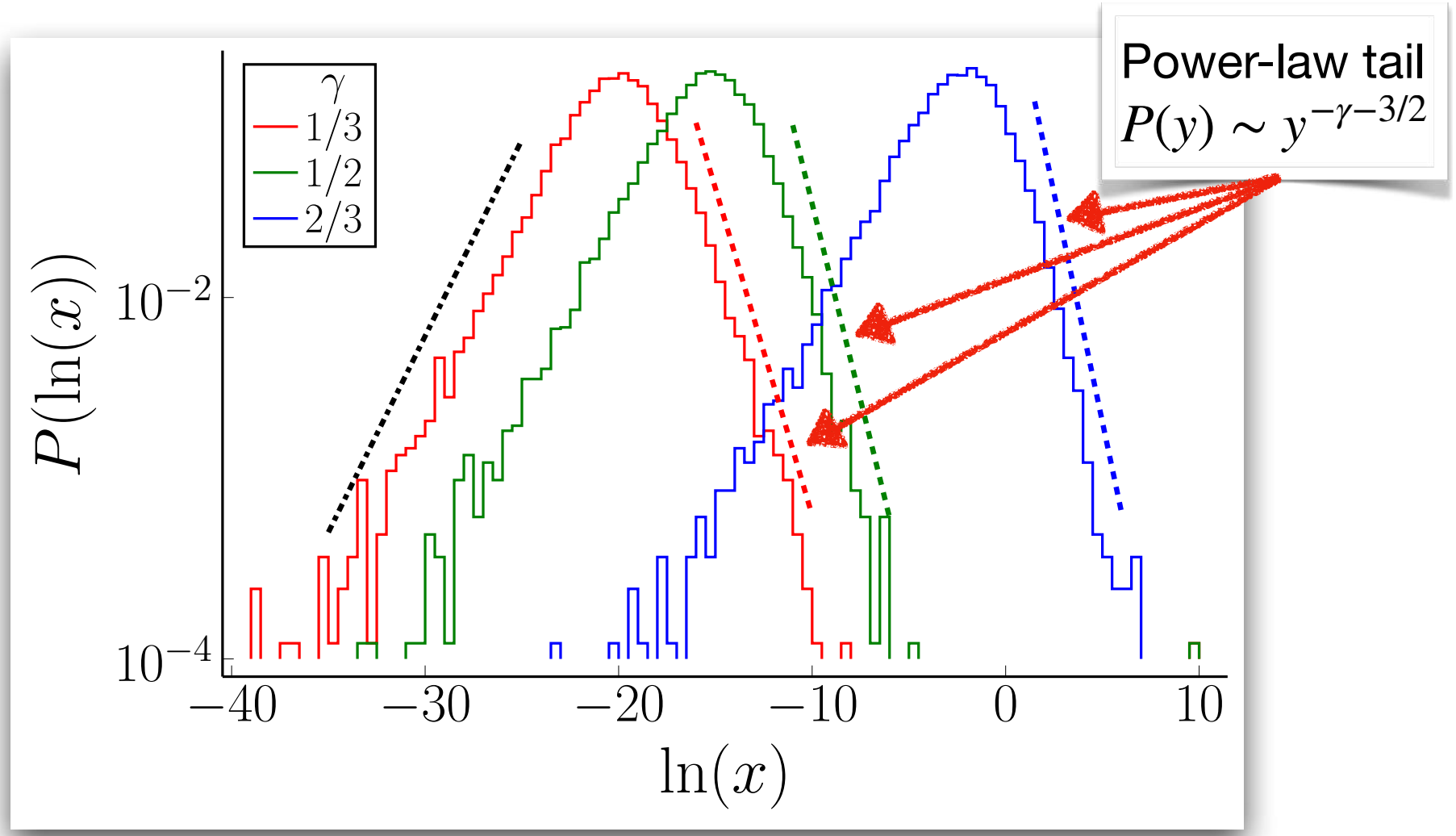
Local density of states

Distribution of $y_i = \mathrm{Im}\, G_{ii}$ at $E = 0$



Wavefunction amplitudes

Distribution of $|\psi_i|^2$ at $\lambda = 0$

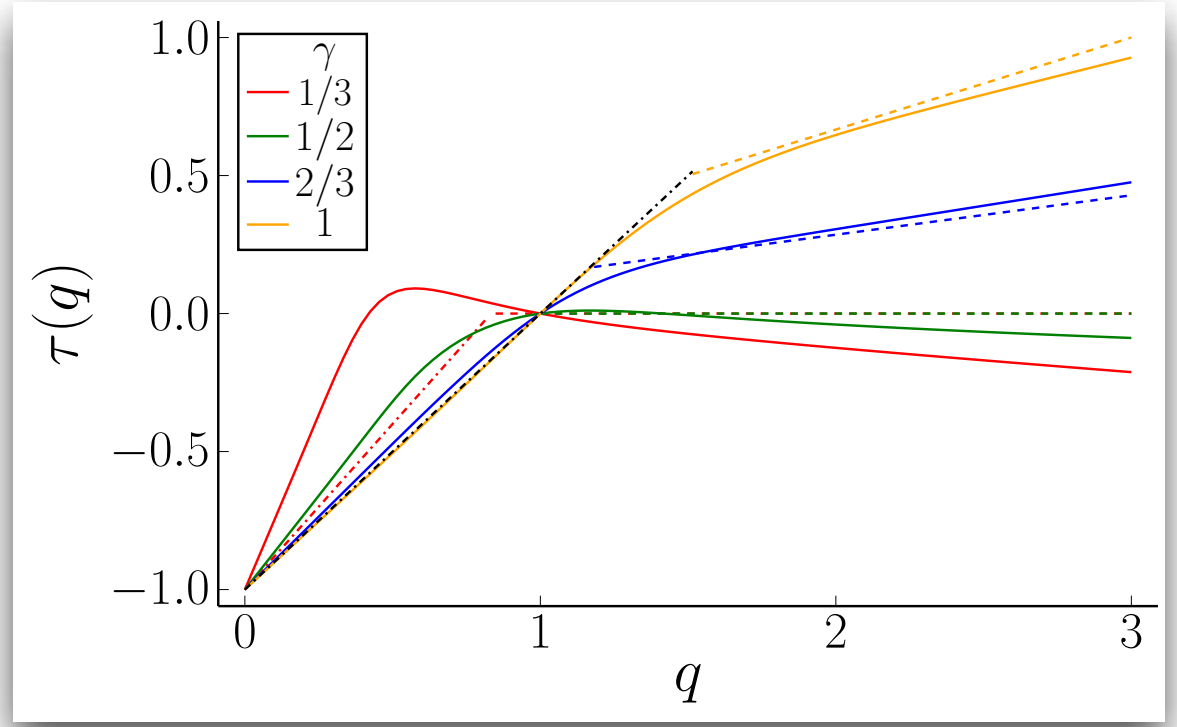


Multifractal exponents

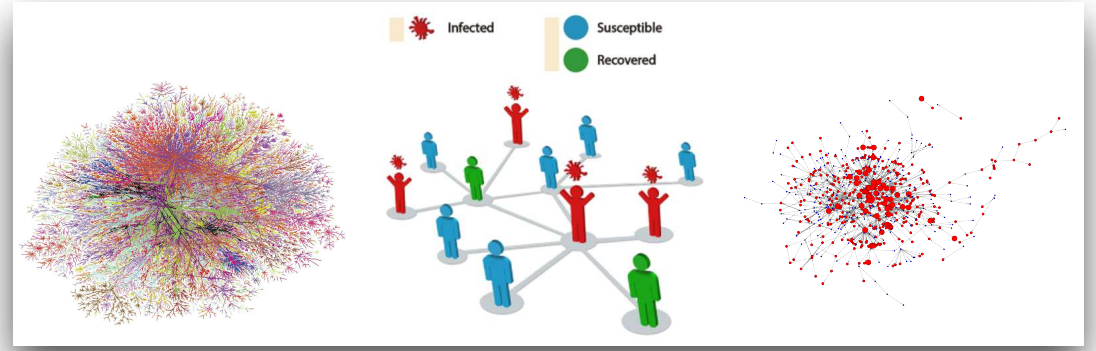
$$E = 0, \gamma < 1/2$$

Eigenvectors localised with multifractal features:

$$\tau = \begin{cases} \dfrac{q}{\gamma+1/2} - 1 \\ 0 \end{cases}$$



For $\gamma > 1/2$, extended non-ergodic: $\tau = \begin{cases} q-1 \\ q(\gamma-1/2)/(\gamma+1/2) \end{cases}$

Summary & Outlook

- Heterogeneous networks can “break” Wigner law even for $c \to \infty$
- Localisation transitions, multifractal eigenvectors, with on-site disorder for $|E| < W/2$, else at spectral divergence
- Nontrivial differences between A and L ($\gamma \to \gamma + \frac{1}{2}$), connections to classical trajectory biasing?
- Statistical rather than spatial localisation, can occur in other models on deep minima
- Origin of M-F symmetry?
- “Strong” multifractality?